REPLY TO: "COMMENTS ON 'ON THE FORMATION OF ENERGY SPECTRA IN SYNCHROTRON SOURCES'" BY E. TADEMARU, C. E. NEWMAN AND F. C. JONES

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Tademaru, Newman and Jones (1971), hereinafter referred to as paper TNJ, raise a number of objections to the results of Melrose (1969), hereinafter referred to as paper M. In particular it is argued in paper TNJ that the solution (M2) is unphysical because it contains a flux in energy space. The boundary conditions leading to the solution (M2) are not clearly stated in paper M and the physical reasoning behind these boundary conditions is omitted. Here we rectify these omissions.

The objections raised to the solution (M2) in paper TNJ are dependent on the assumption that the only physical processes involved are those whose effects are included in Equation (M1). In fact the boundary conditions imposed to obtain (M2) require the existence of other physical processes whose effects are simulated by the boundary condition. The boundary conditions imposed in (M2) require that some other acceleration mechanism leads to a flux in energy space from $E<E_l$. The boundary condition imposed to obtain the solution (M2) includes the requirement that in the stationary state there be no flux in energy space. We discuss the reasons for appealing to such additional physical processes and argue that known processes are likely to lead to the effects simulated by the boundary conditions imposed to obtain (M2).

1. The Solution (M2)

Firstly let us rederive the solution (M2) and clarify the boundary condition imposed. Equation (M1) can be written in the form

$$\frac{\partial}{\partial t} N(E, t) + \frac{\partial}{\partial E} \boxed{F}(E, t) = 0, \tag{1}$$

where

$$\boxed{F}(E, t) = F(E, t) + F_0(t), \tag{2}$$

$$F(E, t) = -D \frac{\partial}{\partial E}[E^{n+1} N(E, t)] - (SE^2 - AE^n) N(E, t). \tag{3}$$

The quantity $\boxed{F}(E, t)$ is the flux in energy space. The flux contains an arbitrary time-dependent constant $F_0(t)$. Formally the time-dependent constant can arise from the

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boundary conditions imposed. In practice any boundary condition which leads to such a term must simulate the effect of physical processes not included in (M1).

The general stationary solution of (1) follows from the solutions of

$$F(E) = 0, \qquad F(E) = -D, \tag{4}$$

and are given by

$$N_1(E) = E^{-(n+1)+A/D} \exp\left[-\frac{SE^{2-n}}{(2-n)D}\right], \tag{5}$$

$$N_2(E) = N_1(E) \int^{E} \mathrm{d}x \left[x^{n+1} N_1(x)\right]^{-1}, \tag{6}$$

respectively. The general solution is

$$N(E) = C_1 N_1(E) + C_2 N_2(E), \tag{7}$$

where C_1 and C_2 are arbitrary constants. The solution (7) (for $n=1$) with $C_2=0$ is obtained by Manley and Olbert (1969).

Consider the case $n=1$. In the neighbourhood of $E=0$ (5) and (6) become

$$N_1(E) \approx E^{-2+A/D}, \qquad N_2(E) \approx -\frac{D}{D-A} E^{-1}.$$

If the total number of particles is to be finite, i.e. if we have

$$\int_0^\infty \mathrm{d}E\, N(E) < \infty,$$

then the only solution is given by (7) with $C_2=0$ and furthermore this solution is only normalizable for $A>0$. Otherwise, for example for $D>A$, the only solution is the trivial solution

$$N(E) \propto \delta(E)$$

with all the particles at $E=0$. At $E=0$ all processes included in (M1) cease to have any effect on the energy of the particles. For $D>A$, and for $A>D$ dependent on the initial conditions, this trivial solution is the formal solution to the problem as posed. To see how this unusual result arises consider any initial distribution $N(E, t=0)$. Near $E=0$ $EN(E, t=0)$ must go to zero as E goes to zero for $N(E, t=0)$ to be normalizable. Suppose that we have

$$EN(E, t=0) \propto E^{\Delta} \qquad (\Delta > 0, E \approx 0).$$

Then for $n=1$ the flux given by (3) (there are no boundary conditions and so $F_0(t)=0$) reduces to

$$F(E, t=0) \propto \left[-D(1+\Delta) + A\right] E^{\Delta} \qquad (E \approx 0),$$

where we ignore the effect of the synchrotron losses for $E \approx 0$. For $D > A$, $\Delta > 0$ this flux is always from high to low energies. There is a flux of particles towards $E = 0$ for

$$D(1 + \Delta) > A.$$

For $D > A$ this effect near $E = 0$ inevitably leads to all the particles at $E = 0$ after a sufficiently long time irrespective of what occurs at high energies. Clearly this is not directly relevant to the formation of energy spectra in synchrotron sources.

This formal difficulty is overcome in paper M by imposing the boundary condition

$$N(E) = 0 \qquad (E \leqq E_l). \tag{8}$$

This boundary condition artificially prevents particles reaching $E = 0$; of course such a boundary condition can be justified only by arguing that physical processes not included in (M1) do prevent particles from reaching $E = 0$. With the boundary condition (8), (7) reduces to

$$N(E) = \begin{cases} C_2 \left[N_2(E) - \dfrac{N_1(E)\, N_2(E_l)}{N_1(E_l)} \right] & (E > E_l) \\ 0 & (E < E_l), \end{cases} \tag{9}$$

which is identical to the solution (M2).

In paper TNJ it is suggested that this solution is unphysical because it contains a finite flux in energy space. However (9) is time-independent, normalizable (corresponds to a finite constant number of particles), conserves the total energy in particles and the boundary condition (8) includes no sources or sinks of particles. Any solution satisfying these conditions cannot contain a finite flux of particles in energy space [such a flux implies a flow from a source to a sink if there is no change in the number of particles or in the energy in particles].

The solution (9) includes a discontinuity in $\partial N(E)/\partial E$ at $E = E_l$

$$\lim_{E \to E_l} \left[\frac{\partial N(E)}{\partial E} \right] = C_2 E_l^{n+1}, \qquad \frac{\partial N(E)}{\partial E} \equiv 0 \qquad (E < E_l).$$

The flux $\boxed{F}(E)$ for $E < E_l$ is identically zero and we *define* the flux across $E = E_l$ to be zero because there is no source or sink at $E = E_l$. Thus we *define* the flux to be

$$\boxed{F}(E) = \begin{cases} F(E) + D E_l^{n+1} \lim\limits_{E \to E_l} \left[\dfrac{\partial N(E)}{\partial E} \right] & (E > E_l), \\ 0 & (E < E_l), \end{cases} \tag{10}$$

i.e. for $E > E_l$ the constant term in (2) is defined to be

$$F_0 = D E_l^{n+1} \lim_{E \to E_l} \left[\frac{\partial N(E)}{\partial E} \right] = C_2 D$$

in order for the flux across $E = E_l$ to be zero. The solution (9) then arises from the boundary conditions

$$N(E) = 0 \qquad (E \leqq E_l), \qquad \boxed{F}\,(E) = 0\,. \tag{11}$$

As already argued, for $n = D > A$ it is necessary to invoke the existence of additional physical processes to obtain any sensible solution. We now argue that the boundary condition (8) plausibly simulates the effect of physical processes which prevent the particles reaching $E = 0$.

2. Formation of Energy Spectra

The actual processes involved in the formation of the energy spectra in synchrotron sources remain unknown. Probably the only way in which these processes can be identified is by theoretical consideration of the various possible forms of the acceleration mechanisms involved and theoretical consideration of the effect such mechanisms have on energy spectra. Any combination of otherwise plausible acceleration mechanisms and initial conditions which lead to spectra of the form observed make the assumed acceleration mechanism all the more plausible. In paper M it is argued that the flat spectra observed can be explained by a combination of an acceleration mechanism of the form included in (M1) with $n = 1$ and synchrotron losses.

Let us repeat the arguments given in (M1) from the point of view of the flux in energy space and add arguments justifying the boundary condition (11) imposed above.

Particles can be accelerated to mildly relativistic energies very effectively by various forms of microturbulence in longitudinal waves. For example, Tsytovich (1966) considers acceleration by electron plasma waves and Lacombe and Mangeney (1969) consider acceleration by ion-acoustic waves. Such microturbulence is very effectively produced by counterstreaming motions and current instabilities. One expects these instabilities to arise in any region in which energy in any macroscopic form, for example, in mass motions, is available for conversion into the energy in particles. Such acceleration mechanisms correspond to $n = -1$ in (M1). Such mechanisms are effective in producing a flux of particles from low (thermal) energies to mildly relativistic energies, see Gurevich (1960) and Tsytovich (1966).

At higher energies the only effective acceleration mechanisms known are those which correspond to $n = 1$ in (M1). Suppose that at $E = E_l$ the systematic acceleration by longitudinal waves

$$\langle \mathrm{d}E/\mathrm{d}t \rangle = A^1 E^{-1}$$

is equal to that due to a mechanism with $n = 1$

$$\langle \mathrm{d}E/\mathrm{d}t \rangle = AE\,,$$

i.e. that

$$A^1 E_l^{-1} = AE_l\,.$$

Then the acceleration by longitudinal waves can be regarded as an injection mechanism providing a flux of particles from $E < E_l$ to $E > E_l$. For $E > E_l$ the details of the origin of this flux can be disregarded by imposing the boundary condition that there is a flux of particles across $E = E_l$.

At $E > E_l$ suppose that the initial ($t = 0$) energy spectrum is of the form

$$N(E, t = 0) \propto E^{-\gamma} \qquad E_l < E < E_2 \tag{12}$$

over some finite energy range. According to (3) with $n = 1$ the acceleration mechanism leads to a flux

$$F(E, t = 0) \propto [(\gamma - 2) D + A - SE] E^{-\gamma}, \tag{13}$$

which is to be added to the assumed flux acros $E = E_l$. For $\gamma > 2$ this flux (13) is from low to high energies. The synchrotron losses can be ignored provided that we have

$$E_2 < \frac{(\gamma - 2) D + A}{S}. \tag{14}$$

As shown by Kardashev (1962) the net effect of the flux (13) corresponds to E_2 increasing linearly in time for, see (M14),

$$2\gamma - 3 + A/D > 0, \qquad t \ll D^{-1}. \tag{15}$$

At this stage there is no major change in the shape of the spectrum.

Eventually E_2 reaches a value at which synchrotron become important, i.e. where (14) is no longer satisfied. This leads to a change in sign of the flux (13) at sufficiently high energies, i.e. to a flux from high to low energies due to synchrotron losses. [At the very highest energies the spectrum must tail off, i.e. γ effectively becomes very large, so that, according to (13), there is a flux of particles to still higher energies despite the synchrotron losses. As pointed out in paper M this effect, due to diffusion in energy, is well known in the context of Coulomb interactions in a thermal plasma with a truncated Maxwellian tail, see MacDonald *et al.* (1957).] Thus there is a net flux from both low and from high energies into the region

$$E_l \lesssim E \lesssim E_m, \quad E_m = \frac{A}{S}.$$

The spectrum in this region flattens.

With the problem as posed the total number of particles at $E > E_l$ increases and the spectrum flattens until there is flux from high to low energies which balances the assumed flux from low to high energies across $E = E_l$. With the processes at $E < E_l$ simulated by a net flux across $E = E_l$ the appropriate boundary conditions at $E = E_l$ are just the conditions (11). These lead to the solution (M2) as already indicated.

A number of objections may be raised regarding the approach to this stationary state. For example, secular changes in the acceleration mechanisms, expansion of the source, escape of particles and many other effects are neglected in our discussion. However, we believe that the solution (M2) is a reasonable approximation to the solution of the physical problem as posed, i.e. (M2) is the stationary solution when a constant injection of particles at $E < E_l$, simulating the effect of acceleration mechanisms operating at $E < E_l$, is balanced by a net flux from high to low energies due to a combination of the acceleration mechanism and synchrotron losses.

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