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A Classical Counterpart to Double Compton Scattering (*).

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Summary. — It is shown that the classical interaction between a charged particle and a radiation field (Thomson scattering) involves a hierarchy of processes analogous to that for photon-electron scattering (Compton scattering) in quantum electrodynamics. Statements that the double and multiple Compton effects are intrinsically quantum-mechanical processes are incorrect; it is shown that double Thomson scattering is the classical counterpart of the double Compton effect. However, it is only the inverse of double Comptori scattering (two photons scattered into one photon) which has a classical counterpart; the direct process does not. Familiar examples of the interaction of three waves in nonlinear plasma theory are shown to be particular cases of double Thomson scattering.

1. – Introduction.

The double Compton effect [1,2] involves the scattering of a photon into two photons, and the inverse process, by a free electron. This and other multiple Compton effects tend to be regarded as intrinsically quantum-mechanical processes. Indeed, HEITLER [3] states that these multiple processes « are typical quantum effects » and « are a valuable test of quantum-electrodynamics ». This impression probably arises from the fact that the cross-section for double

(*) To speed up publication, the author of this papcr has agreed to not receive the proofs for correction.

(1) F. MANDL and T. H. R. SKYRME: *Proc. Roy. Soc.*, **A 215**, 497 (1952).

(2) J. M. JAUCH and F. ROHRLICH: *The Theory of Photons and Electrons* (Reading, Mass., 1955).

(3) W. HEITLER: *Quantum Theory of Radiation*, 2nd ed. (Oxford, 1944).

Compton scattering depends explicitly on Planck's constant for all values of the energies of the photons involved. This is to be contrasted with the cross-section for single Compton scattering (Klein-Nishina formula) which reduces to the classically derived Thomson cross-section for soft photons.

The fact that the cross-section for double Compton scattering depends explicitly on Planck's constant implies only that one cannot ascribe a cross-section to any classical counterpart in a meaningful way. This does not necessarily mean that there is no classical counterpait to the double Compton effect.

The purpose of this paper is to show that the interaction between a charged particle and a radiation field according to classical physics involves a hierarchy of processes analogous to the hierarchy of processes for photon-electron scattering according to quantum electrodynamics. We refer to the classical process as (single, double, multiple) Thomson scattering and to the quantum-mechanical process as Compton scattering. The equivalence of single Compton scattering for soft photons and single Thomson scattering is well known. Here it is shown explicitly that (the inverse of) double Compton scattering reduces to its classical counterpart of double Thomson scattering for soft photons (energies much less than the rest energy of the electron in the centre-of-mass frame).

A classical treatment of the scattering of waves is complementary to that based on quantum electrodynamics. The classically derived results are not valid for hard photons but this restriction is only of practical relevance in vacuo. On the other hand, when the influence of any ambient medium is significant one is usually in the classical regime and a quantum-mechanical treatment is irrelevant as well as impractical. In Sect. **6** below it is shown that certain familiar nonlinear plasma processes can be regarded as examples of double Thomson scattering.

Our treatment of Thomson scattering starts with the perturbations in the orbit of a charged particle moving through a fluctuating electromagnetic field. In Sect. 2 the perturbations are expanded in powers of the field strengths. The power radiated by the accelerated charge is calculated in Sect. **3**; the first-order perturbations lead to Thomson scattering, the second-order to double Thomson scattering, and so on. In Sect. 4 the classically derived results are written in semi-classical notation to facilitate comparison with the Compton-scattering formulae; this comparison is carried out in Sect. 5.

2. – Perturbation expansion.

In this Section the perturbations in the orbit of a charged particle (charge q, mass m, CGS units) due to the presence of a fluctuating electromagnetic field are evaluated up to and including terms quadratic in the field strengths. The actual quantities required are the perturbations in the (Fourier-transformed) current density:

$$(1)\qquad \begin{cases} j_i(k) = \lim\limits_{V,T\to\infty} \int\limits_{-T/2}^{T/2} \mathrm{d}t \int\limits_V \mathrm{d}^3\boldsymbol{r}\, j_i(\boldsymbol{r}, t) \exp\left[-i\left(\boldsymbol{k}\cdot\boldsymbol{r}(t) - \omega t\right)\right], \\ j_i(\boldsymbol{r}, t) = q v_i(t)\, \delta^3\left(\boldsymbol{r} - \boldsymbol{r}(t)\right), \end{cases}$$

where k denotes $(\boldsymbol{k},\omega)$, $\boldsymbol{r} = \boldsymbol{r}(t)$ describes the orbit of the particle and

$$\boldsymbol{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}(t)$$

is its velocity.

In (1) and throughout the truncation in the Fourier transforms is included explicitly. Because 8-functions are defined by

$$\lim_{V,T\to\infty} \int\limits_{-T/2}^{T/2} \mathrm{d}t \int\limits_V \mathrm{d}^3\boldsymbol{r} \exp\left[\pm i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right] = (2\pi)^4 \delta(\omega)\, \delta^3(\boldsymbol{k}),$$

they satisfy the identities

$$(2)\qquad \begin{cases} [\delta(\omega)]^2 = \lim\limits_{T\to\infty} \dfrac{T}{2\pi}\, \delta(\omega), \\ [\delta^3(\boldsymbol{k})]^2 = \lim\limits_{V\to\infty} \dfrac{V}{(2\pi)^3}\, \delta^3(\boldsymbol{k}). \end{cases}$$

Otherwise the truncation could be ignored.

The equation of motion reads

$$(3)\qquad \begin{cases} \dfrac{\mathrm{d}p_i(t)}{\mathrm{d}t} = \displaystyle\int \frac{\mathrm{d}^4k_1}{(2\pi)^4} F_i\left(\boldsymbol{v}(t), k_1\right) \exp\left[i\left(\boldsymbol{k}_1\cdot\boldsymbol{r}(t) - \omega_1 t\right)\right], \\ F_i\left(\boldsymbol{v}(t), k_1\right) = \dfrac{q}{\omega_1}\left[\left(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v}(t)\right)\delta_{ij} + k_{1i} v_j(t)\right] E_j(k_1) \end{cases}$$

with $\boldsymbol{E}(k_1)$ the Fourier transform of the fluctuating electric field. The zeroth-order solution is rectilinear motion. We write the zeroth-order motion in the form

$$(4)\qquad \begin{cases} \boldsymbol{v}^{(0)} = \text{const}, \qquad \boldsymbol{r}^{(0)}(t) = \boldsymbol{v}^{(0)} t, \\ \boldsymbol{v}^{(0)} = \dfrac{\boldsymbol{p}^{(0)} c^2}{E^{(0)}}, \qquad E^{(0)} = \left[m^2 c^4 + |\boldsymbol{p}^{(0)}|^2 c^2\right]^{\frac{1}{2}}. \end{cases}$$

The first-order equation of motion reads

$$(5)\qquad \frac{\mathrm{d}p_i^{(1)}(t)}{\mathrm{d}t} = \int \frac{\mathrm{d}^4k_1}{(2\pi)^4} F_i(\boldsymbol{v}^{(0)}, k_1) \exp\left[-i(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v}^{(0)})t\right].$$

Integration with respect to t yields $\boldsymbol{p}^{(1)}(t)$. One has

$$v_i^{(1)}(t) = \frac{c^2}{E^{(0)}}\pi_{ij}(\boldsymbol{v}^{(0)})p_j^{(1)}(t)\;,$$

$$\pi_{ij}(\boldsymbol{v}) = \delta_{ij} - \frac{v_i v_j}{c^2}\;,$$

and further integration gives $\boldsymbol{r}^{(1)}(t)$. The second-order equation of motion

$$\frac{\mathrm{d}p_i^{(2)}}{\mathrm{d}t} = \int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\left[v_s^{(1)}(t)\frac{\partial}{\partial v_s}F_i(\boldsymbol{v}, k_1) - i\boldsymbol{k}_1\cdot\boldsymbol{r}^{(1)}(t)F_i(\boldsymbol{v}^{(0)}, k_1)\right]\cdot \exp[-i(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v}^{(0)})t] \tag{6}$$

can then be integrated to find $\boldsymbol{p}^{(2)}(t)$. One has

$$v_i^{(2)}(t) = \frac{c^2}{E^{(0)}}\pi_{ij}(\boldsymbol{v}^{(0)})p_j^{(2)}(t) - \frac{c^2 v_i^{(0)}}{2(E^{(0)})^2}\pi_{rs}(\boldsymbol{v}^{(0)})p_r^{(1)}(t)p_s^{(1)}(t) - \\ - \frac{c^2}{(E^{(0)})^2}v_r^{(0)}p_r^{(1)}(t)\pi_{ij}(\boldsymbol{v}^{(0)})p_j^{(1)}(t)\;,$$

and further integration gives $\boldsymbol{r}^{(2)}(t)$. The results of this expansion are written down in Appendix A.

Expanding (1) the first- and second-order current densities read ($\boldsymbol{v}\equiv\boldsymbol{v}^{(0)}$)

$$j_i^{(1)}(k) = \lim_{T\to\infty} q\int_{-T/2}^{T/2}\mathrm{d}t\,[v_i^{(1)}(t) - i\boldsymbol{k}\cdot\boldsymbol{r}^{(1)}(t)]\exp[i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})t] = \\ = \frac{2\pi q^2}{m}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{1}{\omega_1}A_{ij}(\boldsymbol{v}; k, k_1)E_j(k_1)\,\delta\{(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) - (\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v})\} \tag{7}$$

and

$$j_i^{(2)}(k) = \lim_{T\to\infty} q\int_{-T/2}^{T/2}\mathrm{d}t\Big[v_i^{(2)}(t) - i\boldsymbol{k}\cdot\boldsymbol{r}^{(2)}(t)v_i - \\ - i\boldsymbol{k}\cdot\boldsymbol{r}^{(1)}(t)v_i^{(1)}(t) - v_i\{\boldsymbol{k}\cdot\boldsymbol{r}^{(1)}(t)\}^2\Big]\exp[i(\omega - \boldsymbol{k}\cdot\boldsymbol{v})t] = \\ = \frac{2\pi q^3}{m^2}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\frac{1}{\omega_1\omega_2}A_{ijl}(\boldsymbol{v}; k, k_1, k_2)\cdot \\ \cdot E_j(k_1)E_l(k_2)\,\delta\{(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) - (\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v}) - (\omega_2 - \boldsymbol{k}_2\cdot\boldsymbol{v})\}\;, \tag{8}$$

respectively. Equations (7) and (8) define the tensors A_{ij} and A_{ijl}, respectively. Explicit expressions for these tensors are written down in Appendix A.

The tensors A_{ij} and A_{ijl} satisfy a number of symmetry relations. For A_{ij}

one has

$$A_{ij}(\boldsymbol{v}; k, k_1) = -A^*_{ij}(\boldsymbol{v}; -k, -k_1) = A_{ji}(\boldsymbol{v}; -k_1, -k). \tag{9}$$

The first of these equations is just the reality condition for Fourier transforms with * denoting complex conjugation (the minus sign is due to the factor ω_1^{-1} being extracted in (7)). For A_{ijl} one has

$$A_{ijl}(\boldsymbol{v};, k, k_1, k_2) = A^*_{ijl}(\boldsymbol{v}; -k, -k_1, -k_2) = \\ = A_{ilj}(\boldsymbol{v}; k, k_2, k_1) = A_{jil}(\boldsymbol{v}; -k_1, -k, k_2), \tag{10}$$

and further symmetries by combining any two of these. Again the first of these is the reality condition. The second is implicit in the definition (8).

The second of (9) and the last of (10) turn out to be satisfied when the tensors are evaluated explicitly. In a more sophisticated approach these symmetries should emerge naturally; see AL'TSHUL' and KARPMAN [4] for a discussion of symmetries in a related context. The importance of these symmetries is that they ensure that the direct, inverse and crossed processes are described by the same basic expressions, see Sect. 4 belcw.

3. - Power generated in scattered waves.

Given an extraneous current one can include it as a source term in the wave equation, find the electric field which it generates and identify the power radiated as minus the work done by the current against this electric field. The case where an ambient medium with dielectric tensor $\varepsilon_{ij}(k) \neq \delta_{ij}$ is present is no more difficult than for a vacuum ($\varepsilon_{ij} = \delta_{ij}$). We include the effects of an ambient medium and then specialize to a vacuum.

The inhomogeneous wave equation reads

$$\left[\frac{|\boldsymbol{k}|^2 c^2}{\omega^2} \quad - \varepsilon_{ij}(k)\right] E_j(k) = \frac{-4\pi i}{\omega} j_1(k), \qquad \boldsymbol{k} = |\boldsymbol{k}|\boldsymbol{\varkappa}. \tag{11}$$

We write solutions of the homogeneous wave equation for waves in any mode a in the form

$$\begin{cases} \boldsymbol{E}^\sigma(k) = \boldsymbol{e}^\sigma|\boldsymbol{E}^\sigma(\boldsymbol{k})|2\pi\delta(\omega - \omega^\sigma), & \omega^\sigma \equiv \omega^\sigma(\boldsymbol{k}) = -\omega^\sigma(-\boldsymbol{k}), \\ \boldsymbol{e}^\sigma \equiv \boldsymbol{e}^\sigma(\boldsymbol{k}) = [\boldsymbol{e}^\sigma(-\boldsymbol{k})]^*, & [\boldsymbol{e}^\sigma]^* \cdot \boldsymbol{e}^\sigma = 1. \end{cases} \tag{12}$$

(4) A. AL'TSHUL' and V. I. KARPMAN: *Sov.* Phys. JETP, **20, 1043 (1965).**

The dispersion relations $w = \omega^\sigma$, the polarization vector **e"** and the ratio

$$\left[\frac{W_E}{W_T}\right]^\sigma = \frac{W_E^\sigma(\boldsymbol{k})}{W_T^\sigma(\boldsymbol{k})}$$

can be found as functions of $\varepsilon_{ij}(k)$ ([5]). The elcctric-energy density $W_E^\sigma(\boldsymbol{k})$ is defined by

$$(13)\quad \left\{\begin{aligned} W_E^\sigma &= \lim_{V,T\to\infty} \frac{1}{VT} \int_{-T/2}^{T/2} \mathrm{d}t \int_V \mathrm{d}^3\boldsymbol{r}\, \frac{|\boldsymbol{E}^\sigma(\boldsymbol{r}, t)|^2}{8\pi} = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, W_E^\sigma(\boldsymbol{k})\,,\\ W_E^\sigma(\boldsymbol{k}) &= \lim_{V\to\infty} \frac{|\boldsymbol{E}^\sigma(\boldsymbol{k})|^2}{8\pi V}\,,\end{aligned}\right.$$

$W_T^\sigma(\boldsymbol{k})$ refers to the total energy density in waves.

The time-averaged power radiated into waves in the mode a due to a given extraneous current reduces to ([6-7])

$$(14)\quad \left\{\begin{aligned} \bar{P}^\sigma &= -\lim_{V,T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{d}t \int_V \mathrm{d}^3\boldsymbol{r}\, [\boldsymbol{E}(\boldsymbol{r}, t).\boldsymbol{j}(\boldsymbol{r}, t)]^\sigma = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\, P^\sigma(\boldsymbol{k})\,,\\ P^\sigma(k) &= \lim_{T\to\infty} \frac{4\pi}{T} \left[\frac{W_E}{W_T}\right]^\sigma |\boldsymbol{e}^\sigma\cdot\boldsymbol{j}^*(\boldsymbol{k}, \omega^\sigma)|^2\,.\end{aligned}\right.$$

On inserting (7) or (8) in (14) we assume that the initial waves have random phases and average over the phases. Denoting this average by a bar, one has, for example,

$$(15)\quad \left\{\begin{aligned} &\overline{E_j^{\sigma_1^*}(k_1)\, E_r^{\sigma_1}(k_1')} = e_j^{\sigma_1^*} e_r^{\sigma_1} 4\pi W_E^{\sigma_1}(\boldsymbol{k}_1)\, 2\pi\delta(\omega_1 - \omega^{\sigma_1})(2\pi)^4\, \delta^4(k_1 - k_1')\,,\\ &\boldsymbol{e}^{\sigma_1} \equiv \boldsymbol{e}^{\sigma_1}(\boldsymbol{k}_1)\,, \qquad \omega^{\sigma_1} \equiv \omega^{\sigma_1}(\boldsymbol{k}_1)\,.\end{aligned}\right.$$

After averaging over phases, (7) in (14) gives

$$(16)\quad \left\{\begin{aligned} P^\sigma(\boldsymbol{k}) &= 4\, \frac{(2\pi)^3 q^4}{m^2} \left[\frac{W_E}{W_T}\right]^\sigma \int \frac{\mathrm{d}^3\boldsymbol{k}_1}{(2\pi)^3} \left[\frac{W_E}{W_T}\right]^{\sigma_1} \cdot\\ &\qquad \cdot \frac{|e_i^{\sigma^*} e_j^{\sigma_1} A_{ij}|^2}{(\omega^{\sigma_1})^2}\, W_T^{\sigma_1}(\boldsymbol{k}_1)\, \delta\{(\omega^\sigma - \boldsymbol{k}\cdot\boldsymbol{v}) - (\omega^{\sigma_1} - \boldsymbol{k}_1\cdot\boldsymbol{v})\}\,,\\ A_{ij} &\equiv A_{ij}(\boldsymbol{v};\, \mathrm{k}, \mathrm{w}";\, \boldsymbol{k}_1, \omega^{\sigma_1})\end{aligned}\right.$$

([5]) D. B. MELROSE: *Astrophys. Space Sci.*, **2**, 171 (1968).
([6]) V. N. TSYTOVICH: *Sov. Phys. Usp.*, **9**, 805 (1967).
([7]) V. N. TSYTOVICH: *Nonlinear Effects in Plasma* (New York, 1970).

The square of the δ-function is rewritten using (2). Inserting (8) ni (14) gives

$$(17)\quad \begin{cases} P^\sigma(\boldsymbol{k}) = 32\,\dfrac{(2\pi)^4 q^6}{m^3}\left[\dfrac{W_E}{W_T}\right]^\sigma \displaystyle\int \dfrac{\mathrm{d}^3\boldsymbol{k}_1}{(2\pi)^3}\,\dfrac{\mathrm{d}^3\boldsymbol{k}_2}{(2\pi)^3}\left[\dfrac{W_E}{W_T}\right]^{\sigma}\left[\dfrac{W_E}{W_T}\right]^{\sigma_2}\cdot \\ \cdot\,\dfrac{|e_i^{\sigma *} e_j^{\sigma_1} e_l^{\sigma_2} A_{ijl}|^2}{(\omega^{\sigma_1})^2(\omega^{\sigma_2})^2}\, W_T^{\sigma_1}(\boldsymbol{k}_1)\, W_T^{\sigma_2}(\boldsymbol{k}_2)\,\delta\{(\omega^\sigma - \boldsymbol{k}\cdot\boldsymbol{v}) - (\omega^{\sigma_1} - \boldsymbol{k}_1\cdot\boldsymbol{v}) - (\omega^{\sigma_2} - \boldsymbol{k}_2\cdot\boldsymbol{v})\}\,, \\ A_{ijl} \equiv A_{ijl}(\boldsymbol{v};\, \boldsymbol{k}, \omega^\sigma;\, \boldsymbol{k}_1, \omega^{\sigma_1};\, \boldsymbol{k}_2, \omega^{\sigma_2})\,. \end{cases}$$

Equation (16) describes single Thomson scattering, while (18) describes double Thomson scattering. In the presence of a medium one cannot, a priori, justify neglecting certain further nonlinear currents over and above those calculated in Sect. 2. Indeed, for scattering by electrons of waves with wavelength greater than the Debye length, additional terms in $\boldsymbol{j}^{(1)}(k)$, and so in A_{ij}, associated with the shielding of the electron nearly cancel the term due to Thompson scattering [6,7].

In vacuo the only currents to first and second order are those calculated in Sect. 2. The properties of electromagnetic waves in vacuo correspond to

$$(18)\qquad |\omega^\sigma| = |\boldsymbol{k}|\, c\,, \qquad \left[\frac{W_E}{W_T}\right]^\sigma = \frac{1}{2}\,.$$

The polarization is arbitrary transverse polarization. The average over phases gives the polarization tensor

$$p_{ij}(\boldsymbol{k}) = \overline{e_i^*(\boldsymbol{k})\, e_j(\boldsymbol{k})}\,.$$

For unpolarized radiation, which is the case most often discussed in this context, one has

$$P_{ij}(\boldsymbol{k}) = \delta_{ij} - \varkappa_i \varkappa_j\,.$$

Rewriting (16) and (17) for unpolarized electromagnetic waves in vacuo involves inserting (18) for each wave (k, k, and k,), summing over final polarizations and averaging over initial polarizations. This gives

$$(19)\quad \begin{cases} \overline{|e_i^{\sigma *} e_j^{\sigma_1} A_{ij}|^2} = \frac{1}{2}(\delta_{ir} - \varkappa_i\varkappa_r)(\delta_{js} - \varkappa_{1j}\varkappa_{1s}) A_{ij} A^*_{rs}\,, \\ \overline{|e_i^{\sigma *} e_j^{\sigma_1} e_l^{\sigma_2} A_{ijl}|^2} = \frac{1}{4}(\delta_{ir} - \varkappa_i\varkappa_r)(\delta_{js} - \varkappa_{1j} - \varkappa_{1s})(\delta_{lt} - \varkappa_{2l}\varkappa_{2t}) A_{ijl} A^*_{rst}\,, \end{cases}$$

in (16) and (17), respectively.

4. – Semi-classical description.

To facilitate comparison with formulae for Compton scattering we rewrite the classical formulae (16) and (17) in semi-classical notation. This involves introducing a constant $\hbar$ with the dimensions of action and regarding the waves as a collection of quasi-particles with energy $\hbar\omega^\sigma$, momentum $\hbar\boldsymbol{k}$ and distribution function

$$N^\sigma(\boldsymbol{k}) = \frac{W_T^\sigma(\boldsymbol{k})}{\hbar\omega^\sigma}\,. \tag{20}$$

The power radiated $P^a(k)$ then becomes

$$P^a(k) = \lim_{V\to\infty} V\hbar\omega^\sigma \frac{\partial N^\sigma(\boldsymbol{k})}{at}\,. \tag{21}$$

To avoid the appearance of the volume V we integrate (16) and (17) over a distribution of scattering particles with distribution function $f(\boldsymbol{p})$ normalized according to

$$\lim_{V\to\infty} V\int d^3p\, f(p) = \text{number of scattering particles.}$$

Equation (16) becomes

$$\frac{\partial N^\sigma(\boldsymbol{k})}{\partial t} = \int \mathrm{d}^3\boldsymbol{p}\, f(\boldsymbol{p}) \int \frac{\mathrm{d}^3\boldsymbol{k}_1}{(2\pi)^3}\, w^{\sigma\sigma_1}(\boldsymbol{p};\, \boldsymbol{k}, \boldsymbol{k}_1) N^{\sigma_1}(\boldsymbol{k}_1) \tag{22}$$

with

$$w^{\sigma\sigma_1}(\boldsymbol{p};\, \boldsymbol{k}, \boldsymbol{k}_1) = \frac{4(2\pi)^3 q^4}{m^2} \left[\frac{W_E}{W_T}\right]^\sigma \left[\frac{W_E}{W_T}\right]^{\sigma_1} \frac{|e_i^{\sigma *} e_j^{\sigma_1} A_{ij}|^2}{\omega^\sigma \omega^{\sigma_1}} \cdot \delta\{(\omega^\sigma - \boldsymbol{k}\cdot\boldsymbol{v}) - (\omega^{\sigma_1} - \boldsymbol{k}_1\cdot\boldsymbol{v})\}\,. \tag{23}$$

Equation (17) becomes

$$\frac{\partial N^\sigma(\boldsymbol{k})}{\partial t} = \int \mathrm{d}^3\boldsymbol{p}\, f(\boldsymbol{p}) \int \frac{\mathrm{d}^3k_1}{(2\pi)^3} \frac{\mathrm{d}^3k_2}{(2\pi)^3}\, w^{\sigma\sigma_1\sigma_2}(\boldsymbol{p};\, \boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) N^{\sigma_1}(\boldsymbol{k}_1) N^{\sigma_2}(\boldsymbol{k}_2) \tag{24}$$

with

$$w^{\sigma\sigma_1\sigma_2}(\boldsymbol{p};\, \boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = 32\, \frac{(2\pi)^4 q^6 \hbar}{m^4} \frac{|e_j^{\sigma *} e_j^{\sigma_1} e_l^{\sigma_2} A_{ijl}|^2}{\omega^\sigma \omega^{\sigma_1} \omega^{\sigma_2}} \cdot \delta\{(\omega^\sigma - \boldsymbol{k}\cdot\boldsymbol{v}) - (\omega^{\sigma_1} - \boldsymbol{k}_1\cdot\boldsymbol{v}) - (\omega^{\sigma_2} - \boldsymbol{k}_2\cdot\boldsymbol{v})\}\,. \tag{25}$$

The quantity $w^{\sigma\sigma_1}(\boldsymbol{p};\, \boldsymbol{k}, \boldsymbol{k}_1)$ can be interpreted as the probability per unit time that a particle with momentum in the range dp at $\boldsymbol{p}$ scatter a quasi-particle in the mode σ_1 in the range $d\boldsymbol{k}_1$ at $\boldsymbol{k}_1$ into a quasi-particle in the mode σ in the range dk at k. Because of the symmetry properties **(9)** this probability is equal to that of the inverse process. Likewise the crossed process involving the simultaneous emission of two quasi-particles (this being possible in a medium but not in vacuo) is described by (23) with the δ-function appropriately modified, see KOVRIZHNYKH [8] for example.

The quantity $w^{\sigma\sigma_1\sigma_2}(\boldsymbol{p};$ k, $\boldsymbol{k}_1$, k,) has an analogous interpretation. Because of the symmetry properties (10) the inverse and crossed processes are described by the same probability. The probability (25) depends explicitly on $\hbar$, which is an arbitrary constant in the semi-classical approach, and so is not **a** well-defined quantity. Similarly, any cross-section derived from (25) is not a well-defined quantity.

Equations (22) and (24) are incomplete because they do not include the effects of the inverse, induced and crossed processes. It is possible to include all these related processes within the framework of the semi-classical approximation by identifying $\hbar$ with Planck's constant and using the Einstein coefficients [5-8]. Including the inverse process in (22), one has

$$\frac{\partial N^\sigma(\boldsymbol{k})}{\partial t} = \int d^3\boldsymbol{p}\, f(\boldsymbol{p}) \int \frac{d^3\boldsymbol{k}_1}{(2\pi)^3}\, w^{\sigma\sigma_1}(\boldsymbol{p};\boldsymbol{k},\boldsymbol{k}_1)\{N^{\sigma_1}(\boldsymbol{k}_1) - N^\sigma(\boldsymbol{k})\}\,, \tag{26}$$

which remains independent of $\hbar$, with a similar equation with (σ, k) and (σ_1, $\boldsymbol{k}_1$) interchanged. Equation (26) ensures that the number of quasi-particles is conserved:

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{\partial N^\sigma(\boldsymbol{k})}{\partial t} + \int \frac{d^3\boldsymbol{k}_1}{(2\pi)^3}\frac{\partial N^{\sigma_1}(\boldsymbol{k}_1)}{\partial t} = 0\ .$$

We ignore the induced processes and the crossed process of double emission.

Including the effects of the inverse process (24) leads to

$$\frac{\partial N^\sigma(\boldsymbol{k})}{\partial t} = \int d^3\boldsymbol{p}\, f(\boldsymbol{p}) \int \frac{d^3\boldsymbol{k}_1}{(2\pi)^3}\frac{d^3\boldsymbol{k}_2}{(2\pi)^3}\, w^{\sigma\sigma_1\sigma_2}(\boldsymbol{p};\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\cdot \\ \cdot\{N^{\sigma_1}(\boldsymbol{k}_1)N^{\sigma_2}(\boldsymbol{k}_2) - N^\sigma(\boldsymbol{k})N^{\sigma_1}(\boldsymbol{k}_1) - N^\sigma(\boldsymbol{k})N^{\sigma_2}(\boldsymbol{k}_2) - N^\sigma(\boldsymbol{k})\}\,, \tag{27}$$

$$\frac{\partial N^{\sigma_1}(\boldsymbol{k}_1)}{\partial t} = \int d^3\boldsymbol{p}\, f(\boldsymbol{p}) \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{d^3\boldsymbol{k}_2}{(2\pi)^3}\, w^{\sigma\sigma_1\sigma_2}(\boldsymbol{p};\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\cdot \\ \cdot\{-N^{\sigma_1}(\boldsymbol{k}_1)N^{\sigma_2}(\boldsymbol{k}_2) + N^\sigma(\boldsymbol{k})N^{\sigma_1}(\boldsymbol{k}_1) + N^\sigma(\boldsymbol{k})N^{\sigma_2}(\boldsymbol{k}_2) + N^\sigma(\boldsymbol{k})\}\,, \tag{28}$$

plus a further equation derived from (28) by interchanging (σ_1, $\boldsymbol{k}_1$) with (σ_2, k,).

The « classical » counterpart to the direct double Compton effect corresponds

(8) L. M. KOVRIZHNYKH: *Sov. Phys.* JETP, **21**, 744 (1965).

to $N^{\sigma_1}(\boldsymbol{k}_1)=0$, $N^{\sigma_2}(\boldsymbol{k}_2)=0$ in (27) and (28). It is not possible to rewrite the resulting equations in a purely classical way (independent of $\hbar$). This indicates that although the inverse double Compton effect has a classical counterpart described by (17), the direct process has no classical counterpart.

Explicit expressions for the probabilities (23) and (25) can be written down for electromagnetic waves in vacuo with the aid of the results of Sect. **2** and **3.** For unpolarized radiation single Thomson scattering is described by the probability [9]

$$
(29)\quad \begin{cases} w(\boldsymbol{p};\,\boldsymbol{k},\boldsymbol{k}_1)=\dfrac{(2\pi)^3q^4}{m^2}\dfrac{\overline{X}_T}{\omega\omega_1}\,\delta\{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})-(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})\}\,,\\[2mm] \omega=|\boldsymbol{k}|\,c\,,\qquad \omega_1=|\boldsymbol{k}_1|\,c\,,\\[2mm] \overline{X}_T=\dfrac{1}{2}\left[1+\left\{1-\dfrac{m^2c^4}{E^2}\dfrac{(\omega\omega_1/c^2-\boldsymbol{k}\cdot\boldsymbol{k}_1)}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})}\right\}^2\right], \end{cases}
$$

where we have used eq. (A.3). The analcgous expression for double Thomson scattering is very cumbersome but simplifies considerably in the rest frame of the scattering particle. In the rest frame, for unpolarized radiation, we find

$$
(30)\quad \begin{cases} w(\boldsymbol{p}=0;\,\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)=\dfrac{(2\pi)^4q^6\hbar}{m^4c^2}\dfrac{\overline{X}_{DT}}{\omega\omega_1\omega_2}\,\delta(\omega-\omega_1-\omega_2)\,,\\[2mm] \overline{X}_{DT}=\dfrac{c^2}{2}\left[|\boldsymbol{K}|^2\left\{\dfrac{1+(\boldsymbol{\varkappa}\cdot\boldsymbol{\varkappa}_2)^2}{\omega_1^2}+\dfrac{1+(\boldsymbol{\varkappa}\cdot\boldsymbol{\varkappa}_1)^2}{\omega_2^2}+\dfrac{1+(\boldsymbol{\varkappa}_1\cdot\boldsymbol{\varkappa}_2)^2}{\omega^2}\right\}-\right.\\[2mm] \left.-\left\{\left(\dfrac{\boldsymbol{\varkappa}_1\cdot\boldsymbol{K}}{\omega_1}\right)^2+\left(\dfrac{\boldsymbol{\varkappa}_2\cdot\boldsymbol{K}}{\omega_2}\right)^2+\left(\dfrac{\boldsymbol{\varkappa}\cdot\boldsymbol{K}}{\omega}\right)^2\right\}-\left\{\dfrac{\boldsymbol{\varkappa}\cdot\boldsymbol{\varkappa}_2\boldsymbol{\varkappa}_1\cdot\boldsymbol{K}}{\omega_1}+\dfrac{\boldsymbol{\varkappa}\cdot\boldsymbol{\varkappa}_1\boldsymbol{\varkappa}_2\cdot\boldsymbol{K}}{\omega_2}-\dfrac{\boldsymbol{\varkappa}_1\cdot\boldsymbol{\varkappa}_2\boldsymbol{\varkappa}\cdot\boldsymbol{K}}{\omega}\right\}^2\right],\\[2mm] \boldsymbol{K}=\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2\,, \end{cases}
$$

where we have used eq. **(A.6).**

5. – Comparison with quantum electrodynamics.

In this Section we write down known formulae for the single and double Compton effects [1,2] in the form of scattering probabilities and compare these with the semi-classical formulae (29) and (30) with $\hbar$ identified as Planck's constant.

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(9) D. B. MELROSE: *Astrophys. Space Sci.*, **10**, 186 (1971).

It is convenient to introduce 4-vectors $p = (\boldsymbol{p}, E)$, $k = (\boldsymbol{k}, \omega)$ etc., with

$$p\cdot p = \frac{E^2}{c^2} - |\boldsymbol{p}|^2 = m^2 c^2 ,$$

$$k\cdot k = \frac{\omega^2}{c^2} - |\boldsymbol{k}|^2 = 0 .$$

For single Compton scattering $(k_1 \to k)$ by a particle with initial 4-momentum p, the final 4-momentum is

$$p' = p + \hbar k - \hbar k_1 .$$

We write the final energy as

$$E' = [m^2 c^4 + |\boldsymbol{p} + \hbar\boldsymbol{k} - \hbar\boldsymbol{k}_1|^2 c^2]^{\frac{1}{2}} . \tag{31}$$

One can form only two nontrivial independent invariant's, which we write in the dimensionless form

$$\begin{cases} a = \dfrac{\hbar p\cdot k}{m^2 c^2} = -\dfrac{\hbar p' k_1}{m^2 c^2} = \dfrac{\hbar E}{m^2 c^4}(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) , \\[2ex] a_1 = \dfrac{\hbar p\cdot k_1}{m^2 c^2} = -\dfrac{\hbar p' k}{m^2 c^2} = \dfrac{\hbar E}{m^2 c^4}(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v}) \end{cases} \tag{32}$$

with

$$k\cdot k_1 = \frac{\omega\omega_1}{c^2} - \boldsymbol{k}\cdot\boldsymbol{k}_1 = \frac{m^2 c^2}{\hbar^2}(a - a_1) .$$

The probability for single Compton scattering of unpolarized radiation reads [2]

$$\begin{cases} w(\boldsymbol{p};\, \boldsymbol{k}, \boldsymbol{k}_1) = \dfrac{(2\pi)^3 q^4}{m^2}\, \dfrac{\overline{X}_c}{\omega\omega_1}\, \delta\left(\dfrac{E'}{\hbar} - \dfrac{E}{\hbar} - \omega + \omega_1\right) , \\[2ex] \overline{X} = \dfrac{1}{2}\left[\dfrac{a}{a_1} + \dfrac{a_1}{a} + 2\left(\dfrac{1}{a} - \dfrac{1}{a_1}\right) + \left(\dfrac{1}{a} - \dfrac{1}{a_1}\right)^2\right] = \\[2ex] \qquad = \dfrac{1}{2}\left[1 + \left\{1 - \dfrac{m^2 c^4}{E^2}\, \dfrac{\omega\omega_1/c^2 - \boldsymbol{k}\cdot\boldsymbol{k}_1}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v})}\right\}^2 + \right. \\[2ex] \qquad\qquad \left. + \left(\dfrac{\hbar c^2}{E}\right)^2 \dfrac{(\omega\omega_1/c^2 - \boldsymbol{k}\cdot\boldsymbol{k}_1)^2}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})(\omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v})}\right] . \end{cases} \tag{33}$$

The argument of the 8-function in **(33)** for $a, a_1 \ll 1$ reduces to

$$\frac{E'}{\hbar}-\frac{E}{\hbar}-\omega+\omega_1=-\{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})-(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})\}-$$
$$-\frac{\hbar}{2E}\{(\boldsymbol{k}\cdot\boldsymbol{v}-\boldsymbol{k}_1\cdot\boldsymbol{v})^2-|\boldsymbol{k}-\boldsymbol{k}_1|^2c^2\}+\dots .$$

To lowest order in $\hbar$ this reproduces the 8-function in **(29).** Indeed, the generalization of the 6-function in **(29)** to that in **(33)** could be argued on a semi-classical basis. The distinctive contribution of quantum electrodynamics is the term

$$\overline{X}_C-\overline{X}_T=\frac{1}{2}\left(\frac{\hbar c^2}{E}\right)^2\frac{(\omega\omega_1/c^2-\boldsymbol{k}\cdot\boldsymbol{k}_1)^2}{(\omega-\boldsymbol{k}\cdot\boldsymbol{v})(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})}\,.$$

The cross-section for double Compton scattering is evaluated explicitly by MANDL and SKYRME ([1]). In this case one has

$$(34)\qquad p'=p+\hbar k-\hbar k_1-\hbar k_2\,,\qquad E'=[m^2c^4+|\boldsymbol{p}+\hbar\boldsymbol{k}-\hbar\boldsymbol{k}_1-\hbar\boldsymbol{k}_2|^2c^2]^{\frac{1}{2}}$$

The six dimensionless invariants

$$(35)\qquad \begin{cases} a_1=\dfrac{\hbar p\cdot k_1}{m^2c^2}\,, & a_2=\dfrac{\hbar p\cdot k_2}{m^2c^2}\,, \quad a_3=-\dfrac{\hbar p\cdot k}{m^2c^2}\,,\\[2ex] b_1=-\dfrac{\hbar p'\cdot k_1}{m^2c^2}\,, & b_2=-\dfrac{\hbar p'\cdot k_2}{m^2c^2}\,, \quad b_3=\dfrac{\hbar p'\cdot k}{m^2c^2}\end{cases}$$

are related by

$$\sum_{i=1}^{3}a_i=\sum_{i=1}^{3}b_i\,.$$

The result found by MANDL and SKYRME, for unpolarized radiation, can be written as the probability

$$(36)\qquad \begin{cases} w(\boldsymbol{p};\,\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)=\dfrac{(2\pi)^4q^6\hbar}{m^4}\,\dfrac{\overline{X}_{DC}}{\omega\omega_1\omega_2}\,\delta\left(\dfrac{E'}{\hbar}-\dfrac{E}{\hbar}-\omega+\omega_1+\omega_2\right),\\[2ex] \overline{X}_{DC}=\dfrac{1}{2}\Big[-(\alpha\beta-\gamma)^2+2(\alpha\beta-\gamma)(\alpha+\beta)-8\alpha\beta+4\gamma-x(\alpha^2+\beta^2)+\\ \quad+(\alpha\beta-\gamma)(\alpha+\beta)x-\dfrac{2x}{AB}\{x^2(\delta-1)-2\delta\}+2x(x+1)\left(\dfrac{1}{A}+\dfrac{1}{B}\right)-\\ \qquad\qquad-2\{2x+\delta(1-x)\}\left(\dfrac{\alpha}{B}+\dfrac{\beta}{A}\right)-\varrho\{\alpha\beta+\gamma(1-x)\}\Big]\,,\\[2ex] \alpha=\displaystyle\sum_{i=1}^{3}\frac{1}{a_i}\,,\qquad \beta=\sum_{i=1}^{3}\frac{1}{b_i}\,,\qquad \gamma=\sum_{i=1}^{3}\frac{1}{a_ib_i}\,,\\[2ex] A=a_1a_2a_3\,,\qquad B=b_1b_2b_3\,,\qquad \delta=\displaystyle\sum_{i=1}^{3}a_ib_i\,,\\[2ex] \varrho=\displaystyle\sum_{i=1}^{3}\left(\frac{a_i}{b_i}+\frac{b_i}{a_i}\right),\qquad x=\sum_{i=1}^{3}a_i=\sum_{i=1}^{3}b_i\,.\end{cases}$$

The δ-function in (36) reduces to its classical counterpart, see (25), for $|a_i|$, $|b_i| \ll 1$. It is shown in Appendix B that $\bar{X}_{DC}$ for $|a_i|$, $|b_i| \ll 1$ reduces to $\bar{X}_{DT}$ (see (30)) to lowest order in an expansion in terms proportional to $\hbar$ in the rest frame of the scattering particle.

Thus formally, double Thomson scattering is the classical counterpart to double Compton scattering. However, it is possible to describe only the (spontaneous) scattering of two photons into one photon in a classical way.

6. – Discussion.

The existence of a regime in which double Compton scattering can be treated classically is only of academic interest for waves in vacuo. Measurement [10] of double Compton scattering is only practical for hard photons where the cross-section is 1/137 times that for single Compton scattering. For soft photons, that is in the classical regime, the ratio of the cross-sections is smaller.

One interesting aspect of our classical treatment is that the inverse of double Compton scattering has a classical counterpart, while the direct process does not. We should add that the existence of inverse double Thomson scattering but not of the direct process in classical physics does not violate the second law of thermodynamics, which, in the form of Kirchhoff's law, requires that an emission process and its associated «inverse» or «absorptive» process must balance in thermal equilibrium. The relevant «inverse» or «absorptive» process to Thomson scattering here is induced Thomson scattering. Including this but not direct Thomson scattering in (27) gives

$$(37)\qquad \frac{\partial N^{\sigma}(\boldsymbol{k})}{\partial t} = \int \mathrm{d}\boldsymbol{p} \int \frac{\mathrm{d}\boldsymbol{k}_1}{(2\pi)^3}\,\frac{\mathrm{d}\boldsymbol{k}_2}{(2\pi)^3}\, w^{\sigma\sigma_1\sigma_2}(\boldsymbol{p};\, \boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)\cdot \Big[\{N^{\sigma_1}(\boldsymbol{k}_1)N^{\sigma_2}(\boldsymbol{k}_2) - N^{\sigma}(\boldsymbol{k})N^{\sigma_1}(\boldsymbol{k})_1 - N^{\sigma}(\boldsymbol{k})N^{\sigma_2}(\boldsymbol{k}_2)\} f(\boldsymbol{p}) + {} + N^{\sigma}(\boldsymbol{k})N^{\sigma_1}(\boldsymbol{k}_1)N^{\sigma_2}(\boldsymbol{k}_2)\hbar(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot\frac{\partial f(\boldsymbol{p})}{\partial \boldsymbol{p}}\Big].$$

This can be rewritten in a purely classical form. In thermal equilibrium

$$(38)\qquad \left\{\begin{aligned} & N^{\sigma}(\boldsymbol{k}) = \frac{T}{\hbar\omega^{\sigma}}, \qquad N^{\sigma_1}(\boldsymbol{k}_1) = \frac{T}{\hbar\omega^{\sigma_1}}, \qquad N^{\sigma_2}(\boldsymbol{k}_2) = \frac{T}{\hbar\omega^{\sigma_2}}, \\ & f(\boldsymbol{p}) \propto \exp[-E/T], \\ & \hbar(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\cdot\frac{\partial f(\boldsymbol{p})}{\partial \boldsymbol{p}} = \frac{-\hbar(\omega^{\sigma} - \omega^{\sigma_1} - \omega^{\sigma_2})}{T} f(\boldsymbol{p}), \end{aligned}\right.$$

where T is the temperature in erg, (37) gives zero as required.

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[10] P. E. CAVANAGH: *Phys. Rev.*, 87, 1131 (1952).

Double Thomson scattering is familiar in a different form in nonlinear plasma physics. The processes described as the coalescence of two waves into one and the corresponding decay of one wave into two waves ([6,8]) can be regarded as resulting from inverse double Thomson scattering with

$$\boldsymbol{K} = \boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2 = 0 \; .$$

The probability (25) is nonzero in this case if one or more of the three waves has longitudinal rather than transverse polarization. For a distribution of scattering electrons with velocities much less than $\omega/|\boldsymbol{k}|$ for all waves involved, these coalescence and decay processes are described by (27) and (28) with the replacement of

$$\int \mathrm{d}^3\boldsymbol{p} f(\boldsymbol{p})\, w^{\sigma\sigma_1\sigma_2}(\boldsymbol{p};\, \boldsymbol{k},\, \boldsymbol{k}_1,\, \boldsymbol{k}_2)$$

$$n_e(2\pi)^3\,\delta^3(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, w^{\sigma\sigma_1\sigma_2}(\boldsymbol{p} = 0;\, \boldsymbol{k},\, \boldsymbol{k}_1,\, \boldsymbol{k}_2) \; .$$

In particular, emission at twice the local plasma frequency from the active solar corona ([11,12]) and the scattering of transverse waves in a microturbulent plasma ([5,6,13]) can both be regarded as special cases of double Thomson scattering.

APPENDIX A

To carry out the expansions indicated in Sect. **2** we introduce the following simplifying notation:

$$\text{(A.1)} \qquad \begin{cases} \tilde{\omega} = \omega - \boldsymbol{k}\cdot\boldsymbol{v}\,, \qquad \tilde{\omega}_1 = \omega_1 - \boldsymbol{k}_1\cdot\boldsymbol{v}\,, \qquad \tilde{\omega}_2 = \omega_2 - \boldsymbol{k}_2\cdot\boldsymbol{v}\,, \\ f_{ij}(\boldsymbol{v}, k) = \tilde{\omega}\delta_{ij} + k_i v_j\,, \qquad g_{ij}(\boldsymbol{v}, k) = \left(\delta_{ir} - \dfrac{v_i v_r}{c^2}\right) f_{rj}(\boldsymbol{v}, k)\,, \\ a_{ij}(\boldsymbol{v}, k, k_1) = f_{ir}(\boldsymbol{v}, k)\, g_{rj}(\boldsymbol{v}, k_1) = g_{ir}(\boldsymbol{v}, k)\, f_{rj}(\boldsymbol{v}, k_1)\,, \end{cases}$$

where $\boldsymbol{v}$ refers to the zeroth-order velocity.

([11]) M. R. KUNDU: *Solar Radio Astronomy* (New York, 1965).
([12]) D. F. SMITH: *Adv. Astron. Astrophys.*, 7, 147 (1970).
([13]) S. A. COLGATE, E. P. LEE and M. N. ROSENBLUTH: *Astrophys. Journ.*, 162, 649 (1970).

Solving *(5)* gives

(A.2)
$$\begin{cases} \begin{bmatrix} v_i^{(1)}(t) \\ r_i^{(1)}(t) \end{bmatrix} = \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \exp\left[-i\tilde{\omega}_1 t\right] \begin{bmatrix} v_i^{(1)}(k_1) \\ r_i^{(1)}(k_1) \end{bmatrix}, \\ v_i^{(1)}(k_1) = \dfrac{iq}{m\gamma\omega_1\tilde{\omega}_1} g_{ij}(\boldsymbol{v}, k_1) E_j(k_1), \\ r_i^{(1)}(k_1) = \dfrac{i}{\tilde{\omega}_1} v_i^{(1)}(k_1), \qquad \gamma = \dfrac{E}{mc^2}. \end{cases}$$

In (7) one has

(A.3)
$$A_{ij}(\boldsymbol{v}; k, k_1) = \frac{i}{m\gamma} \frac{a_{ij}(\boldsymbol{v}, k, k_1)}{\tilde{\omega}\tilde{\omega}_1} \qquad (\tilde{\omega} = \tilde{\omega}_1).$$

Solving (6) gives

(A.4)
$$\begin{cases} \begin{bmatrix} v_i^{(2)}(t) \\ r_i^{(2)}(t) \end{bmatrix} = \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \exp\left[-i\tilde{\omega}' t\right] \begin{bmatrix} v_i^{(2)}(k_1, k_2) \\ r_i^{(2)}(k_1, k_2) \end{bmatrix}, \\ \tilde{\omega}' = \tilde{\omega}_1 + \tilde{\omega}_2, \qquad r_i^{(2)}(k_1, k_2) = \dfrac{i}{\tilde{\omega}'} v_i^{(2)}(k_1, k_2), \\ v_i^{(2)}(k_1, k_2) = \dfrac{q^2}{2m^2\gamma^2\omega_1\omega_2}\left[\left(\delta_{ij} - \dfrac{v_i v_j}{c^2}\right)\dfrac{k_{1s} g_{sl}}{\tilde{\omega}_2\tilde{\omega}'} - \right. \\ \quad - \left(\delta_{si} - \dfrac{v_s v_i}{c^2}\right)\dfrac{k_{1s} g_{jl}}{\tilde{\omega}_2\tilde{\omega}'} + \dfrac{k_{1s} g_{sl} g_{ij}}{\tilde{\omega}_2^2\tilde{\omega}'} + \dfrac{v_i}{c^2}\dfrac{g_{rl} f_{rj}}{\tilde{\omega}_1\tilde{\omega}_2} + \dfrac{v_t}{c^2}\dfrac{f_{tl} g_{ij}}{\tilde{\omega}_1\tilde{\omega}_2} + \\ \quad \left. + \dfrac{1}{2}\dfrac{v_t}{c^2}\dfrac{f_{tj} g_{il}}{\tilde{\omega}_1\tilde{\omega}_2} + \text{symmetrized terms}\right] E_j(k_1) E_l(k_2), \end{cases}$$

where the symmetry arises from

$$v_i(k_1, k_2) = v_i(k_2, k_1),$$

and amounts to simultaneously interchanging (j,*k*,,*o*,)with (*l*, $\boldsymbol{k}_2$, *o*,)*.* In *(A.4)* we write

$$g_{rj} \equiv g_{rj}(\boldsymbol{v}, k_1), \qquad g_{rl} \equiv g_{rl}(\boldsymbol{v}, k_2)$$

and so on. In (8) one has

(A.5)
$$A_{ijl}(\boldsymbol{v}; k, k_1, k_2) = \frac{1}{2}\left[\frac{g_{ji} k_{1t} g_{tl}}{\tilde{\omega}^2\tilde{\omega}_2} + \frac{g_{li} k_{2t} g_{tj}}{\tilde{\omega}^2\tilde{\omega}_1} - \frac{k_t g_{tl} g_{ij}}{\tilde{\omega}_2^2\tilde{\omega}_1} - \frac{k_t g_{tj} g_{il}}{\tilde{\omega}_1^2\tilde{\omega}} - \right.$$
$$- \frac{k_{1t} g_{ti} g_{jl}}{\tilde{\omega}_2^2\tilde{\omega}_1} - \frac{k_{2t} g_{ti} g_{lj}}{\tilde{\omega}_1^2\tilde{\omega}} + \frac{v_s}{c^2}\frac{1}{\tilde{\omega}_1\tilde{\omega}_2\tilde{\omega}}\{g_{rl} f_{rj} f_{si} + f_{sl} g_{rj} f_{ri} + f_{sj} g_{rl} f_{ri}\} -$$
$$\left. - v_i \frac{k_t g_{tl} k_t g_{tj}}{\tilde{\omega}_1^2\tilde{\omega}_2^2} + v_j \frac{k_{1t} g_{tl} k_{1s} g_{si}}{\tilde{\omega}^2\tilde{\omega}_2^2} + v_l \frac{k_{2t} g_{tj} k_{2s} g_{si}}{\tilde{\omega}^2\tilde{\omega}_1^2}\right].$$

In the limit $v = 0$, A_{ijl} reduces to

$$(A.6)\qquad A_{ijl}(\boldsymbol{v}=0;\,k,k_1,k_2)=\frac{1}{2}\left[-\frac{K_j}{\omega_1}\,\delta_{il}-\frac{K_l}{\omega_2}\,\delta_{ij}+\frac{K_i}{\omega}\,\delta_{jl}\right]-\\-\frac{1}{2}\left[\frac{k_{1j}}{\omega_1}\,\delta_{il}+\frac{k_{2l}}{\omega_2}\,\delta_{ij}+\frac{k_i}{\omega}\,\delta_{il}\right],\qquad \boldsymbol{K}=\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2\,.$$

Appendix B

For $|a_i|$, $|b_i|\ll 1$ expansions of $\overline{X}_{DC}$, given by *(36)* in small quantities, is formally equivalent to an expansion in $\hbar$. Writing

$$(B.1)\qquad \begin{cases} a_1=\dfrac{\hbar E}{m^2c^4}\,(\omega_1-\boldsymbol{k}_1\cdot\boldsymbol{v})\,, \qquad b_1=-a_1(1-\varDelta_1)\,,\\[2ex] a_1\varDelta_1=-\dfrac{\hbar^2K\cdot k_1}{m^2c^2}\,,\\[2ex] \mathrm{K}=(\boldsymbol{K},\varOmega)=(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2,\,\omega-\omega_1-\omega_2) \end{cases}$$

and so on, one has

$$(B.2)\qquad x=\sum a_i=\frac{1}{2}\sum a_i\varDelta_i=\frac{\hbar^2K\cdot K}{2m^2c^2}\,.$$

The lowest-order nonvanishing terms are

$$(B.3)\qquad \lim_{\hbar\to 0}\overline{X}_{DC}=-\left\{-\frac{x}{A}\sum\varDelta_i+\frac{1}{A}\sum a_i\varDelta_i^2+\sum\frac{\varDelta_i}{a_i}\right\}^2+\\+\left(\sum\frac{\varDelta_i}{a_i}\right)^2-\frac{2x}{A}\sum\varDelta_i+2\left(\alpha\sum\varDelta_i-\sum\frac{\varDelta_i}{a_i}\right)^2-\\-10\left(-\frac{x}{A}\sum\varDelta_i+\frac{1}{A}\sum a_i\varDelta_i^2\right)-\frac{2}{A^2}\,(x-\alpha A)\sum a_i^2\varDelta_i^2-\\-2\alpha^2\sum\varDelta_i^2+\frac{2x^3}{A^2}-\frac{4\alpha x^2}{A}-4\alpha^2x\,.$$

To lowest order one can rewrite **(B.3)** in the form

$$(B.4)\qquad \lim_{\hbar\to 0}\overline{X}_{DC}=-\left\{-\frac{x}{A}\sum\varDelta_i+\frac{1}{A}\sum a_i\varDelta_i^2+\sum\frac{\varDelta_i}{a_i}\right\}^2-\\-\sum\frac{\varDelta_i^2}{a_i^2}-\frac{4x}{A}\sum\varDelta_i-\frac{2x}{A^2}\sum a_i^2\varDelta_i^2+\frac{2x^3}{A^2}-4\alpha^2x+\frac{4\alpha x^2}{A}\,.$$

For $\boldsymbol{v}=0$ (B.4) is equal to $\bar{X}_{DT}$ defined by (30). To show this we rewrite $\bar{X}_{DT}$ using, for $\boldsymbol{v}=0$,

$$\text{(B.5)}\qquad \begin{cases} x=-\dfrac{\hbar^2\,|\boldsymbol{K}|^2}{2m^2c^2}, \qquad a_1=\dfrac{\hbar\omega_1}{mc^2}, \qquad \Delta_1=\dfrac{\hbar\varkappa_1\cdot\boldsymbol{K}}{mc}, \\[2ex] \varkappa_1\cdot\varkappa_2=1-\dfrac{a_3x}{A}+\dfrac{a_3^2\Delta_3}{A}, \text{ etc.} \end{cases}$$

In (30) one has

$$\text{(B.6)}\qquad \begin{cases} \left(\dfrac{\varkappa\cdot\varkappa_2}{\omega_1}\right)^2+\left(\dfrac{\varkappa\cdot\varkappa_1}{\omega_2}\right)^2+\left(\dfrac{\varkappa_1\cdot\varkappa_2}{\omega}\right)^2=\left[\dfrac{\hbar}{mc^2}\right]^2\cdot \\[2ex] \qquad\qquad \cdot\left\{\alpha^2+\dfrac{3x^2}{A^2}+\dfrac{1}{A^2}\sum a_i^2\Delta_i^2-\dfrac{2\alpha x}{A}+\dfrac{2}{A}\sum\Delta_i-\dfrac{4x^2}{A^2}\right\}, \\[2ex] \dfrac{\varkappa\cdot\varkappa_2\,\varkappa_1\cdot\boldsymbol{K}}{\omega_1}+\dfrac{\varkappa\cdot\varkappa_1\,\varkappa_2\cdot\boldsymbol{K}}{\omega_2}-\dfrac{\varkappa_1\cdot\varkappa_2\,\varkappa\cdot\boldsymbol{K}}{\omega}= \\[2ex] \qquad\qquad =\dfrac{1}{c}\left\{-\dfrac{x}{A}\sum\Delta_i+\dfrac{1}{A}\sum a_i^2\Delta_i+\sum\dfrac{\Delta_i}{a_i^2}\right\}, \\[2ex] \left(\dfrac{\varkappa_1\cdot\boldsymbol{K}}{\omega_1}\right)^2+\left(\dfrac{\varkappa_2\cdot\boldsymbol{K}}{\omega_2}\right)^2+\left(\dfrac{\varkappa\cdot\boldsymbol{K}}{\omega}\right)^2=\dfrac{1}{c^2}\sum\dfrac{\Delta_i^2}{a_i^2}. \end{cases}$$

Collecting terms in $\bar{X}_{DT}$ one finds

$$\text{(B.7)}\qquad \bar{X}_{DT}=\lim_{\hbar\to 0}\bar{X}_{DC}.$$

This proof applies for $\boldsymbol{v}=0$. For $\boldsymbol{v}\neq 0$ the generalization of $\bar{X}_{DT}$ is given by (B.4) with (B.1) because (B.4) is a Lorentz invariant.

● RIASSUNTO (*)

Si dimostra che l'interazione classica fra una particella carica ed un campo di radiazione (scattering di Thomson) coinvolge una gerarchia di processi analoga a quella per lo scattering fotone-elettrone (scattering di Compton) nell'elettrodinamica quantistica. L'affermazione che gli effetti Compton doppi e multipli sono intrinsecamente processi quantomeccanici è sbagliata; si mostra che lo scattering di Thomson doppio è la controparte classica dell'effetto Compton doppio. Però solo il processo inverso dello scattering di Compton doppio (due fotoni diffusi in un solo fotone) ha una controparte classica; il processo diretto non lo ha. Si dimostra che i familiari esempi dell'interazione di tre onde nella teoria non lineare del plasma sono casi particolari dello scattering di Thomson doppio.

(*) *Traduzione a cura della Redazione.*

Классический аналог для двойного комптоновского рассеяния.

Резюме (*). — Показывается, что классическое взаимодействие между заряженной частицей и полем излучения (томсоновское рассеяние) включает иерархию процессов, аналогичных процессу для фотон-электронного рассеяния (комптоновское рассеяние) в квантовой электродинамике. Утверждения, что двойные и многократные комптоновские эффекты являются существенно квантовомеханическими процессами, являются неверными. Показывается, что двойное томсоновское рассеяние представляет классический аналог двойного комптоновского эффекта. Однако, это утверждение справедливо только для процесса, обратного двойному комптоновскому рассеянию (два фотона рассеялись в один фотон), который имеет классический аналог; а для прямого процесса нет. Показывается, что аналогичные примеры взаимодействия трех волн в нелинейной теории плазмы представляют частные случаи двойного томсоновского рассеяния.

(*) *Переведено редакцией.*

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