Plasma Physics, Vol. 14, pp. 1035 to 1046. Pergamon Press 1972. Printed in Northern Ireland

SYMMETRY PROPERTIES OF NONLINEAR RESPONSES IN A PLASMA

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(Received **21** *April* **1972)**

Abstract—The symmetry properties of tensors describing the nonlinear responses of a collisionless magnetoactive plasma are discussed, concentrating on the generalization of the Onsager relations. The Onsager relations generalize into two sets of relations called time-reversal-invariance relations and crossing symmetries. For the parts of the tensors of practical relevance the following properties are found. The tensor describing the relation between the induced current and a product of n vector potentials changes sign by $(-)^{n+1}$ under reversal of the signs of the frequencies and of the sense of the background magnetic field, and is completely symmetric under permutations of the indices with associated permutations of the arguments. These symmetry properties underlie the reciprocal and crossing symmetries of coefficients describing the consequent nonlinear process.

1. INTRODUCTION

IN PRINCIPLE, the response of a collisionless plasma to an arbitrary electromagnetic disturbance can be calculated, using the Vlasov equation, to all orders of nonlinearity. For each order of nonlinearity there is a tensor which characterizes the response of the medium. We shall define the tensors to relate the induced current (the response) to the vector potential (the disturbance) in Fourier transformed space. In practice, the linear response is of major significance and the first, and sometimes the second, nonlinear response is of interest. Besides being of little practical significance, the still higher order responses are associated with processes that occur more and more slowly; the collisionless assumption and so the use of the Vlasov equation must become invalid for sufficiently high order responses in any plasma.

The tensor describing the linear response (conventionally chosen to be the dielectric tensor) satisfies three well-known sets of relations. These are the reality conditions, the causal relations (the Kramers–Kronig relations) and the reciprocal relations (Onsager relations). It is apparent on physical grounds that the tensors describing the nonlinear responses must satisfy relations analogous to these. Indeed the reality conditions are trivially satisfied and the causal relations can be imposed or derived, depending on one's approach, for all the tensors. There is an additional type of symmetry for the nonlinear responses; this arises from permuting dummy indices and arguments in the product of the amplitudes of the disturbances in the defining relation. However, the generalization of the set of relations known as the reciprocal relations in the linear case is not obvious.

The new result found in this paper is that the Onsager relations generalize into two sets of relations for the higher rank tensors. These sets are called time-reversal-invariance relations and crossing symmetries respectively. Neither set can be written down in closed form, at least not in any simple way. In practice one is concerned with only the non-resonant contributions to the tensors, i.e. with the non-dissipative higher order responses (even non-linear Landau damping is due to the dissipative part of the linear response). For the non-resonant contributions, the two sets of relations are given by (26) and (31) below respectively.

In Section 2 we define the tensors and write down the relations which either are obviously satisfied or can be readily imposed. In Section 3 the tensors are derived following the method used by AL'TSHUL' and KARMAN (1965). The remaining symmetry properties are deduced in Section 4, and in Section 5 we discuss them and their implications.

2. THE TENSORS

The induced current $\mathbf{j}(k)$, with k = (k, ω), which flows due to the presence of an electromagnetic disturbance described by a vector potential $\mathbf{A}(k)$ (with the gauge condition such that the scalar potential vanishes) can be expended in powers of $\mathbf{A}(k)$. We write this expansion in the form

$$j_i(k) = \sum_{n=1}^{\infty} \int d\lambda^{(n)} \kappa_{ij_1\ldots j_n}(k, k_1, \ldots, k_n) A_{j_1}(k_1) \ldots A_{j_n}(k_n), \tag{1}$$

with

$$d\lambda^{(n)} = \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} (2\pi)^4 \, \delta^4(k - k_1 - \ldots - k_n). \tag{2}$$

The nth order term in (1) defines the (n + 1)th rank tensor $\kappa_{ij_1\ldots j_n}$. This tensor describes the complete response to the complete disturbance. The induced charge density is related to j by the equation of charge continuity so that any quantity describing the response can be written in terms of j. Likewise $\mathbf{A}(k)$ contains both the electric and magnetic fields in the disturbance:

$$\mathbf{E}(k) = \frac{i\omega}{c} \mathbf{A}(k), \qquad \mathbf{B}(k) = i\mathbf{k} \times \mathbf{A}(k). \tag{3}$$

The argument k in $\kappa_{ij_1\ldots j_n}$ is redundant, i.e. $\kappa_{ij_1\ldots j_n}$ is physically meaningful only for

$$k = \sum_{r=1}^{n} k_r.$$

However, it is convenient to retain this argument explicitly. The linear tensor $\kappa_{ij}(k, k_1)$ is related to the dielectric tensor $\varepsilon_{ij}(k)$ by

$$\varepsilon_{ij}(k) = \delta_{ij} + \frac{4\pi c}{\omega^2} \kappa_{ij}(k, k), \tag{4}$$

where δ_{ij} is the unit tensor.

Equation (1) is the Fourier transform of a real equation, and so each tensor satisfies reality conditions

$$\kappa_{ij_1\ldots j_n}(k, k_1, \ldots, k_n) = \kappa^*_{ij_1\ldots j_n}(-k, -k_1, \ldots, -k_n), \tag{5}$$

where $-k_r = (-\mathbf{k}_r, -\omega_r)$ and where * denotes complex conjugation.

One can freely permute dummy indices $j_1, \ldots, j_n$ and associated arguments in (1), consequently the only physically meaningful part of each tensor satisfies

$$\kappa_{ij_1\ldots j_n}(k, k_1, \ldots, k_n) = \kappa_{i(j_1\ldots j_n)}(k, k_1, \ldots, k_n), \tag{6}$$

where brackets ( ) around n indices indicate that one take $1/n!$ of the sum of all permutations of the indices and associated arguments. Specifically, for $n = 2$, equation (6) reads

$$\kappa_{ijl}(k, k_1, k_2) = \tfrac{1}{2}[\kappa_{ijl}(k, k_1, k_2) + \kappa_{ilj}(k, k_2, k_1)]. \tag{6'}$$

The causal requirement is that the response be subsequent to the disturbances. In view of (6) all time-orderings of the disturbances are already included. Consequently, one can impose the causal requirements in the form (see comments at the end of Section 5 however)

$$\kappa_{ij_1\ldots j_n}(k, k_1, \ldots, k_n) = \left(\frac{i}{2\pi}\right)^n \int \frac{d\omega_1' \ldots d\omega_n'}{(\omega_1 - \omega_1' + i0)\ldots(\omega_n - \omega_n' \ldots i0)} \kappa_{ij_1\ldots j_n}(k', k_1', \ldots, k_n'), \quad (7)$$

with $k_r' = (\mathbf{k}_r, \omega_r')$, $r = 1, \ldots, n$, and $k' = \left(\mathbf{k}, \sum_{r=1}^{n} \omega_r'\right)$. The quantity $i0$ is an infinitesimal positive imaginary constant which is equivalent to a prescription for evaluation the singular integral, see MONTGOMERY and TIDMAN (1964) for example.

3. DERIVATION OF THE TENSORS

We follow AL'TSHUL' and KARPMAN (1965) in deriving the tensors for a collisionless magnetoactive plasma starting from Liouville's equation. We generalize the derivation of Al'tshul' and Karpman by including relativistic effects (this simplifies the formal calculation).

The Hamiltonian for a particle with charge q and rest mass m moving in the presence of a constant background magnetic field $\mathbf{B}_0$ and fluctuating electromagnetic fields is given by

$$H(\mathbf{p}, \mathbf{r}, t) = \left[m^2c^4 + \left(\mathbf{p} - \frac{q}{c}\mathbf{A}_0 - \frac{q}{c}\mathbf{A}(\mathbf{r}, t)\right)^2 c^2\right]^{1/2}, \quad (8)$$

where

$$\mathbf{A}_0 = \tfrac{1}{2}\mathbf{B}_0 \times \mathbf{r}$$

is the vector potential for the background field. In equation (8), and throughout, $\mathbf{p}$ the canonical momentum and not the kinetic momentum. In the collisionless approximation Liouville's equation factorized into a set of identical equations describing the evolution of the single particle distribution function $f(\mathbf{p}, \mathbf{r}, t)$ with the single particle Hamiltonian given by (8). One has

$$\frac{\partial f}{\partial t}(\mathbf{p}, \mathbf{r}, t) = [H(\mathbf{p}, \mathbf{r}, t), f(\mathbf{p}, \mathbf{r}, t)], \quad (9)$$

where

$$[A, B] = \frac{\partial A}{\partial \mathbf{r}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{r}} \quad (10)$$

is the Poisson bracket.

We solve equation (9) by a perturbation procedure in the interaction picture. The unperturbed Hamiltonian reads

$$H^{(0)}(\mathbf{p}, \mathbf{r}) = \left[m^2c^4 + \left(\mathbf{p} - \frac{q}{c}\mathbf{A}_0\right)^2 c^2\right]^{1/2} \quad (11)$$

The unperturbed distribution function $f(\mathbf{p})$ is assumed to be independent of space and time. Time independence requires that $f(\mathbf{p})$ be invariant under $\mathbf{p}, \mathbf{B}_0 \to -\mathbf{p}, -\mathbf{B}_0$.

In the interaction picture one solves the unperturbed (Hamilton's) equations of motion exactly and then perturb about the zeroth order trajectory. The solution of the unperturbed equations of motion can be written in the form

$$p_i(t) = \alpha_{ij}(t, t_0)p_j(t_0) + \beta_{ij}(t, t_0)r_j(t_0), \tag{12a}$$

$$r_i(t) = \gamma_{ij}(t, t_0)p_j(t_0) + \eta_{ij}(t, t_0)r_j(t_0), \tag{12b}$$

where t_0 is an arbitrary initial time. We write down explicit expressions for the coefficients in (12) in the Appendix. The interaction Hamiltonian is defined by

$$H_I(\mathbf{p}(t_0), \mathbf{r}(t_0), t) = H(\mathbf{p}(t), \mathbf{r}(t), t) - H^{(0)}(\mathbf{p}(t), \mathbf{r}(t)), \tag{13}$$

where $\mathbf{p}(t)$ and $\mathbf{r}(t)$ are re-expressed in terms of $\mathbf{p}(t_0)$ and $\mathbf{r}(t_0)$ using (12). The distribution function in the interaction picture is defined by

$$F(\mathbf{p}(t_0), \mathbf{r}(t_0), t) = f(\mathbf{p}(t), \mathbf{r}(t), t) \tag{14}$$

in like manner.

The variables in the interaction picture are $\mathbf{p}(t_0)$, $\mathbf{r}(t_0)$ (and rand t). Poisson brackets are to be taken with respect to these variables. [The transformation from $\mathbf{p}(t)$, $\mathbf{r}(t)$ to $\mathbf{p}(t_0)$, $\mathbf{r}(t_0)$ is a canonical transformation so that the numerical value of Poisson brackets is unaffected by the change, see GOLDSTEIN (1959) for example.] We are free to choose $t_0 = -\infty$; we write $\mathbf{p}(-\infty) = \mathbf{p}_0$, $\mathbf{r}(-\infty) = \mathbf{r}_0$.

Leaving the variables $\mathbf{p}_0$, $\mathbf{r}_0$ understood, equation (9) becomes, in the interaction picture,

$$\frac{\partial F(t)}{\partial t} = [H_I(t), F(t)]. \tag{15}$$

Expanding $F(t)$ in powers of H_I, with $F^{(0)}$ the unperturbed distribution function, one finds

$$\begin{aligned} F(t) &= F^{(0)} + \sum_{n=1}^{\infty} F^{(n)}(t), \\ F^{(1)}(t) &= \int_{-\infty}^{t} \mathrm{d}t_1 [H_I(t_1), F^{(0)}], \\ F^{(n)}(t) &= \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n [H_I(t_1), [H_I(t_2), \ldots, [H_I(t_n), F^{(0)}] \ldots]]. \end{aligned} \tag{16}$$

The induced current density is found by multiplying by the single particle current density

$$q\left[\frac{\partial H}{\partial \mathbf{p}}(\mathbf{p}, \mathbf{r}, t)\right]_{\mathbf{p}=\mathbf{p}(t), \mathbf{r}=\mathbf{r}(t)} \delta^3(\mathbf{r} - \mathbf{r}(t))$$

by $F^{(n)}(t)$, integrating over phase space and summing over n. [For simplicity we include the contribution of only one species of particle.] It is convenient to write the result in the form (the interaction energy)

$$\frac{-1}{c}\,\mathbf{j}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t)$$
$$=\sum_{n=0}^{\infty}\int d^3\mathbf{p}_0\,d^3\mathbf{r}_0 H_I(t)\int_{-\infty}^{t}dt_1\ldots\int_{-\infty}^{t_{n-1}}dt_n[H_I(t_1),\ldots,[H_I(t_n),F^{(0)}]\ldots]$$
$$=\sum_{n=0}^{\infty}\int d^3\mathbf{p}_0\,d^3\mathbf{r}_0 F^{(0)}\int_{-\infty}^{t}dt_1\ldots\int_{-\infty}^{t_{n-1}}dt_n[\ldots[H_I(t),H_I(t_1)],\ldots,H_I(t_n)], \quad (17)$$

where we partially integrate to reverse the order of the Poisson brackets.

The nth order induced current can be picked out of (17) by writing

$$H_I(t)=\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{-q}{c}\right)^n\int d^3\mathbf{r}\,\frac{\partial^n H^{(0)}}{\partial p_{0j_1}\ldots\partial p_{0j_n}}\,A\quad,t)\ldots A\quad,t)\,d^3(r-\mathbf{r}(t)), \quad (18)$$

and picking out all terms which contain a product of n + 1 A's in (17). The nth order current includes one term which involves no time-integral (n = 0 in (17)) and an (n + 1)th derivative of $H^{(0)}$, terms with one time integral and a product of an nth derivative and a first derivative of $H^{(0)}$, and so on down to a single term involving n time-integrals and a product of n + 1 first derivatives of $H^{(0)}$.

We refer to this hierarchy of terms as the nth contracted term, the (n − 1)th contracted terms and so on down to the uncontracted term, respectively. The rth contracted terms (r ≤ n) consist of terms involving an (r + 1)th derivative of $H^{(0)}$ multiplied by n first derivatives of $H^{(0)}$, and so on down to a product of r second derivatives of $H^{(0)}$ and n + 1 − 2r (if this greater than or equal to zero) first derivatives of $H^{(0)}$. Each of these contracted terms satisfy slightly different symmetry properties. For example $\partial^r H^{(0)}/p_{0j_1}\ldots\partial p_{0j_r}$ is completely symmetric under permutations of $j_1\ldots j_r$.

A consideration of the symmetry properties of all contracted terms is a formidable task in general. However, one can establish the symmetry properties by induction including only the uncontracted term for arbitrary n. The symmetry properties of the contracted terms are related in a simple way to the symmetry properties of lower order uncontracted terms.

For example, consider the first contracted terms and the doubly contracted term for n = 2. From (17) and (18) the first contracted terms read

$$\frac{q}{2}\left(\frac{q}{c}\right)^2\int d^3\mathbf{p}_0\,d^3\mathbf{r}_0 F^{(0)}\int_{-\infty}^{t}dt_1\int d^3\mathbf{r}_1$$
$$\times\left\{\left[\frac{\partial^2 H^{(0)}}{\partial p_{0i}\,\partial p_{0j}}\,\delta^3(\mathbf{r}-\mathbf{r}(t)),\frac{\partial H^{(0)}}{\partial p_{0l}}\,\delta^3(\mathbf{r}_1-\mathbf{r}(t_1))\right]A_j(\mathbf{r},t)A_l(\mathbf{r}_1,t_1)+(j\leftrightarrow l)\right.$$
$$\left.+\left[\frac{\partial H^{(0)}}{\partial p_{0i}}\,\delta^3(\mathbf{r}-\mathbf{r}(t)),\frac{\partial^2 H^{(0)}}{\partial p_{0j}\,\partial p_{0l}}\,\delta^3(\mathbf{r}_1-\mathbf{r}(t_1))\right]A_j(\mathbf{r}_1,t_1)A_l(\mathbf{r}_1,t_1)\right\}$$

and the doubly contracted term reads

$$\frac{q}{6}\left(\frac{q}{c}\right)^2\int d^3\mathbf{p}_0\,d^3\mathbf{r}_0 F^{(0)}\frac{\partial^3 H^{(0)}}{\partial p_{0i}\,\partial p_{0j}\,\partial p_{0l}}\,d^3(r-\mathbf{r}(t))A_j(\mathbf{r},t)A_l(\mathbf{r},t).$$

After Fourier transforming, the total tensor $\kappa_{ijl}(k, k_1, k_2)$ can be written in the form

$$\kappa_{ijl}(k, k_1, k_2) = \kappa^{(u)}_{ijl}(k, k_1, k_2) + \{\kappa_{\underline{ij}l}(k - k_1, k_2) + \kappa_{\underline{il}j}(k - k_2, k_1) + \kappa_{i\underline{jl}}(k, k_1 + k_2)\} + \kappa_{\underline{ijl}},$$

where the first term is the uncontracted term (u), the three terms in brackets are the first contracted terms and the final (constant) term is the doubly contracted term. The contracted indices are joined by a line. Each contracted term is symmetric under permutations of the contracted indices because of the multiple derivatives of $H^{(0)}$. Thus, for example, one has

$$\kappa_{\underline{il}j}(k - k_2, k_1) = \kappa_{\underline{li}j}(k - k_2, k_1)$$

and otherwise $\kappa_{\underline{il}j}$ has symmetry properties analogous to those of the uncontracted linear term $\kappa^{(u)}_{ij}$ with j and l treated as a single index.

The completely contracted term for arbitrary n is a constant (independent of k, k,, ... , k,) and satisfies

$$\kappa_{\underline{ij_1\ldots j_n}} = \kappa_{\underline{(ij_1\ldots j_n)}} \tag{19}$$

trivially. For this term one readily establishes the symmetry property

$$\kappa_{\underline{ij_1\ldots j_n}}|_{\mathbf{B}_0} = (-)^{n+1}\kappa_{\underline{ij_1\ldots j_n}}|_{-\mathbf{B}_0} \tag{20}$$

by using the fact that $F^{(0)}$ is invariant under $\mathbf{p}_0, \mathbf{B}_0 \to -\mathbf{p}_0, -\mathbf{B}_0$.

In Section 4 we consider the extension of the symmetry properties (19) and (20) to the tensor as a whole. For this purpose we need consider only the uncontracted part, using induction to establish the result for the contracted parts.

4. THE SYMMETRY PROPERTIES

We consider only the uncontracted part of the tensor $\kappa_{ij_1\ldots j_n}$, omitting the superscript (u) for simplicity. Starting from (17) this part is given by

$$j_i(\mathbf{r}, t) = q\left(\frac{-q}{c}\right)^n \int d^3\mathbf{p}_0\, d^3\mathbf{r}_0 F^{(0)} \int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{n-1}} dt_n \times \int d^3r_1 \ldots d^3r_n \left[\ldots\left[\frac{\partial H^{(0)}}{\partial p_{0i}}\delta^3(\mathbf{r} - \mathbf{r}(t)), \frac{\partial H^{(0)}}{\partial p_{0j_1}}\delta^3(\mathbf{r}_1 - \mathbf{r}(t_1))\right], \ldots, \frac{\partial H^{(0)}}{\partial p_{0j_n}}\delta^3(\mathbf{r}_n - \mathbf{r}(t_n))\right] A_{j_1}(\mathbf{r}_1, t_1) \ldots A_{j_n}(\mathbf{r}_n, t_n). \tag{21}$$

Fourier transforming, one has

$$j_i(k) = q\left(\frac{-q}{c}\right)^n \int d^3\mathbf{p}_0\, d^3\mathbf{r}_0 F^{(0)} \int_{-\infty}^{\infty} dt \int d^3\mathbf{r} \exp\left[-i(\mathbf{k}\cdot\mathbf{r} - \omega t)\right] \times \int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t_{n-1}} dt_n \int d^3\mathbf{r}_1 \ldots \int d^3\mathbf{r}_n \int \frac{d^4k_1}{(2\pi)^4} \ldots \int \frac{d^4k_n}{(2\pi)^4} \times \exp\left[i\sum_{s=1}^{n}(\mathbf{k}_s\cdot\mathbf{r}_s - \omega_s t_s)\right] A_{j_1}(k_1) \ldots A_{j_n}(k_n) \times \left[\ldots\left[\frac{\partial H^{(0)}}{\partial p_{0i}}\delta^3(\mathbf{r} - \mathbf{r}(t)), \frac{\partial H^{(0)}}{\partial p_{0j_1}}\delta^3(\mathbf{r}_1 - \mathbf{r}(t_1))\right], \ldots, \frac{\partial H^{(0)}}{\partial p_{0j_n}}\delta^3(\mathbf{r}_n - \mathbf{r}(t_n))\right]. \tag{22}$$

In evaluating (22) we follow AL'TSHUL' and KARPMAN (1965). Firstly rewrite the time-integrals in terms of

$$\tau_1 = t - t_1, \ldots, \quad \tau_n = t - t_n,$$

and the space-integrals in terms of

$$\mathbf{r}_1' = \mathbf{r} - \mathbf{r}_1, \ldots, \quad \mathbf{r}_n' = \mathbf{r} - \mathbf{r}_n.$$

The time-integrals then run over the ranges $\tau_{r-1} < \tau_r < \infty$ (r $\geq$ 2), $0 < \tau_1 < \infty$. Extending the lower limits to $-\infty$ one defines the complete Fourier transform $\tilde{\kappa}_{ij_1\ldots j_n}(k, k_1, \ldots, k_n)$ by writing

$$\begin{aligned}(2\pi)^4\, \delta^4(k - k_1 - \ldots - k_n)\tilde{\kappa}_{ij_1\ldots j_n}(k, k_1, \ldots, k_n) \\ = q\left(\frac{-q}{c}\right)^n \int \mathrm{d}^3\mathbf{p}_0\, \mathrm{d}^3\mathbf{r}_0 F^{(0)} \int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3\mathbf{r} \int_{-\infty}^{\infty} \mathrm{d}\tau_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}\tau_n \int \mathrm{d}^3\mathbf{r}_1' \ldots \int \mathrm{d}^3\mathbf{r}_n' \\ \times \exp[-i(k \cdot \mathbf{r} - \omega t)] \exp\left[i\sum_{s=1}^{n}(\mathbf{k}_s \cdot \mathbf{r} - \mathbf{k}_s\mathbf{r}_s' - \omega_s t + \omega_s\tau_s)\right] \\ \times \left[\ldots\left[\frac{\partial H^{(0)}}{\partial p_{0i}}\,\delta^3(\mathbf{r} - \mathbf{r}(t)), \frac{\partial H^{(0)}}{\partial p_{0j_1}}\,\delta^3(\mathbf{r}_1' - \mathbf{r}(\tau_1))\right], \ldots, \frac{\partial H^{(0)}}{\partial p_{0j_n}}\,\delta^3(\mathbf{r}_n' - \mathbf{r}(\tau_n))\right].\end{aligned} \tag{23}$$

The tensor $\kappa_{ij_1\ldots j_n}$, which results by writing (22) in the form (1), is then given by

$$\kappa_{ij_1\ldots j_n}(k, k_1, \ldots, k_n) = \left(\frac{i}{2\pi}\right)^n \int \frac{\mathrm{d}\omega_1' \ldots \mathrm{d}\omega_n' \tilde{\kappa}_{ij_1\ldots j_n}(k', k_1', \ldots, k_n')}{(\omega - \omega' + i0)\left(\sum_{s=2}^{n}(\omega_s - \omega_s') + i0\right) \ldots (\omega_n - \omega_n' + i0)}, \tag{24}$$

with $\omega' = \sum_{s=1}^{n} \omega_s'$.

The symmetry property (6) needs to be imposed. Having done so, the resulting expression does indeed satisfy causal relations in the form (7).

We are concerned primarily with the non-resonant contribution. This is obtained from (24) by retaining only the principal value parts of each of the singular integrals. That is, writing

$$\int \frac{\mathrm{d}\omega'}{\omega - \omega' + i0} f(\omega') = \oint \frac{\mathrm{d}\omega'}{\omega - \omega'} f(\omega') - i\pi f(\omega),$$

one retains only the principal parts and ignores the semi-residues. The singly, doubly, etc. parts are those in which one, two, etc., respectively, of the integrals are replaced by their semi-residues.

Let us denote the non-resonant part by a prefix N. Thus we write

$$N\kappa_{ij_1\ldots j_n}(k, k_1, \ldots, k_n) = \left(\frac{i}{2\pi}\right)^n \oint \frac{\mathrm{d}\omega_1' \ldots \mathrm{d}\omega_n'}{\omega - \omega'} \left[\frac{1}{n!}\sum_{p} \frac{\tilde{\kappa}_{ij_{1'}\ldots j_{n'}}(k', k_{1'}', \ldots, k_{n'}')}{\left(\sum_{s=2'}^{n'}(\omega_s - \omega_s')\right) \ldots (\omega_{n'} - \omega_{n'}')}\right], \tag{25}$$

where the sum is over all permutations of 1' to n' amongst 1 to n.

Time-reversal invariance

The time-reversal-invariance property of $\tilde{\kappa}_{ij_1\ldots j_n}$ follows from the fact that the equations of motion are invariant under $\mathbf{p}_0$, $\mathbf{B}_0$, $t \to -\mathbf{p}_0$, $-\mathbf{B}_0$, $-t$. It is straightforward to show that the symmetry property (see the Appendix)

$$N\kappa_{ij_1\ldots j_n}(\mathbf{k}, \omega; \mathbf{k}_1, \omega_1; \ldots; \mathbf{k}_n, \omega_n)|_{\mathbf{B}_0}$$
$$= (-)^{n+1}N\kappa_{ij_1\ldots j_n}(\mathbf{k}_1 - \omega; \mathbf{k}_1, -\omega_1; \ldots; \mathbf{k}_n, -\omega_n)|_{-\mathbf{B}_0} \quad (26)$$

is satisfied. The doubly, and all even multiply, resonant contributions satisfy (26) while the singly, and all odd multiply, resonant contribution satisfy an analogous relation with $(-)^{n+1}$ replaced by $(-)^n$. It is easily shown that the contracted terms satisfy the same symmetry properties as the uncontracted term.

The crossing symmetry

The final set of relations are consequences of the symmetry properties satisfied by the Poisson brackets. It is well known that one has the anti-symmetry property

$$[A, B] + [B, A] = 0, \quad (27)$$

and the Jacobi identity

$$[[A, B], C] + [[C, A], B] + [[B, C], A] = 0. \quad (28)$$

We have been unable to find a suitable discussion of the symmetry properties relevant when an arbitrary number of quantities is included. For four quantities one has

$$[[[A, B], C], D] + [[[B, A], D], C] +$$
$$[[[C, D], A], B] + [[[D, C], B], A] = 0, \quad (29)$$

but proceeding by trial and error to find the higher order symmetry properties rapidly becomes impractical.

We can write down the symmetry properties which result from the sequence of Poisson brackets in closed form. One has

$$\tilde{\kappa}_{(ij_1)j\ldots j_n} = 0,$$
$$\tilde{\kappa}_{[ij_1j_2]j_3\ldots j_n} = 0,$$
$$\tilde{\kappa}_{[ij_1j_2j_3]j_4\ldots j_n} = 0, \quad (30)$$

$$\tilde{\kappa}_{[ij_1\ldots j_n]} = 0,$$

where we omit the arguments (which are to be permuted with the associated indices). Square brackets [ ] around n indices indicate that one takes $1/n!$ of the sum of all even permutations minus the sum of all odd permutations. The first of (30) is just the condition (27), the second together with he first leads to (28), and the third with the first and the second leads to (29). However, it becomes increasingly difficult to identify the essentially new features which result as one proceeds down the sequence of identities (30).

For $n = 1$, 2 and 3 we have been able to establish the symmetry property

$$N\kappa_{ij_1\ldots j_n}(k, k_1, \ldots, k_n) = N\kappa_{(ij_1\ldots j_n)}(k, k_1, \ldots, k_n). \tag{31}$$

The result is plausible for arbitrary n but we have been unable to construct a proof by induction.

The proof for $n = 1$ is trivial. For n = 2 one has

$$N\kappa_{ijl}(k, k_1, k_2) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \oint \mathrm{d}\omega_1' \oint \mathrm{d}\omega_2' \frac{1}{\omega_1 + \omega_2 - (\omega_1' + \omega_2')} \times \left\{\frac{\tilde{\kappa}_{ijl}(k', k_1', k_2')}{\omega_2 - \omega_2'} + \frac{\tilde{\kappa}_{ilj}(k', k_2', k_1')}{\omega_1 - \omega_1'}\right\}$$

From (30) one has

$$\tilde{\kappa}_{ijl}(k', k_1', k_2') = -\tilde{\kappa}_{jil}(-k_1', -k', k_2'),$$

$$\tilde{\kappa}_{ilj}(k', k_2', k_1') = -\tilde{\kappa}_{jil}(-k_1', -k', k_2') + \tilde{\kappa}_{jli}(-k_1', k_2', -k').$$

Inserting these identities, collecting terms and relabelling the variables of integration, one has

$$N\kappa_{ijl}(k, k_1, k_2) = N\kappa_{jil}(-k_1, -k, k_2). \tag{32}$$

For $n = 2$, (31) reads

$$N\kappa_{ijl}(k, k_1, k_2) = \tfrac{1}{6}[N\kappa_{ijl}(k, k_1, k_2) + N\kappa_{ilj}(k, k_2, k_1) + N\kappa_{jil}(-k_1, -k, k_2) + N\kappa_{jli}(-k_1, k_2, -k) + N\kappa_{lji}(-k_2, k_1, -k) + N\kappa_{lij}(-k_2, -k, k_1)]. \tag{33}$$

The result (32) together with (6') establishes (33). The proof for $n = 3$ is lengthy but follows in the same manner.

Our proof applies only to the uncontracted part of the tensor. The extension of (33) to include the first contracted terms is straightforward; we already show that the doubly contracted term satisfies **(33)**, see (19). For $n \geq 3$ the proof that the contracted terms also satisfy (31) follows by induction.

5. DISCUSSION

We discuss three subjects related to the symmetry properties derived above. Firstly we discuss the expressions for the tensors derived using kinetic theory. Secondly, we use the above results to rederive the Onsager relations. Thirdly we discuss the significance of the results derived.

Kinetic theory approach

A number of authors, notably TSYTOVICH (1967, 1970) and SAGDEEV and GALEEV (1969), have discussed nonlinear processes in a plasma deriving the nonlinear response using kinetic theory. In particular TSYTOVICH (1967,1970) derives a tensor $S_{ijl}(k, k_1,$ k,) which relates the induced current to the product $E_j(k_1)E_l(k_2)$ of electric amplitudes. In the absence of a background magnetic field a derivation from the Vlasov equation gives

$$S_{ijl}(k, k_1, k_2) = -\frac{q^3}{\omega_1\omega_2}\int \mathrm{d}^3\mathbf{p}\, \frac{v_i}{\omega - \mathbf{k}\cdot\mathbf{v}} \{(\omega_1 - \mathbf{k}_1\cdot\mathbf{v})\delta_{jr} + v_j k_{1r}\} \frac{\partial}{\partial p_r} \times \left[\frac{1}{\omega_2 - \mathbf{k}_2\cdot\mathbf{v}} \{(\omega_2 - \mathbf{k}_2\cdot\mathbf{v})\delta_{ls} + v_l k_{2s}\} \frac{\partial f}{\partial p_s}(\mathbf{p})\right], \tag{34}$$

where we retain only the contribution from one species of particle. The tensor S_{ijl} is related to κ_{ijl} by

$$S_{ijl}(k, k_1, k_2) = -\frac{c^2}{\omega_1\omega_2}\kappa_{ijl}(k, k_1, k_2). \tag{35}$$

On imposing the symmetry property (6′), viz.

$$\bar{S}_{ijl}(k, k_1, k_2) = \tfrac{1}{2}[S_{ijl}(k, k_1, k_2) + S_{ilj}(k, k_2, k_1)],$$

the non-resonant contribution to the resulting tensor should satisfy (33), which, with (35), reads

$$N\bar{S}_{ijl}(k, k_1, k_2) = \frac{1}{6}\Big[NS_{ijl}(k, k_1, k_2) + NS_{ilj}(k, k_2, k_1) - \frac{\omega}{\omega_1}NS_{jil}(-k_1, -k, k_2) - \frac{\omega}{\omega_1}NS_{jli}(-k_1, k_2, -k) - \frac{\omega}{\omega_2}NS_{lij}(-k_2, -k, k_1) - \frac{\omega}{\omega_2}NS_{lji}(-k_2, k_1, -k)\Big]. \tag{36}$$

A straightforward, but laborious, comparison shows that (36) is indeed satisfied.

The more conventional kinetic theory derivation of the tensors obscures the existence of the symmetry property (31).

The Onsager relations

Let us rederive the Onsager relations for the dielectric tensor (4) using the above results. For $n = 1$, (31) gives

$$N\kappa_{ij}(k, k_1) = N\kappa_{ji}(-k_1, -k),$$

which, with $k = k_1$, shows that the non-resonant contribution is the hermitian part. A separation into hermitian and anti-hermitian parts being unique, the resonant contribution must be the anti-hermitian part. Time reversal invariance, see (26), then implies that the hermitian part is even under $\omega, \boldsymbol{B}_0 \to -\omega, -\boldsymbol{B}_0$, while the anti-hermitian part is odd under this transformation. Using the reality condition (5) one can express these statements by the single equation

$$\kappa_{ij}(\mathbf{k}, \omega)|_{\mathbf{B}_0} = \kappa_{ji}^*(\mathbf{k}, -\omega)|_{-\mathbf{B}_0} = \kappa_{ji}(-\mathbf{k}, \omega)|_{-\mathbf{B}_0}. \tag{37}$$

The dielectric tensor (4) also satisfies (37). Equation (37) is the usual expression of the Onsager reciprocal relations.

Implications of the symmetry properties

The relevance of the symmetry properties for the tensors describing the nonlinear responses is that they underlie symmetry properties of the coefficients describing nonlinear processes in a plasma. The symmetry properties of such coefficients were discussed by AL'TSHUL' and KARPMAN (1965) and by SAGDEEV and GALEEV (1969). KOVRIZHNYKH (1965) wrote down a number of the symmetry properties explicitly.

We cite just one specific example to illustrate the symmetry properties. Suppose one calculates a coefficient describing the coalescence of a Langmuir wave (I) and an ion sound wave (s) into a transverse wave (t) in an isotropic plasma. If the coefficient is chosen appropriately it should describe not only the process $l + s \to t$ but also the inverse process $t \to l + s$ (the latter following essentially from the reality condition). However, the processes $t + s \leftrightarrow l$ are also possible. The coefficient describing these processes should be related in a simple way to that for $l + s\ t,\ t$. The symmetry property (31) (for n = 2 in this case) ensures that this is so—see KOVRIZNHYKH (1965).

It should be emphasized that the result (31) has been established for $n \leq 3$; it is speculated that a proof for $n \geq 4$ could be constructed with sufficient perseverance and ingenuity. For $n \geq 4$ an interesting effect, on which the author can offer no constructive comment, arises. This is that one has $n \geq 4$ apparently independent k-vectors; space is only three dimension so that these cannot in fact be independent.

A referee, citing MARTIN (1967), points out that the imposition of the causal relations (7) is not necessarily justified when $\mathbf{A}(\mathbf{k}, \omega)$ is identified with the total field. The symmetry properties (26) and (31) are not dependent on the assumption that causality obtain; they follow under the assumption that the non-resonant parts be given by interpreting integrals such as that in (34) as principal value integrals.

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APPENDIX

The coefficients in (12a, b) are found by solving the equations of motion

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\partial H^{(0)}}{\partial \mathbf{p}}, \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H^{(0)}}{\partial \mathbf{r}}, \tag{38}$$

with $H^{(0)}$ given by (11) and with $\mathbf{r} = \mathbf{r}(t_0)$, $\mathbf{p} = \mathbf{p}(t_0)$ at $t = t_0$. Setting up axes with the 3-axis along $\mathbf{B}_0$, p_3 and the energy $E = H^{(0)}$ are constants of the motion. Explicit expression for the coefficients in (12a, b) read:

$$\alpha_{ij}(t, t_0) = \begin{pmatrix} \frac{1}{2}(1 + \cos\Omega(t - t_0)) & \frac{\varepsilon}{2}\sin\Omega(t - t_0) & 0 \\ -\frac{\varepsilon}{2}\sin\Omega(t - t_0) & \frac{1}{2}(1 + \cos\Omega(t - t_0)) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\beta_{ij}(t, t_0) = \frac{|q|B_0}{2c}\begin{pmatrix} \frac{1}{2}\sin\Omega(t - t_0) & \frac{\varepsilon}{2}(1 - \cos\Omega(t - t_0)) & 0 \\ -\frac{\varepsilon}{2}(1 - \cos R(t - t_0)) & \frac{1}{2}\sin R(t - t_0) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_{ij}(t, t_0) = \frac{2c}{|q|B_0}\begin{pmatrix} \frac{1}{2}\sin\Omega(t - t_0) & \frac{\varepsilon}{2}(1 - \cos\Omega(t - t_0)) & 0 \\ -\ (1 - \cos\Omega(t - t_0)) & \frac{1}{2}\sin\Omega(t - t_0) & 0 \\ 0 & 0 & \frac{1}{2}p_3\Omega(t - t_0) \end{pmatrix}, \tag{39}$$

$$\eta_{ij}(t, t_0) = \alpha_{ij}(t, t_0),$$

with

$$B_0 = |\mathbf{B}_0|, \qquad \varepsilon = \frac{|q|}{q}, \qquad \Omega = \frac{|q|B_0 c}{H^{(0)}}.$$

Inserting (12) to find $\mathbf{p}(t_0)$, $\mathbf{r}(t_0)$ in terms of $\mathbf{p}(t)$, $\mathbf{r}(t)$, the appropriate coefficients are $\alpha_{ij}(t_0, -t)$ and so on. In establishing the time-reversal-invariance properties in Section 4, we replace $\mathbf{p}(t_0)$, $\mathbf{r}(t_0)$ by $-\mathbf{p}'(-t')$, $\mathbf{r}'(-t')$ and $\mathbf{p}(t)$,$\mathbf{r}(t)$ by $-\mathbf{p}'(-t_0')$, $\mathbf{r}'(-t_0')$ respectively. The variables in the primed interaction picture are $\mathbf{p}'(t_0')$, $\mathbf{r}'(t_0')$. [The transformation from $\mathbf{p}(t_0)$, $\mathbf{r}(t_0)$ to $\mathbf{p}'(t_0')$, $\mathbf{r}'(t_0')$ is a canonical transformation.] One has

$$\alpha_{ij}(t, t_0)|_{\mathbf{B}_0} = \alpha_{ij}(t', t_0')|_{-\mathbf{B}_0},$$
$$\beta_{ij}(t, t_0)|_{\mathbf{B}_0} = -\beta_{ij}(t', t_0')|_{-\mathbf{B}_0},$$
$$\gamma_{ij}(t, t_0)|_{\mathbf{B}_0} = -\gamma_{ij}(t', t_0')|_{-\mathbf{B}_0}.$$