A RAZIN-TSYTOVICH EFFECT FOR BREMSSTRAHLUNG

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Abstract. It is shown that bremsstrahlung from electrons with Lorentz factor $\gamma \gg 1$ is suppressed for $\omega < \gamma\omega_p$ in a plasma with plasma frequency ω_p compared with emission in vacuo. For $\omega \ll \gamma\omega_p$ the ratio of the power emitted per unit frequency in the plasma to that in vacuo varies as ω^2.

This suppression effect is analogous to the suppression of synchrotron radiation in a plasma (Razin-Tsytovich effect). It is argued that such suppression is a characteristic property of emission by relativistic particles in a plasma.

1. Introduction

It is well known that synchrotron radiation in an isotropic plasma (plasma frequency ω_p) by electrons with Lorentz factor $\gamma \gg 1$ is suppressed relative to emission in vacuo for $\omega < \gamma\omega_p$, see Ginzburg and Syrovatski (1965) for example. The purposes of the present article are to point out that a similar effect occurs at $\omega < \gamma\omega_p$ for bremsstrahlung emitted by relativistic electrons and to argue that the suppression is a characteristic property of emission by relativistic particles in a plasma.

The suppression effect for synchrotron radiation has been called the Tsytovich effect, the Razin effect and the Razin-Tsytovich effect. The variety of names may be ascribed to the lack of availability of the earliest references cited. The references are to Tsytovich (1951) and Razin (1957); these have been cited by Razin (1960), Gailitis and Tsytovich (1963), Ginzburg and Syrovatski (1964, p. 75) and by Scheuer (1965). Of these two I have read only the dissertation by Tsytovich (1951). Tsytovich considered the effect of a medium with refractive index greater than unity on gyromagnetic radiation; he did not consider the effect of a plasma in the reference cited.

In Section 2 below, bremsstrahlung by relativistic electrons is treated using the straight-line approximation; the effects of an arbitrary medium are included. The case of an isotropic plasma is considered in Section 3 where it is found that the ratio of the power radiated per electron per unit frequency, P_ω say, in the plasma to that in vacuo reduces to

$$(P_\omega)_{\omega_p \neq 0}/(P_\omega)_{\omega_p = 0} \simeq (1 + \gamma^2\omega_p^2/\omega^2)^{-1} \tag{1}$$

for $\gamma \gg 1$, $\omega \gg \omega_p$. In section 4 a general argument based on transforming to the instantaneous rest frame of the particle is used to infer that the suppression effect for $\omega < \gamma\omega_p$ is independent of the specific radiation process.

2. Relativistic Bremsstrahlung

Waves, in an arbitrary mode σ say, are described by three wave properties; these are the dispersion relation $\omega = \omega^\sigma(\mathbf{k})$, the (unimodular) polarization vector $\mathbf{e}^\sigma(\mathbf{k})$ and

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the ratio $W_E^\sigma(\mathbf{k})/W_T^\sigma(\mathbf{k})$ of electric to total energy in the waves. The energy radiated, E^σ say, over an arbitrarily long time-interval into waves in the mode σ due to the work done by an extraneous current density whose Fourier transform is $\mathbf{j}(\mathbf{k}, \omega)$ involves only these wave properties. One has

$$E^\sigma = 4\pi \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{W_E}{W_T}\right]^\sigma |\mathbf{e}^{\sigma *}\cdot\mathbf{j}(\mathbf{k}, \omega^\sigma)|^2 , \tag{2}$$

where arguments $\mathbf{k}$ are left understood.

Equation (2) would describe bremsstrahlung once the appropriate current were identified. For an electron (charge $-e$, rest mass m) with its orbit described by $\mathbf{r}=\mathbf{r}(t)$ and with $\mathbf{v}(t)=d\mathbf{r}(t)/dt$ its instantaneous velocity, one has

$$\mathbf{j}(\mathbf{k}, \omega) = - e \int dt\, \mathbf{v}(t)\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r}(t))} . \tag{3}$$

For bremsstrahlung due to passage near an ion (charge Ze, mass infinite) the orbit can be found by solving the equation of motion

$$\frac{d\mathbf{p}(t)}{dt} = - \frac{Ze^2}{[r(t)]^3}\mathbf{r}(t) . \tag{4}$$

For the present purposes it suffices to make the straight-line approximation. In this approximation one solves (4) using a perturbation expansion with the zeroth order motion assumed to be rectilinear. The motion is confined to a plane, the 1–3 plane say. Let the zeroth order velocity $\mathbf{v}$ be along the 3-axis and let $\mathbf{b}$ be along the 1-axis with $b=|\mathbf{b}|$ the impact parameter. The zeroth order orbit can then be described by

$$\mathbf{r}(t) = \mathbf{b} + \mathbf{v}t , \tag{5}$$

where the ion is assumed to be at the origin.

The first order equation of motion leads to a rate of change of the velocity given by

$$\frac{d\mathbf{v}^{(1)}(t)}{dt} = - \frac{Ze^2}{m\gamma[n^2 + v^2t^2]^{3/2}}\left(\mathbf{b} + \frac{\mathbf{v}t}{\gamma^3}\right) \tag{6}$$

Integrating once gives

$$\mathbf{v}^{(1)}(t) = - \frac{Ze^2}{m\gamma[b^2 + v^2t^2]^{1/2}}\left(\frac{\mathbf{b}t}{b^2} - \frac{\mathbf{v}}{v^2\gamma^2}\right). \tag{7}$$

One could integrate again to find $\mathbf{r}^{(1)}(t)$ but this is unnecessary.

The first order terms in the current density (3) give

$$\begin{aligned}\mathbf{j}^{(1)}(\mathbf{k}, \omega) &= - e \int dt [\mathbf{v}^{(1)}(t) - \mathbf{v}i\mathbf{k}\cdot\mathbf{r}^{(1)}(t)]\, e^{i(\omega - \mathbf{k}\cdot\mathbf{v})t} e^{-i\mathbf{k}\cdot\mathbf{b}} \\ &= - e \int dt \left[\mathbf{v}^{(1)}(t) + \frac{\mathbf{v}\mathbf{k}\cdot\mathbf{v}^{(1)}t)}{\omega - \mathbf{k}\cdot\mathbf{v}}\right] e^{i(\omega - \mathbf{k}\cdot\mathbf{v})t} e^{-i\mathbf{k}\cdot\mathbf{b}} ,\end{aligned} \tag{8}$$

where a partial integration is performed.

On inserting (7) in (8), the integral is of the form, setting $\omega' = \omega - \mathbf{k}\cdot\mathbf{v}$,

$$\int \mathrm{d}t\mathbf{v}^{(1)}(t)\, e^{i\omega' t} = \frac{2Ze^2}{m\gamma v^2}\left[iK_1\left(\frac{b\omega'}{v}\right)\frac{\mathbf{b}}{b} + \frac{1}{\gamma^2}K_0\left(\frac{b\omega'}{v}\right)\frac{\mathbf{v}}{v}\right], \tag{9}$$

where K_ν is a modified Bessel function. For $b|\omega'| \gg v$ the modified Bessel functions drop off exponentially. For $b|\omega'| \ll v$ one has

$$K_1\left(\frac{b\omega'}{v}\right) \simeq \frac{v}{b|\omega'|} \gg K_0\left(\frac{b\omega'}{v}\right) \quad (b|\omega'| \ll v). \tag{10}$$

It is convenient to approximate (9) by neglecting the term involving K_0 and by replacing K_1 by $b|\omega'|/v$ for $b|\omega'| < v$ and by zero for $b|\omega'| > v$.

Thus an approximate expression for the energy radiated in waves in an arbitrary mode σ due to a single encounter between a relativistic electron and an ion is given by

$$E^\sigma = \frac{16\pi Z^2 e^6}{m^2\gamma^2 v^2 b^4}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{W_\mathrm{E}}{W_\mathrm{T}}\right]^\sigma \left|\frac{\mathbf{e}^{\sigma *}\cdot\mathbf{b}}{\omega^\sigma - \mathbf{k}\cdot\mathbf{v}} + \frac{\mathbf{e}^{\sigma *}\cdot\mathbf{v}\mathbf{k}\cdot\mathbf{b}}{(\omega^\sigma - \mathbf{k}\cdot\mathbf{v})^2}\right|^2, \tag{11}$$

where it is left understood that integrand is to be replaced by zero for $|\omega^\sigma - \mathbf{k}\cdot\mathbf{v}| > v/b$. (The result would apply for $\omega^\sigma < \mathbf{k}\cdot\mathbf{v}$ but would be invalid for $\omega^\sigma \simeq \mathbf{k}\cdot\mathbf{v}$ where any Cerenkov emission would occur.)

3. Isotropic Medium

In an isotropic medium with dielectric constant $\mathscr{E}(\omega)$, transverse waves have the following wave properties:

$$\frac{kc}{\omega} = n(\omega) = \mathscr{E}^{1/2}(\omega), \qquad \frac{W_\mathrm{E}}{W_\mathrm{T}} = \frac{1}{2n(\omega)\dfrac{\partial}{\partial\omega}[\omega n(\omega)]}. \tag{12}$$

On summing over the two transverse states of polarization (11) reduces to

$$E = \frac{16\pi Z^2 e^6}{m^2\gamma^2 v^2 b^4}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{W_\mathrm{E}}{W_\mathrm{T}}\frac{1}{(\omega - \mathbf{k}\cdot\mathbf{v})^4} \times \\ \times \left[(\omega - \mathbf{k}\cdot\mathbf{v})^2 b^2 - \frac{(\omega^2 - k^2 v^2)}{k^2}(\mathbf{k}\cdot\mathbf{b})^2\right]. \tag{13}$$

The angular integrals in (13) can then be performed for $n(\omega)v/c < 1$.

Writing

$$E = \int_0^\infty \mathrm{d}\omega E_\omega, \tag{14}$$

the energy radiated per unit frequency range per encounter with impact parameter

b reduces to

$$E_\omega = \frac{8}{3\pi} \frac{Z^2 e^6}{m^2 c^3 v^2 b^2} \left[\frac{n(\omega)(1 - v^2/c^2)}{1 - n^2(\omega) v^2/c^2} \right]. \tag{15}$$

On writing $n^2(\omega) = 1 - \omega_p^2/\omega^2$ in (15) and assuming $\omega \gg \omega_p$, $\gamma \gg 1$ one obtains the result (1).

However, it is necessary to consider the power radiated quasi-continuously due to frequent encounters because the limit of validity of (15) due to the approximation (9) may be dependent on the properties of the medium. Let there be N_α ions of species α (charge $Z_\alpha e$) per unit volume. The number of encounters per unit time with impact parameter in the range db at b with ions of species α is $2\pi v N_\alpha b$ db. The average power radiated per unit frequency per electron reduces to

$$P_\omega = \frac{16}{3} \frac{\sum_\alpha N_\alpha Z_\alpha^2 e^6}{m^2 c^3 v} \left[\frac{n(\omega)(1 - v^2/c^2)}{1 - n^2(\omega) v^2/c^2} \right] \ln \left(\frac{b_{\max}}{b_{\min}} \right), \tag{16}$$

where the sum is over all ionic species.

The integral over impact parameters is cut off at

$$b_{\max} \simeq \frac{v}{\omega - \mathbf{k} \cdot \mathbf{v}} \tag{17}$$

in accord with the approximation made to the modified Bessel functions. The minimum impact parameter may be taken as, see Jackson (1962, p. 515),

$$b_{\min} \simeq \frac{\hbar}{mv}. \tag{18}$$

In vacuo, standard arguments on any emission by relativistic particles lead one to conclude that most of the power is emitted at angle which correspond to $\omega - \mathbf{k} \cdot \mathbf{v} \simeq \omega/\gamma^2$. In an isotropic plasma it may be plausible that one has, analogously, $\omega - \mathbf{k} \cdot \mathbf{v} \simeq \omega/\xi^2$, where

$$\xi = \left[1 - n^2(\omega) \frac{v^2}{c^2} \right]^{-1/2} \tag{19}$$

corresponds to an effective Lorentz factor see, for example, Scheuer (1965). However, any effect of the medium on $b_{\max}$ appears only in the logarithm in (16). This can be ignored so that the result (1) follows from (16).

Thus, relativistic bremsstrahlung in a plasma differs insignificantly from that in vacuo for $\omega \gg \gamma \omega_p$, but for $\omega \lesssim \gamma \omega_p$ the power per unit frequency per electron differs from the corresponding power in vacuo in accord with (1).

4. General Nature of the Suppression

Any physical interpretation of why suppression of synchrotron radiation should occur is usually based on a verbalization of the result that, in appropriate places in the formulae, v/c in vacuo is replaced by $n(\omega)v/c$ in the medium, see Ginzburg and Syrovatski (1965), and Scheuer (1965) for example. On the basis of this one would not expect suppression to occur for bremsstrahlung or for suppression to be a general effect; e.g. on applying the prescription to (16), starting from $n(\omega)=1$, further insight would be required to guess that (16) is the correct result for $n(\omega)\neq 1$. Below it is argued that suppression is a characteristic effect of a plasma on radiation by relativistic particles. The argument is based on a Lorentz transformation to the instantaneous rest frame.

Let K be the laboratory frame and K' be the instantaneous rest frame of the particle. Because $\omega^2-k^2c^2=\omega_p^2$ is an invariant (when the refractive index is $[1-\omega_p^2/\omega^2]^{1/2}$) the form of the refractive index is the same in all inertial frames. Emission at ω, θ in K and ω', θ' in K' are related by

$$\begin{aligned} &\omega' = \gamma\omega\left[1-\beta n(\omega)\cos\theta\right], \qquad \omega = \gamma\omega'\left[1+\beta n(\omega')\cos\theta'\right],\\ &\omega' n(\omega')\cos\theta' = \gamma\omega\left[-\beta + n(\omega)\cos\theta\right]\\ &\omega n(\omega)\cos\theta = \gamma\omega'\left[\beta + n(\omega')\cos\theta'\right],\\ &\omega' n(\omega')\sin\theta' = \omega n(\omega)\sin\theta, \end{aligned} \tag{20}$$

with $\gamma=(1-\beta^2)^{1/2}$ and $n(\omega)=(1-\omega_p^2/\omega^2)^{1/2}$. Note that one has

$$\frac{\partial\omega}{\partial\omega'} = \gamma\left(1+\frac{\beta\cos\theta'}{n(\omega')}\right) = \left[\gamma\left(1-\frac{\beta\cos\theta}{n(\omega)}\right)\right]^{-1}. \tag{21}$$

For $n(\omega')>\beta\,(\omega'>\gamma\omega_\mathrm{p})$ or for $n(\omega)>\beta\,(\omega>\gamma\omega_\mathrm{p})$ the frequency ω of emission in K is an increasing function of ω' in K'. Emission at $\omega=\gamma\omega_\mathrm{p}$ in K corresponds to the lowest possible frequency, i.e. to $\omega'=\omega_\mathrm{p}$, in K'. Consequently, emission at $\omega_\mathrm{p}<\omega<\gamma\omega_\mathrm{p}$ in K must correspond to emission at $\gamma\omega_\mathrm{p}>\omega'>\omega_\mathrm{p}$ in K' from angles θ' for which $\partial\omega/\partial\omega'$ is negative. Emission at $\omega\ll\gamma\omega_\mathrm{p}$ in K arises from a very restricted range of angles $\cos\theta'\simeq-1$ in K'.

In the instantaneous rest frame the power radiated is not strongly dependent on angle of emission. The power radiated at $\omega<\gamma\omega_\mathrm{p}$ in K must be a decreasing function of decreasing ω for the following two reasons. Firstly, only an increasingly small fraction of the power generated as a function of angle in K' corresponds to decreasing $\omega<\gamma\omega_\mathrm{p}$ in K. Secondly, the strong forward beaming of radiation characteristic of emission by relativistic electrons in vacuo no longer obtains, i.e. the Lorentz transformation from K' to K predicts no substantial boost to the power for $\omega<\gamma\omega_\mathrm{p}$ in K.

Besides synchrotron radiation and bremsstrahlung, inverse Compton radiation is the only other emission process of interest for relativistic particles in astrophysical contexts. Let ω_i be the initial frequency of the photon; in inverse Compton scattering the final frequency is $\omega\sim\gamma^2\omega_\mathrm{i}$. Suppression in this case is not relevant because $\omega<\gamma\omega_\mathrm{p}$ is inconsistent with $\omega_\mathrm{i}>\omega_\mathrm{p}$ and $\gamma\gg 1$.

Note added in proof: It has been pointed out to me that the suppression effect for synchroton radiation was mentioned in a review article on cosmic rays by Ginzburg (1953).

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