PLASMA EMISSION PROCESSES IN A MAGNETOACTIVE PLASMA

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Abstract

Plasma emission (i.e. emission at about the plasma frequency and twice this frequency) is treated taking into account the effects of the magnetic field on the electron plasma waves, on the conversion processes, and on the escaping radiation. The expected degrees of polarization of the fundamental and second harmonic are calculated in the weak field limit. The results are used to estimate the magnetic field strength B at the 80 MHz level from the observed polarization of type III bursts; the result $B < 0 \cdot 04$ G is smaller than previous estimates. The possible importance of electron-cyclotron waves in an application to type I bursts is noted.

I. Introduction

In the context of solar radio astronomy the nonlinear plasma processes in the thermal plasma whereby energy in electron plasma waves is converted into energy in transverse electromagnetic waves are known as the plasma emission processes. Plasma emission is the accepted emission for types II, III, and V solar radio bursts and remains a possible mechanism for type I bursts. For bursts of types II, III, and V the background magnetic field is known to be weak (from the observed low degree of circular polarization). The theory of plasma emission mechanisms has been developed in detail for the case of zero background magnetic field (see Wild, Smerd, and Weiss (1963) and Kundu (1965) for summaries of earlier treatments, and Tidman and Dupree (1965), Melrose (1970), and Smith (1970) for alternative approaches to the theory).

The purpose of the present paper is to develop the theory of plasma emission processes in the case when a background magnetic field is present. Ginzburg and Ozernoy (1966) and Tidman, Birmingham, and Stainer (1966) dealt with plasma emission processes when the effect of the magnetic field on the electron plasma waves was included. However, the magnetic field affects both the nonlinear conversion processes themselves and also the properties of the escaping radiation (which must be in the magnetoionic modes). The latter effect results in a net degree of polarization, as is well known. All three effects of the magnetic field are taken into account here.

For the purpose of discussion here we restrict the use of the term "magnetoionic" to refer only to those waves (of magnetoionic theory) which can escape directly to infinity. These waves are waves in the ordinary mode (o-mode) at (angular) frequencies $\omega > \omega_p$, where ω_p is the plasma frequency, and waves in the extraordinary

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mode (x-mode) at $\omega > \omega_{\mathrm{x}}$, that is, above the cutoff frequency, with

$$\omega_{\mathrm{x}} = \tfrac{1}{2}\Omega_{\mathrm{e}} + \tfrac{1}{2}(4\omega_{\mathrm{p}}^2 + \Omega_{\mathrm{e}}^2)^{\frac{1}{2}}, \tag{1}$$

where Ω_{e} is the electron gyrofrequency. By "electron plasma" waves we refer to the longitudinally polarized waves with frequency in the range $\omega_{\mathrm{p}} \lesssim \omega \lesssim (\omega_{\mathrm{p}}^2 + \Omega_{\mathrm{e}}^2)^{\frac{1}{2}}$ for $\Omega_{\mathrm{e}} < \omega_{\mathrm{p}}$. The wave properties are discussed in Appendix I.

The only application considered explicitly in this paper is to the case where the magnetic field is weak ($\Omega_{\mathrm{e}} \ll \omega_{\mathrm{p}}$). The expected degrees of polarization of the fundamental and of the second harmonic are calculated and the results are applied to the interpretation of the polarization of type III bursts. However, one of our primary motivations for the development of the theory reported here was to treat plasma emission in the case where the magnetic field is strong, e.g. for $\Omega_{\mathrm{e}} \sim \omega_{\mathrm{p}}$. The condition $\Omega_{\mathrm{e}} \sim \omega_{\mathrm{p}}$ may apply in the region where type I bursts are generated and probably does apply for the dekametric radiation from Jupiter. For $\Omega_{\mathrm{e}} \sim \omega_{\mathrm{p}}$ alternative forms of plasma emission become possible in which electron-cyclotron waves (see Stix 1962, p. 40) or waves in the Bernstein modes (Stix 1962, Section 9–16; Bekefi 1966, p. 237) replace electron plasma waves. The Bernstein modes are thought to play an important role in emission from laboratory plasmas with strong magnetic fields (see e.g. review by Crawford 1965). Plasma emission processes involving such cyclotron waves could be regarded as candidate mechanisms for type I bursts and for the Jovian dekametric radiation. Although not done here, the theory presented below could be applied in the treatment of such plasma emission.

In Section II we present a general theory of the relevant conversion processes and in Section III we derive simplified formulae that are applicable to plasma emission processes. The degree of circular polarization in the weak field limit is calculated in Section IV. We discuss applications of the theory in Section V.

II. Nonlinear Processes

We begin by summarizing the semiclassical theory of the scattering of waves and the decay and coalescence of waves in a magnetoactive plasma. This generalizes the theory presented by Tsytovich (1966, 1970), Melrose (1970), and Smith (1970) for the case of an unmagnetized plasma. For the method of calculation we refer to Tsytovich and Shvartsburg (1965, 1966) and Melrose and Sy (1972).

(a) *Notation*

Particles of species α are described by their charge $q_\alpha = \epsilon_\alpha |q_\alpha|$, rest mass m_α, gyrofrequency $\Omega_\alpha = |q_\alpha| B/\gamma m_\alpha c$, Lorentz factor γ, momentum components $p_\perp = m_\alpha \gamma v_\perp$ and $p_\parallel = m_\alpha \gamma v_\parallel$ perpendicular and parallel to the background magnetic field $\boldsymbol{B}$ respectively, number density n_α, and distribution function $f_\alpha(p_\perp, p_\parallel)$ normalized by

$$\int \mathrm{d}^3\boldsymbol{p}\; f_\alpha(\boldsymbol{p}) = 2\pi \int_{-\infty}^{\infty} \mathrm{d}p_\parallel \int_0^\infty \mathrm{d}p_\perp\, p_\perp f_\alpha(p_\perp, p_\parallel) = 1\,. \tag{2}$$

The subscripts α are omitted where no confusion should result.

Waves in the mode σ are described by their wave vector $\boldsymbol{k}$, dispersion relation $\omega = \omega^\sigma(\boldsymbol{k}) = -\omega^\sigma(-\boldsymbol{k})$, unimodular polarization vector $\boldsymbol{e}^\sigma(\boldsymbol{k}) = \boldsymbol{e}^{\sigma *}(-\boldsymbol{k})$ (the asterisk denoting complex conjugation), ratio of electrical to total energy $W_E^\sigma(\boldsymbol{k})/W_T^\sigma(\boldsymbol{k})$, and occupation number $N^\sigma(\boldsymbol{k})$ of photons so that

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\,\hbar\,\omega^\sigma(\boldsymbol{k})\,N^\sigma(\boldsymbol{k})$$

is the energy density, where $\hbar$ is Planck's constant divided by 2π. The wave properties are discussed in Appendix I. The coordinate axes are chosen so that the 3-axis lies along $\boldsymbol{B}$ with $\boldsymbol{k}$, $\boldsymbol{k}'$, ... written as

$$\boldsymbol{k} = (k_\perp \cos\psi,\, k_\perp \sin\psi,\, k_\parallel) \;\;= k(\sin\theta\cos\psi,\, \sin\theta\sin\psi,\, \cos\theta)\,, \tag{3a}$$

$$\boldsymbol{k}' = (k'_\perp \cos\psi',\, k'_\perp \sin\psi',\, k'_\parallel) = k'(\sin\theta'\cos\psi',\, \sin\theta'\sin\psi',\, \cos\theta')\,, \tag{3b}$$

and so on. Where no confusion should result the arguments $\boldsymbol{k}$ are omitted so that ω^σ, $\boldsymbol{e}^{\sigma'}$, ... mean $\omega^\sigma(\boldsymbol{k})$, $\boldsymbol{e}^{\sigma'}(\boldsymbol{k}')$, ...

Individual processes are described in terms of probabilities. We write $w^{\sigma\sigma'}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')$ for the probability per unit time that a particle $(\alpha,\boldsymbol{p})$ will scatter a photon in the mode σ' in the range $\mathrm{d}^3\boldsymbol{k}'/(2\pi)^3$ at $\boldsymbol{k}'$ into a photon in the mode σ in the range $\mathrm{d}^3\boldsymbol{k}/(2\pi)^3$ at $\boldsymbol{k}$. The probability of the coalescence process $\sigma'+\sigma'' \rightarrow \sigma$ is written as $u^{\sigma\sigma'\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')$. It can be shown (Melrose 1972) that the probabilities are equal to those of the inverse and crossed processes, e.g. one has

$$w_\alpha^{\sigma\sigma'}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}') = w_\alpha^{\sigma'\sigma}(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k})\,, \qquad u^{\sigma\sigma'\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'') = u^{\sigma'\sigma\sigma''}(-\boldsymbol{k}',-\boldsymbol{k},\boldsymbol{k}'')\,. \tag{4}$$

(*b*) *Scattering Processes*

The scattering processes have been treated in a general way by Melrose and Sy (1972). Assuming that all variations are temporal the evolutions of $N^\sigma(\boldsymbol{k})$, $N^{\sigma'}(\boldsymbol{k}')$, and $f_\alpha(\boldsymbol{p})$ due to scattering $\sigma \leftrightarrow \sigma'$ are described by

$$\frac{\partial N^\sigma(\boldsymbol{k})}{\partial t} = n_\alpha \int \mathrm{d}^3\boldsymbol{p} \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3} \times w_\alpha^{\sigma\sigma'}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')[\{N^{\sigma'}(\boldsymbol{k}')-N^\sigma(\boldsymbol{k})\}f_\alpha(\boldsymbol{p}) + N^{\sigma'}(\boldsymbol{k}')\,N^\sigma(\boldsymbol{k})\,\hbar\hat{\mathrm{D}}\{f_\alpha(\boldsymbol{p})\}]\,, \tag{5}$$

$$\frac{\partial f_\alpha(\boldsymbol{p})}{\partial t} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \int \frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3} \times \hbar\hat{\mathrm{D}}\{w_\alpha^{\sigma\sigma'}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')[\{N^{\sigma'}(\boldsymbol{k}')-N^\sigma(\boldsymbol{k})\}f_\alpha(\boldsymbol{p}) + N^{\sigma'}(\boldsymbol{k}')\,N^\sigma(\boldsymbol{k})\,\hbar\hat{\mathrm{D}}\{f_\alpha(\boldsymbol{p})\}]\}\,, \tag{6}$$

where the differential operator $\hat{\mathrm{D}}$ is given by

$$\hat{\mathrm{D}} = \frac{\omega^\sigma - \omega^{\sigma'} - (k_\parallel - k'_\parallel)v_\parallel}{v_\perp}\frac{\partial}{\partial p_\perp} + (k_\parallel - k'_\parallel)\frac{\partial}{\partial p_\parallel}\,. \tag{7}$$

The evolution of $N^{\sigma'}(\boldsymbol{k}')$ is found by interchanging primed and unprimed quantities in equations (5) and (7).

The scattering probability is given by

$$w^{\sigma\sigma'}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}') = \sum_{s=-\infty}^{\infty} \frac{4(2\pi)^3 q^4}{m^2}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^{\sigma}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^{\sigma'} \frac{|\alpha^{\sigma\sigma'}(s,\boldsymbol{k},\boldsymbol{k}',\boldsymbol{v})|^2}{\omega^{\sigma}\,\omega^{\sigma'}} \times \delta\{(\omega^{\sigma}-\omega^{\sigma'})-(k_{\parallel}-k'_{\parallel})v_{\parallel}-s\Omega\}\,, \tag{8}$$

with

$$\alpha^{\sigma\sigma'}(s,\boldsymbol{k},\boldsymbol{k}',\boldsymbol{v}) = e_i^{\sigma *} e_j^{\sigma'}\{\alpha_{ij}^{q}(s,\boldsymbol{k},\boldsymbol{k}',\boldsymbol{v}) + \Gamma_{ijm}(\boldsymbol{k},\omega^{\sigma};\boldsymbol{k}',\omega^{\sigma'})\, V_m^{q}(s,\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{v})\}\,. \tag{9}$$

The quantity α_{ij}^{q} has been written down by Melrose and Sy (1972); this term describes Thomson scattering which is unimportant in the cases discussed below. The other quantities in equation (9) are

$$\Gamma_{ijm}(\boldsymbol{k},\omega;\boldsymbol{k}',\omega') = \frac{8\pi mc^2}{q(\omega-\omega')^2}\,\kappa_{ijl}(\boldsymbol{k},\omega;\boldsymbol{k}',\omega';\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')\,\frac{\lambda_{lm}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')}{\Lambda(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')} \tag{10}$$

and

$$V_m^{q}(s,\boldsymbol{k},\boldsymbol{v}) = \big(\tfrac{1}{2}v_{\perp}\{\exp(\mathrm{i}\epsilon\psi)\,\mathrm{J}_{s-1}(z)+\exp(-\mathrm{i}\epsilon\psi)\,\mathrm{J}_{s+1}(z)\}\,,\; -\tfrac{1}{2}\mathrm{i}\epsilon v_{\perp}\{\exp(\mathrm{i}\epsilon\psi)\,\mathrm{J}_{s-1}(z)-\exp(-\mathrm{i}\epsilon\psi)\,\mathrm{J}_{s+1}(z)\},\; v_{\parallel}\,\mathrm{J}_s(z)\big)\,, \tag{11}$$

with $z = k_{\perp}v_{\perp}/\Omega$. The quantities λ_{ij} and Λ are defined in Appendix I and κ_{ijl} is discussed in Appendix II.

(c) *Coalescence and Decay Processes*

The evolutions of the distribution of photons due to the decay and coalescence processes $\sigma'+\sigma'' \leftrightarrow \sigma$ are unchanged from the case of an unmagnetized plasma:

$$\frac{\partial N^{\sigma}(\boldsymbol{k})}{\partial t} = \int\frac{\mathrm{d}^3\boldsymbol{k}'}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}''}{(2\pi)^3} \times u^{\sigma\sigma'\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')[N^{\sigma'}(\boldsymbol{k}')\,N^{\sigma''}(\boldsymbol{k}'')-N^{\sigma}(\boldsymbol{k})\{N^{\sigma'}(\boldsymbol{k}')+N^{\sigma''}(\boldsymbol{k}'')\}]\,, \tag{12}$$

$$\frac{\partial N^{\sigma'}(\boldsymbol{k}')}{\partial t} = -\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}''}{(2\pi)^3} \times u^{\sigma\sigma'\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')[N^{\sigma'}(\boldsymbol{k}')\,N^{\sigma''}(\boldsymbol{k}'')-N^{\sigma}(\boldsymbol{k})\{N^{\sigma'}(\boldsymbol{k}')+N^{\sigma''}(\boldsymbol{k}'')\}]\,, \tag{13}$$

plus a further equation obtained by interchanging primed and double primed quantities in equation (13).

Likewise the probability is formally unchanged from the case of an unmagnetized plasma; it is

$$u^{\sigma\sigma'\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'') = \frac{8(2\pi)^7\hbar c^4}{\omega^{\sigma}\,\omega^{\sigma'}\,\omega^{\sigma''}}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^{\sigma}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^{\sigma'}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^{\sigma''} \times |\kappa^{\sigma\sigma'\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')|^2\,\delta^3(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'')\,\delta(\omega^{\sigma}-\omega^{\sigma'}-\omega^{\sigma''})\,, \tag{14}$$

with

$$\kappa^{\sigma\sigma'\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'') = e_i^{\sigma *}\, e_j^{\sigma'}\, e_l^{\sigma''}\,\kappa_{ijl}(\boldsymbol{k},\omega^{\sigma};\boldsymbol{k}',\omega^{\sigma'};\boldsymbol{k}'',\omega^{\sigma''})\,. \tag{15}$$

For $\sigma' = \sigma''$ and a single distribution of such waves, the factor 8 in equation (14) is replaced by 4.

III. Approximations

The probabilities (8) and (14) are exact within the framework of a general theory, but in order to treat specific processes one needs to make approximations. In this section we discuss the approximations made in treating plasma emission processes and write down appropriate simplified probabilities.

(a) *Simplifying Assumptions*

The following simplifying assumptions are made.

(i) As discussed in the Introduction we consider processes in which electron plasma waves are converted into those magnetoionic waves that can escape. Simplified properties of the relevant waves are summarized in Appendix I.

(ii) Three conversion processes are considered: scattering by thermal ions and coalescence of electron plasma waves (l-waves) with sound waves (s-waves), both of which lead to emission at the fundamental frequency, and coalescence of two l-waves, which leads to second harmonic emission. Scattering by ions predominates over scattering by electrons for the same reason as in unmagnetized plasmas (see Tsytovich and Shvartsburg 1965, 1966; Tsytovich 1966; Melrose and Sy 1972).

(iii) The probability of scattering by an ion is averaged over a Maxwellian distribution of ions.

(iv) Appropriate simplified forms of the tensor κ_{ijl} are inserted in equations (10) and (14); these forms are discussed in Appendix II.

(v) The phase velocity of the l-waves is assumed to be much less than the velocity of light.

(vi) The probabilities depend on the azimuthal angle between the plane containing $\boldsymbol{k}$ and $\boldsymbol{B}$ and that containing $\boldsymbol{k}'$ and $\boldsymbol{B}$. It is assumed that the l-waves are distributed with axial symmetry about $\boldsymbol{B}$ and the probabilities are averaged over this azimuthal angle. This average is denoted by enclosing the probability in angle brackets.

(vii) Approximations are made to simplify the quantity λ_{ij}/Λ, appearing in equation (10) and the sum over harmonic numbers in equation (8); these approximations are discussed in the following subsections.

(b) *The Quantity λ_{ij}/Λ*

The quantity λ_{ij}/Λ appearing in equation (10) relates the shielding fields around an individual charge to the current density associated with that charge. Scattering by ions results from the nonlinear response of the plasma to the combined effect of these shielding fields and to the fields of the unscattered waves. On physical grounds it is reasonable to expect that the dominant shielding effect is electrostatic. Granted this, the appropriate approximation to λ_{ij}/Λ is

$$\lambda_{ij}(\boldsymbol{k},\omega)/\Lambda(\boldsymbol{k},\omega) \approx k_i k_j/|\boldsymbol{k}|^2 \mathscr{E}^l(\boldsymbol{k},\omega), \tag{16}$$

where

$$\mathscr{E}^l(\boldsymbol{k},\omega) = (k_i k_j/|\boldsymbol{k}|^2)\mathscr{E}_{ij}(\boldsymbol{k},\omega) \tag{17}$$

is the longitudinal part of the dielectric tensor. Mathematically, the approximation

(16) is justified when $|\boldsymbol{k}|^2c^2/\omega^2$ greatly exceeds all components of the dielectric tensor. For scattering of electron plasma waves with phase velocity much less than the velocity of light into magnetoionic waves that can escape, there is a large change in wave number but a small change in frequency. The approximation (16) is a reasonable one in this case. In the terminology used by Tsytovich and Shvartsburg (1965, 1966) and Tsytovich (1966, 1970), the approximation (16) is referred to as the condition that scattering via virtual longitudinal waves predominate.

Combining (16) and the appropriate approximate form of κ_{ijl} (equation (A24) of Appendix II), the quantity Γ_{ijm} from equation (10) reduces to

$$\Gamma_{ijm}(\boldsymbol{k},\omega;\boldsymbol{k}',\omega') = \frac{m_{\mathrm{i}}}{Z_{\mathrm{i}}\,m_{\mathrm{e}}(\omega-\omega')}\left[\frac{\mathscr{E}^{l(\mathrm{e})}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')-1}{\mathscr{E}^{l}(\boldsymbol{k}-\boldsymbol{k}',\omega-\omega')}\right]\tau_{ij}^{(\mathrm{e})}(\omega)\,k_m\,, \tag{18}$$

where m_{i} and $Z_{\mathrm{i}}\,e$ are the mass and charge of the ion respectively. The quantity $\tau_{ij}^{(\mathrm{e})}(\omega)$ is defined by

$$\tau_{ij}^{(\mathrm{e})}(\omega) = \begin{bmatrix} \omega^2/(\omega^2-\Omega_{\mathrm{e}}^2) & -\mathrm{i}\omega\Omega_{\mathrm{e}}/(\omega^2-\Omega_{\mathrm{e}}^2) & 0 \\ \mathrm{i}\omega\Omega_{\mathrm{e}}/(\omega^2-\Omega_{\mathrm{e}}^2) & \omega^2/(\omega^2-\Omega_{\mathrm{e}}^2) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{19}$$

The quantity in square brackets in equation (18) appears squared in the probability, and, as in the case of an unmagnetized plasma (Tsytovich 1966), the applicable approximations are

$$|\{\mathscr{E}^{l(\mathrm{e})}(\boldsymbol{k},\omega)-1\}/\mathscr{E}^{l}(\boldsymbol{k},\omega)|^2 \approx (1+Z_{\mathrm{i}}\,T_{\mathrm{e}}/T_{\mathrm{i}})^{-2}\,, \qquad |\omega| \ll |k_{\parallel}|\,V_{\mathrm{i}}\,, \tag{20a}$$

$$\approx 1\,, \qquad |\omega| \gg |k_{\parallel}|\,V_{\mathrm{i}}\,, \tag{20b}$$

where T_{e} and T_{i} are the electron and ion temperatures respectively and $V_{\mathrm{i}} = (T_{\mathrm{i}}/m_{\mathrm{i}})^{\frac{1}{2}}$ is the thermal velocity of the ions.

(c) *The Sum over Harmonic Numbers*

The sum over s in equation (8) considerably complicates the form of the scattering probability. We consider two relevant opposite limiting cases for which this sum is eliminated and argue that one or other of these two limiting cases suffices for almost all purposes.

On averaging the probability (8) over a Maxwellian distribution of ions, as

$$w_{\mathrm{i}}^{\sigma\sigma'}(\boldsymbol{k},\boldsymbol{k}') = \frac{1}{(2\pi)^{3/2}V_{\mathrm{i}}^3}\int \mathrm{d}^3\boldsymbol{v}\; w_{\mathrm{i}}^{\sigma\sigma'}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')\exp(-v^2/2V_{\mathrm{i}}^2)\,, \tag{21}$$

the integrals can be performed exactly. Only the second term in equation (9), in which $\boldsymbol{v}$ appears only in V_m^q, is retained because the first term in (9) describes Thomson scattering which is negligible for ions. The integrals over the products of Bessel functions which result lead to expressions involving modified Bessel functions $\mathrm{I}_s(\lambda_{\mathrm{i}})$ and their derivatives with arguments

$$\lambda_{\mathrm{i}} = |(\boldsymbol{k}-\boldsymbol{k}')_{\perp}\,V_{\mathrm{i}}/\Omega_{\mathrm{i}}|^2\,, \tag{22}$$

where Ω_{i} is the gyrofrequency of the ions. For the case $\sigma' = l$, that is, when the initial waves are longitudinally polarized, the probability (21) reduces to

$$w_{\mathrm{i}}^{\sigma l}(\boldsymbol{k},\boldsymbol{k}') = \frac{4(2\pi)^{5/2}Z_{\mathrm{i}}^{2}e^{4}}{m_{\mathrm{e}}^{2}\omega^{\sigma}\omega^{l}}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^{\sigma}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^{l}\left|e_i^{\sigma *}\,\tau_{ij}\,\kappa_j'\right|^2\left|\frac{\mathscr{E}^{l(\mathrm{e})}-1}{\mathscr{E}^{l}}\right|^2$$

$$\times\sum_{s=-\infty}^{\infty}\frac{\exp(-\lambda_{\mathrm{i}})\,\mathrm{I}_s(\lambda_{\mathrm{i}})}{|k_{\|}-k_{\|}'|\,V_{\mathrm{i}}}\exp\left(-\frac{(\omega^{\sigma}-\omega^{l}-s\Omega_{\mathrm{i}})^2}{2\,|k_{\|}-k_{\|}'|^2V_{\mathrm{i}}^2}\right), \qquad (23a)$$

with $\boldsymbol{\kappa}' = \boldsymbol{k}'/|\boldsymbol{k}'|$ and $\tau_{ij} \equiv \tau_{ij}^{(\mathrm{e})}(\omega)$. The sum over s in equation (23a) simplifies greatly if $\lambda_{\mathrm{i}} \ll 1$. In this limit the contribution from different values of $|s|$ is a rapidly decreasing function of increasing $|s|$. The contribution from $s = 0$, with $\exp(-\lambda_{\mathrm{i}})\,\mathrm{I}_0(\lambda_{\mathrm{i}}) \approx 1$, dominates. This corresponds to the limiting case of negligibly small ion gyroradii.

The opposite limiting case is when the ion gyroradii are arbitrarily large. In this case high values of $|s|$ dominate in the sum in (23a). However, one can greatly simplify this by noting that in this limit the motion of the ions can be regarded as rectilinear for the purpose of the scattering. Thus in the limit $\lambda_{\mathrm{i}} \gg 1$ one can justify replacing the δ-function in (8) by $\delta\{(\omega^{\sigma}-\omega^{\sigma'})-(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}\}$ and the quantity V_m^q in (9) by v_m. With these approximations the probability (23a) can be replaced by

$$w_{\mathrm{i}}^{\sigma l}(\boldsymbol{k},\boldsymbol{k}') = \frac{4(2\pi)^{5/2}Z_{\mathrm{i}}^{2}e^{4}}{m_{\mathrm{e}}^{2}\omega^{\sigma}\omega^{l}}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^{\sigma}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^{l}\left|e_i^{\sigma *}\,\tau_{ij}\,\kappa_j'\right|^2\left|\frac{\mathscr{E}^{l(\mathrm{e})}-1}{\mathscr{E}^{l}}\right|^2$$

$$\times\frac{1}{|\boldsymbol{k}-\boldsymbol{k}'|\,V_{\mathrm{i}}}\exp\left(-\frac{\omega^{\sigma}-\omega^{l}}{2\,|\boldsymbol{k}-\boldsymbol{k}'|^2V_{\mathrm{i}}^2}\right). \qquad (23b)$$

Comparing equation (23a) for $\lambda_{\mathrm{i}} = 0$ with (23b) one finds that the scattering is not an exponentially small effect only for

$$|\omega^{\sigma}-\omega^{l}| \lesssim |k_{\|}-k_{\|}'|\,V_{\mathrm{i}} \qquad \text{and} \qquad |\omega^{\sigma}-\omega^{l}| \lesssim |\boldsymbol{k}-\boldsymbol{k}'|\,V_{\mathrm{i}}$$

respectively, while the respective ranges of validity are

$$\Omega_{\mathrm{i}} \gg |(\boldsymbol{k}-\boldsymbol{k}')_{\perp}|\,V_{\mathrm{i}} \qquad \text{and} \qquad \Omega_{\mathrm{i}} \ll |(\boldsymbol{k}-\boldsymbol{k}')_{\perp}|\,V_{\mathrm{i}}\,.$$

At their limits of validity the two approximate forms overlap. We have examined the conditions under which the contributions from $|s| = 1$ could dominate over that from $s = 0$ in equation (23a) and have found that, except for special cases of little interest, these conditions are just those for (23b) to be a reasonable approximation, i.e. for large values of $|s|$ to dominate.

(*d*) *Approximate Probabilities*

We now identify the mode σ with one or other of the magnetoionic waves whose properties are described by the refractive index $\mu_{\sigma}(\omega,\theta)$ and by T_{σ} and K_{σ} (see Appendix I). Because the phase velocity of the l-waves is assumed to be much less than the velocity of light one can approximate $|k_{\|}-k_{\|}'|$ in (23a) by $|k_{\|}'| = k'\,|\cos\theta'|$. In view of the exponential in (23a) for $s = 0$, the approximation (20a) is appropriate.

The simplified probability for scattering of l-waves into magnetoionic waves by thermal ions, after averaging over the azimuthal angles, reduces to

$$\langle w_{\mathrm{i}}^{\sigma l}(\boldsymbol{k},\boldsymbol{k}')\rangle = \frac{2(2\pi)^{5/2}Z_{\mathrm{i}}^2 e^4}{m_{\mathrm{e}}^2\,\omega\omega^l(1+Z_{\mathrm{i}}\,T_{\mathrm{e}}/T_{\mathrm{i}})^2}\left[\frac{W_E}{W_{\mathrm{T}}}\right]^l \frac{F^{\sigma l}(\omega,\theta,\theta')}{k'\,|\cos\theta'\,|}\exp\left(-\frac{(\omega-\omega^l)^2}{2k'^2V_{\mathrm{i}}^2\,|\cos\theta'\,|^2}\right), \quad (24)$$

where ω^l and $[W_E/W_{\mathrm{T}}]^l$ are functions of θ' (Appendix I) and

$$F^{\sigma l}(\omega,\theta,\theta') = \frac{1}{(1+T_\sigma^2)\mu_\sigma\,\partial(\omega\mu_\sigma)/\partial\omega} \times\left(\frac{1+Y^2}{2(1-Y^2)^2}(1+a_\sigma^2)\sin^2\theta' + \frac{2Y}{(1-Y^2)^2}a_\sigma\sin^2\theta' + b_\sigma^2\cos^2\theta'\right), \quad (25)$$

with

$$a_\sigma = K_\sigma\sin\theta + T_\sigma\cos\theta, \quad b_\sigma = K_\sigma\cos\theta - T_\sigma\sin\theta, \quad Y = \Omega_{\mathrm{e}}/\omega, \quad X = \omega_{\mathrm{p}}^2/\omega^2. \quad (26)$$

Equation (25) can be re-expressed in terms of T_σ alone using

$$\left.\begin{aligned} a_\sigma &= \frac{\cos\theta - YT_\sigma}{T_\sigma - Y\cos\theta}, \qquad b_\sigma = \frac{\tan\theta\{YT_\sigma-(1-X)\cos\theta\}}{(1-X)(T_\sigma - Y\cos\theta)}, \\ \mu_\sigma\frac{\partial(\omega\mu_\sigma)}{\partial\omega} &= 1+\frac{XY\cos\theta\,T_\sigma}{2(T_\sigma - Y\cos\theta)^2}\left(1+\frac{(1+X)(1-T_\sigma^2)}{(1-X)(1+T_\sigma^2)}\right). \end{aligned}\right\} \quad (27)$$

In writing down the probability (24) we have used equation (23a) with $\lambda_{\mathrm{i}} = 0$. For the opposite limiting case $\lambda_{\mathrm{i}} = \infty$ (equation (23b)), $k'\,|\cos\theta'\,|$ in (24) is replaced by k'.

The probability for the coalescence process $l+l \to \sigma$ is given by inserting the wave properties in equation (14) with κ_{ijl} given by equation (A21) of Appendix II. For l-waves with phase velocities much less than the velocity of light, the requirement $\boldsymbol{k} = \boldsymbol{k}'+\boldsymbol{k}''$ with $|\,\boldsymbol{k}'\,|$ and $|\,\boldsymbol{k}''\,|$ very much greater than $|\,\boldsymbol{k}\,|$ requires $\boldsymbol{k}'' \approx -\boldsymbol{k}'$. This leads to considerable simplification. In particular $\boldsymbol{k}'' = -\boldsymbol{k}'$ reduces the number of independent directions in the problem, and the average probability reduces to

$$\langle u^{\sigma ll}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')\rangle = \frac{(2\pi)^5\hbar e^2\omega_{\mathrm{p}}^2}{m_{\mathrm{e}}^2 c^2\omega^l}\left(\left[\frac{W_E}{W_{\mathrm{T}}}\right]^l\right)^2 G^{\sigma l}(\omega^l,\theta,\theta')\,\delta^3(\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}'')\,\delta(\omega-2\omega^l), \quad (28)$$

with

$$G^{\sigma l}(\omega^l,\theta,\theta') = \frac{\mu_\sigma}{(1+T_\sigma^2)\partial(\omega\mu_\sigma)/\partial\omega} \times\left(\frac{\sin^2\theta\sin^4\theta'}{8(1-Y'^2)^4}\{(3+Y'^2)(a_\sigma+Y')^2+(1+3Y'^2)(a_\sigma Y'+1)^2\} + \frac{\cos^2\theta\sin^2\theta'\cos^2\theta'}{2(1-Y'^2)^2}\{(a_\sigma+Y')^2+(1+a_\sigma Y')^2\}\right.$$

$$+\frac{2\sin\theta\cos\theta\sin^2\theta'\cos^2\theta'}{(1-Y'^2)^2}(a_\sigma+Y')b_\sigma$$

$$+\frac{(1+Y'^2)}{2(1-Y'^2)^2}\sin^2\theta\sin^2\theta'\cos^2\theta'\,b_\sigma^2+\cos^2\theta\cos^4\theta'\,b_\sigma^2\Big), \tag{29}$$

where we have written $Y' = \Omega_e/\omega^l$.

The average probability describing the coalescence of an electron plasma wave and a sound wave into a magnetoionic wave is

$$\langle u^{\sigma ls}(\boldsymbol{k},\boldsymbol{k}_l',\boldsymbol{k}_s')\rangle = \frac{(2\pi)^5\hbar e^2}{2m_e^2}\left(\frac{m_i}{Z_i\,m_e}\right)\frac{(\omega^s)^3}{k_l'^2\,V_e^4}\left[\frac{W_E}{W_T}\right]^l$$

$$\times F^{\sigma l}(\omega^l,\theta,\theta')\,\delta^3(\boldsymbol{k}-\boldsymbol{k}_l'-\boldsymbol{k}_s')\,\delta(\omega-\omega^l-\omega^s)\,, \tag{30}$$

where $F^{\sigma l}$ is given by equation (25) and we assume that $\boldsymbol{k}_l' \approx -\boldsymbol{k}_s'$.

IV. Weak Field Limit

The only case we consider in any detail is the weak field limit $\Omega_e \ll \omega_p$, in which the effect of the magnetic field can be regarded as a perturbation. In this limit the only change from the case of plasma emission in the absence of a magnetic field is that the escaping radiation is partially circularly polarized due to preferential emission of one or other of the magnetoionic modes. We now determine the expected degrees of polarization for both the fundamental and the second harmonic.

In estimating the degree of polarization we assume that the polarization limiting region is far removed from the region of emission and that at the limiting region o-mode and x-mode waves are oppositely circularly polarized. Granted this, the observed degree of polarization (neglecting any preferential absorption of one mode or the other in the intervening region) is the difference in the powers generated in the two modes divided by the sum of the powers. This assumption seems more plausible than the opposite limiting case, considered by Ramaty (1969) for example, in which the degree of polarization is estimated at the source itself, for this would imply that the polarization limiting region is at the source.

(a) *Completely Polarized Fundamental Emission*

Emission at the fundamental can be 100% polarized in the sense of the o-mode. This occurs when emission is restricted to frequencies $\omega < \omega_x$ below the cutoff for the x-mode.

In the weak field limit equation (1) reduces to

$$\omega_x \approx \omega_p+\tfrac{1}{2}\Omega_e\,,$$

while the frequency of l-waves becomes

$$\omega^l(\theta') \approx \omega_p+(\Omega_e^2/2\omega_p)\sin^2\theta'+(3\,V_e^2/2v_\phi^2)\omega_p\,, \tag{31}$$

where $v_\phi = \omega^l/k'$ is the phase velocity. According to the probability (24), emission occurs at $\omega < \omega_x$ for

$$\omega^l\{1+(V_i/v_\phi)\,|\cos\theta'\,|\} \lesssim \omega_x\,,$$

and this occurs for

$$v_\phi \gtrsim \max\{V_e(3\omega_p/\Omega_e)^{\frac{1}{2}},\ V_i(2\omega_p/\Omega_e)\}\,. \tag{32}$$

When the condition (32) is satisfied the escaping radiation should be 100% polarized.

(b) *Weakly Polarized Fundamental Emission*

Assuming the reverse of the inequality (32) to be satisfied, one expects the fundamental to be only weakly polarized because the emission is a mixture of the two modes. For emission at a given angle θ from l-waves propagating at a given angle θ', the degree of polarization in the sense of the o-mode is given by

$$r_c = \{F^{\mathrm{o}l}(\omega,\theta,\theta')-F^{\mathrm{x}l}(\omega,\theta,\theta')\}/\{F^{\mathrm{o}l}(\omega,\theta,\theta')+F^{\mathrm{x}l}(\omega,\theta,\theta')\}\,, \tag{33}$$

with $F^{\sigma l}$ given by equation (25). For $X \lesssim 1$, which is necessarily the case, equation (33) is a rather complicated function of θ and θ'.

The case of most practical interest is for l-wave propagation nearly along the field lines, i.e. for $\sin\theta' \approx 0$. We consider simple limiting cases of (33) for $\sin\theta' = 0$ and $Y < 1-X$ (the reverse of the condition for the emission to be purely in the o-mode). The limiting cases are as follows.

(i) For $\frac{1}{4}Y^2\sin^4\theta \ll (1-X)^2\cos^2\theta$,

$$T_o \approx -\cos\theta/|\cos\theta\,|\,, \qquad K_o \approx XY\sin\theta(1-Y\,|\cos\theta\,|)/(1-X)\,,$$

$$T_x \approx \cos\theta/|\cos\theta\,|\,, \qquad K_x \approx XY\sin\theta(1+Y\,|\cos\theta\,|)/(1-X)\,,$$

and so

$$r_c \approx 2Y(1-X)|\cos\theta\,|/\{Y^2+(1-X)^2\cos^2\theta\}\,, \tag{34}$$

where we assume $1-X$ and Y very much less than unity.

(ii) For $\frac{1}{4}Y^2\sin^4\theta \gg (1-X)^2\cos^2\theta$,

$$T_o \approx -Y\sin^2\theta/(1-X)\cos\theta\,, \qquad K_o \approx XY\sin\theta/(1-X)\,,$$

$$T_x \approx (1-X)\cos\theta/Y\sin^2\theta\,, \qquad K_x \approx XY/(1-X-Y^2)\,,$$

where the value of K_x is actually that for $\cos\theta = 0$. One finds

$$F^{\mathrm{o}l}(\omega,\theta\approx\tfrac{1}{2}\pi,\theta'\approx 0) \approx 1 \gg F^{\mathrm{x}l}(\omega,\theta\approx\tfrac{1}{2}\pi,\theta'\approx 0)\,. \tag{35}$$

Thus for emission nearly perpendicular to the field lines the power is almost all in the o-mode even for $Y \ll 1-X$.

By way of illustration, the angle defined by $\frac{1}{4}Y^2\sin^2\theta = (1-X)^2\cos^2\theta$ is $\theta = 65°$ for $Y = 1-X$ and $\theta = 77°$ for $Y = \frac{1}{2}(1-X)$. For $\sin\theta' = 0$ the power generated varies with θ roughly as $\sin^2\theta$. In view of this the expression (34) would underestimate the degree of polarization. An approximate way of compensating for this underestimate, which is most serious for $|\cos\theta\,| \ll 1$, would be to set $|\cos\theta\,| = 1$ in (34), i.e. to set

$$r_c = 2Y(1-X)/\{Y^2+(1-X)^2\}\,. \tag{36}$$

This applies for $Y < 1-X$. For $Y \geqslant 1-X$, we have $r_c = 1$ because emission in the x-mode is forbidden. Setting $Y = 1-X$ in (36) reproduces this result.

(c) *Polarization of Second Harmonic*

The degree of polarization of the second harmonic for given θ' and θ is

$$r_c = \{G^{ol}(\omega^l, \theta, \theta') - G^{xl}(\omega^l, \theta, \theta')\}/\{G^{ol}(\omega^l, \theta, \theta') + G^{xl}(\omega^l, \theta, \theta')\}, \qquad (37)$$

with $G^{\sigma l}$ given by equation (29). For second harmonic emission $X \approx \frac{1}{4}$. As this is significantly less than unity it is reasonable to approximate the properties of the magnetoionic waves by

$$T_o = -\frac{\cos\theta}{|\cos\theta|} - \frac{Y'\sin^2\theta}{3\cos\theta}, \qquad T_x = \frac{\cos\theta}{|\cos\theta|} - \frac{Y'\sin^2\theta}{3\cos\theta}, \qquad (38)$$

where we set $X = \frac{1}{4}$ and expand in powers of $Y' = \Omega_e/\omega_p$ ($\ll 1$). The expansion is valid for $|\cos\theta| \gg \frac{1}{3}Y'\sin^2\theta \approx \frac{1}{3}Y'$.

Considering only the case $\sin\theta' = 0$, that is, l-waves propagating along the field lines, we find

$$r_c \approx (16 + 11\cos^2\theta)\Omega_e/48\,|\cos\theta|\,\omega_p. \qquad (39a)$$

For $\sin\theta' = 0$ the power generated varies with θ as $\sin^2\theta\cos^2\theta$ in the limit $Y' \approx 0$. The maximum power is at $\cos^2\theta = \sin^2\theta = \frac{1}{2}$. Inserting this value of θ in (39a) gives

$$r_c \approx 0\cdot63\,\Omega_e/\omega_p \qquad (\theta' = 0,\ \theta = \tfrac{1}{4}\pi). \qquad (39b)$$

Thus the second harmonic is weakly polarized in the sense of the o-mode.

V. Discussion

We now consider the interpretation of the polarization of type III bursts and then discuss possible ways in which the above theory could be applied to develop models for emission of type I bursts.

(a) *Type III Bursts*

The vast majority of type III bursts are observed to be only weakly circularly polarized or unpolarized. Kai (1970) used an inequality equivalent to (32) to infer that for a burst at 80 MHz to be unpolarized required $B \ll 0\cdot14$ G. McLean (1971) reported an exceptionally high degree of circular polarization ($\sim 50\%$) in one burst; the handedness was the same as for an associated type I burst, indicating polarization in the sense of the o-mode. It was only the fundamental which was polarized in the event reported by McLean.

Using the above case as an illustration one could infer the field strength in the following way. Setting $Y_0 = 1-X$ as the value at which the emission should be 100% polarized, we have $Y/Y_0 = B/(0\cdot14\text{ G})$ at the 80 MHz level. The value of Y/Y_0 can be found by inserting $r_c = \frac{1}{2}$ in (36), i.e. in

$$r_c = 2YY_0/(Y^2 + Y_0^2).$$

This gives $Y/Y_0 = 0 \cdot 27$ and thus $B \approx 0 \cdot 04$ G. According to equation (39b), absence of comparable observable polarization of the second harmonic emission, $r_c \lesssim 3\%$ say, would imply $B < 1 \cdot 2$ G.

It is apparent that either our calculated degree of polarization of the fundamental overestimates the actual degree of polarization or the field strength in the region where type III bursts are generated is about an order of magnitude smaller ($B \ll 0 \cdot 04$ G) than is usually thought. There is no reason why the field strength should not be so very low in view of the accepted idea that type III bursts result from streams of electrons propagating along neutral sheets in the magnetic structure of the corona. However, Steinberg *et al.* (1971) and Riddle (1972) have shown that the fundamental is observed only after it is refracted strongly towards the radial direction and scattered due to inhomogeneities in the corona. The presence of such inhomogeneities implies that there must be some mode–mode coupling. Now mode–mode coupling would convert radiation which is initially 100% polarized in either mode into a mixture of the modes. If such coupling is a significant factor, it would invalidate the assumptions made in writing down the expression (33) for the degree of polarization. Significant mode–mode coupling would lead to an underestimation of the field strength.

We conclude that the low degree of polarization of the fundamental for type III bursts implies that the field strength is much less than $0 \cdot 04$ G at the 80 MHz level provided that propagation effects do not lead to significant mode–mode coupling. The second harmonic would show significant polarization at the same level only for fields greater than about 1 G.

(*b*) *Type I Bursts*

There have been two suggestions for the emission mechanism of type I bursts. Takakura (1963) developed a model based on plasma emission. This model was discussed by Trakhtengerts (1966) and was used by Kai (1970) in estimating $B > 3 \cdot 5$ G at the 80 MHz level (using essentially the inequality (32) and the observation of the high polarization in the sense of the o-mode). Thus, according to Kai (1970), one requires $\Omega_e/\omega_p \gtrsim 0 \cdot 13$ to account for the polarization of type I bursts using this model. Fung and Yip (1966) developed a model based on coherent gyromagnetic emission; this model requires $\Omega_e/\omega_p > 1$.

Models based on plasma emission remain only semiquantitative. In order to apply the theory developed in this paper to calculate the properties of the emitted radiation one would require more detailed knowledge of the spectrum of electron plasma waves and of the ratio Ω_e/ω_p. The ratio Ω_e/ω_p is important because the waves in the Bernstein modes (i.e. perpendicular electrostatic electron-cyclotron waves) can exist with frequencies slightly greater than Ω_e, $2\Omega_e$, and higher multiples of Ω_e. These waves are readily generated (see e.g. Crawford 1965) and would lead to plasma emission provided their frequency exceeded the plasma frequency, e.g. for $\Omega_e > \omega_p/n$ with $\omega \approx n\Omega_e$, $n = 1, 2, \ldots$. Likewise for $\Omega_e > \frac{1}{2}\omega_p$ ($\Omega_e > \omega_p$) plasma involving parallel electron-cyclotron waves at the second harmonic (the fundamental) becomes possible. Such cyclotron waves are readily generated due to a mild pressure anisotropy (see e.g. Stix 1962, Section 9–13).

The possible role of plasma emission involving electron-cyclotron waves in the generation of type I bursts requires further investigation.

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Appendix I

We summarize the properties of waves in a magnetoactive plasma, firstly in general and then for the specific waves referred to in the text.

(a) General Case

For an arbitrary plasma with dielectric tensor $\mathscr{E}_{ij}(\boldsymbol{k},\omega)$ the wave properties are found by inserting the hermitian part of $\mathscr{E}_{ij}$ in

$$\Lambda_{ij}(\boldsymbol{k},\omega) = (c^2/\omega^2)(k_i k_j - |\boldsymbol{k}|^2\delta_{ij}) + \mathscr{E}_{ij}(\boldsymbol{k},\omega) \tag{A1}$$

and solving

$$\Lambda(\boldsymbol{k},\omega) = |\Lambda_{ij}(\boldsymbol{k},\omega)| = 0\,, \tag{A2}$$

where the modulus signs denote the determinant, for any solution

$$\omega = \omega^\sigma(\boldsymbol{k}) = -\omega^\sigma(-\boldsymbol{k})\,.$$

Constructing $\lambda_{ij}(\boldsymbol{k},\omega)$ from

$$\Lambda_{ij}(\boldsymbol{k},\omega)\,\lambda_{jl}(\boldsymbol{k},\omega) = \Lambda(\boldsymbol{k},\omega)\,\delta_{il}\,, \tag{A3}$$

the polarization vector for the mode σ is given by

$$\lambda_{ij}\big(\boldsymbol{k},\omega^\sigma(\boldsymbol{k})\big) = \lambda_{ss}\big(\boldsymbol{k},\omega^\sigma(\boldsymbol{k})\big)\,e_i^\sigma(\boldsymbol{k})\,e_j^{\sigma*}(\boldsymbol{k})\,. \tag{A4}$$

The ratio of electrical to total energy in the waves also involves the trace λ_{ss} of λ_{ij}:

$$\frac{W_E^\sigma(\boldsymbol{k})}{W_{\mathrm{T}}^\sigma(\boldsymbol{k})} = \left[\frac{\lambda_{ss}(\boldsymbol{k},\omega)}{\omega\,\partial\{\Lambda(\boldsymbol{k},\omega)\}/\partial\omega}\right]_{\omega=\omega^\sigma(\boldsymbol{k})}. \tag{A5}$$

(b) *Magnetoionic Waves*

When the dielectric tensor is taken to be

$$\mathscr{E}_{ij}(\boldsymbol{k},\omega) = \delta_{ij}-(\omega_{\mathrm{p}}^2/\omega^2)\tau_{ij}^{(\mathrm{e})}(\omega)\,, \tag{A6}$$

with $\tau_{ij}^{(\mathrm{e})}$ given by equation (19), the wave modes are referred to as the magnetoionic modes. The dispersion relation is usually expressed in terms of the refractive index $\mu_\sigma(\omega,\theta)$ $(=|\boldsymbol{k}|\,c/\omega^\sigma)$. One can always choose to write the polarization vectors in the form

$$\boldsymbol{e}^\sigma = (K_\sigma\boldsymbol{\kappa}+T_\sigma\boldsymbol{\tau}+\mathrm{i}\boldsymbol{a})/(K_\sigma^2+T_\sigma^2+1)^{\frac{1}{2}}\,, \tag{A7}$$

with

$$\left.\begin{aligned}\boldsymbol{\kappa} &= (\sin\theta\cos\psi,\sin\theta\sin\psi,\cos\theta)\,,\\ \boldsymbol{\tau} &= (\cos\theta\cos\psi,\cos\theta\sin\psi,-\sin\theta)\,,\qquad \boldsymbol{a} = (-\sin\psi,\cos\psi,0)\,.\end{aligned}\right\} \tag{A8}$$

This choice is particularly convenient for magnetoionic waves.

It can be shown that the properties of the magnetoionic waves are found as follows. If the equation

$$T_\sigma^2+\{Y\sin^2\theta/(1-X)\cos\theta\}T_\sigma-1 = 0\,, \tag{A9}$$

with $Y = \Omega_{\mathrm{e}}/\omega$ and $X = \omega_{\mathrm{p}}^2/\omega^2$, is solved for

$$T_{\mathrm{o}} = -\frac{\frac{1}{2}Y^2\sin^2\theta+\Delta}{Y(1-X)\cos\theta} = \frac{Y(1-X)\cos\theta}{\frac{1}{2}Y^2\sin^2\theta-\Delta} = -T_{\mathrm{x}}^{-1}\,, \tag{A10}$$

with

$$\Delta^2 = \tfrac{1}{4}Y^4\sin^4\theta+(1-X)^2Y^2\cos^2\theta\,, \tag{A11}$$

the remaining wave properties are then given by

$$K_\sigma = \frac{XY\sin\theta}{1-X}\,\frac{T_\sigma}{T_\sigma-Y\cos\theta} = \frac{XY\sin\theta(1+YT_\sigma\cos\theta)}{(1-X)(1-Y^2)-XY^2\sin^2\theta}\,, \tag{A12}$$

$$\mu_\sigma^2 = 1-\frac{XT_\sigma}{T_\sigma-Y\cos\theta} = 1-\frac{(1-X)K_\sigma}{Y\sin\theta}\,, \tag{A13}$$

$$\left[\frac{W_E}{W_{\mathrm{T}}}\right]^\sigma = \frac{K_\sigma^2+T_\sigma^2+1}{2(1+T_\sigma^2)\mu_\sigma\,\partial(\omega\mu_\sigma)/\partial\omega}\,. \tag{A14}$$

Physically T_σ is the axial ratio of the ellipse describing the transverse part of the polarization, with the sign of T_σ determining the handedness (positive for right hand), while K_σ describes the longitudinal part of the polarization.

(c) *Electron Plasma Waves*

The waves we refer to as electron plasma waves have a polarization vector $\boldsymbol{e} = \boldsymbol{k}/|\boldsymbol{k}|$ and a frequency given by

$$(\omega^l)^2 = \omega_+^2 + \frac{\omega_p^2(\omega_+^2 - \Omega_e^2)}{\omega_+^2 - \omega_-^2}\left(\frac{V_e}{v_\phi}\right)^2 f(Y_+^2, \theta) + \dots, \tag{A15}$$

where $Y_+ = \Omega_e/\omega_+$,

$$f(Y^2, \theta) = 3\cos^4\theta - \frac{\cos^2\theta\sin^2\theta}{Y^2} - \frac{\sin^4\theta}{Y^2(1-Y^2)} + \frac{(1+3Y^2)\sin^2\theta\cos^2\theta}{Y^2(1-Y^2)} + \frac{\sin^4\theta}{Y^2(1-4Y^2)}, \tag{A16}$$

and

$$\omega_\pm^2 = \tfrac{1}{2}(\omega_p^2 + \Omega_e^2) \pm \tfrac{1}{2}\{(\omega_p^2 + \Omega_e^2)^2 - 4\omega_p^2\,\Omega_e^2\cos^2\theta\}^{\frac{1}{2}}. \tag{A17}$$

The expansion indicated in equation (A15) is in powers of $(V_e/v_\phi)^2$ with $v_\phi = \omega_+/|\boldsymbol{k}|$. The ratio of electrical to total energy can be approximated by

$$[W_E/W_T]^l = (\omega_+^2 - \Omega_e^2)/2(\omega_+^2 - \omega_-^2). \tag{A18}$$

(d) *Sound Waves*

The properties of sound waves used in equation (30) are

$$\omega^s = |\boldsymbol{k}|v_s|\cos\theta|, \qquad \boldsymbol{e}^s = \boldsymbol{k}/|\boldsymbol{k}|, \qquad [W_E/W_T]^s = \tfrac{1}{2}(\omega^s/\omega_{pi})^2, \tag{A19}$$

$$v_s = \omega_{pi}\,\lambda_{De}, \qquad \omega_{pi}^2 = (Z_i\, m_e/m_i)\omega_p^2, \qquad \lambda_{De} = V_e/\omega_p. \tag{A20}$$

Appendix II

The forms of the tensor κ_{ijl} used in the text are given here. Melrose (1972) has discussed the symmetry properties satisfied by κ_{ijl} and Melrose and Sy (1972) have derived a general expression for κ_{ijl} using kinetic theory.

For the case of a cold electronic plasma κ_{ijl} is given by

$$\kappa_{ijl}(\boldsymbol{k},\omega;\boldsymbol{k}',\omega';\boldsymbol{k}'',\omega'')$$

$$= -\frac{e\omega_p^2}{8\pi m_e c^2}\Bigg(\frac{k_r}{\omega'}\tau_{rj}^{(e)}(\omega')\,\tau_{il}^{(e)}(\omega'') + \frac{k_r}{\omega''}\tau_{rl}^{(e)}(\omega'')\,\tau_{ij}^{(e)}(\omega') + \frac{k_r'}{\omega}\tau_{ir}^{(e)}(\omega)\,\tau_{jl}^{(e)}(\omega'')$$

$$- \frac{k_r'}{\omega''}\tau_{rl}^{(e)}(\omega'')\,\tau_{ij}^{(e)}(\omega) + \frac{k_r''}{\omega}\tau_{ir}^{(e)}(\omega)\,\tau_{lj}^{(e)}(\omega') - \frac{k_r''}{\omega'}\tau_{rj}^{(e)}(\omega')\,\tau_{il}(\omega)\Bigg), \tag{A21}$$

where $-e$ is the electronic charge and $\tau_{ij}^{(e)}$ is defined by equation (19). This form is appropriate for

$$\omega,\ |\omega - \Omega_e| \gg |k_\parallel|\,V_e, \qquad \Omega_e \gg k_\perp V_e, \tag{A22}$$

for each of $(\omega, \boldsymbol{k})$, $(\omega', \boldsymbol{k}')$, and $(\omega'', \boldsymbol{k}'')$. Accordingly it is used in deriving equation (28) from (14).

In the case where two of the Fourier components satisfy the conditions (A22) but the third, $(\omega'', \boldsymbol{k}'')$ say, satisfies $\omega'' \ll |k''_\parallel| V_e$, it is convenient to break the symmetry

$$\kappa_{ijl}(\boldsymbol{k}, \omega;\, \boldsymbol{k}', \omega';\, \boldsymbol{k}'', \omega'') = \kappa_{ilj}(\boldsymbol{k}, \omega;\, \boldsymbol{k}'', \omega'';\, \boldsymbol{k}', \omega')\,. \tag{A23}$$

Because this symmetry is used in writing down the expression (10), we need to insert the extra factor 2 in the approximate form

$$2\kappa_{ijl}(\boldsymbol{k}, \omega;\, \boldsymbol{k}', \omega';\, \boldsymbol{k}'', \omega'') = (e\omega''/4\pi m_e c^2)\tau^{(e)}_{ij}(\omega)\, k''_r\{\mathscr{E}^{(e)}_{rs}(\boldsymbol{k}'', \omega'') - \delta_{rs}\} \tag{A24}$$

which is appropriate in this case. In the expression (A24) only the electronic contribution $\mathscr{E}^{(e)}_{rs}$ to the dielectric tensor is retained (with the vacuum contribution δ_{rs} included implicitly and so subtracted). Because one has $|\omega''|$ very much less than $|\omega|$ and $|\omega'|$ by hypothesis and because $\omega = \omega' + \omega''$ and $\boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{k}''$ are implicit, ω and ω' are to be regarded as equal in (A24) except where their difference ω'' appears explicitly. The approximate form (A24) is used in equations (18) and (30).