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A RELATIVISTIC QUANTUM THEORY FOR PROCESSES IN COLLISIONLESS PLASMAS

D. B. MELROSE
Department of Theoretical Physics, Faculty of Science, The Australian National University, Canberra, Australia

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Abstract—A relativistic quantum theory for processes in a plasma is developed in such a way that the theory reduces to that for a classical collisionless plasma in the classical limit and to conventional quantum electrodynamics *in vacuo*. The generalization involves two stages. The first stage is the use of quantum electrodynamics to calculate the linear and non-linear responses of an electron gas. Explicit expressions are derived for the linear response of an electron gas in a static magnetic field, and for the lowest order non-linear response of an electron gas in the absence of a static magnetic field. The second stage involves a generalization of the conventional diagrammatic technique to include the effects of the non-linear responses of the medium. Additional diagrammatic structures called multiple-photon vertices are introduced. Rules are given for drawing the generalized diagrams and for writing down the associated amplitudes. The Cerenkov effect, photon splitting and emission in the presence of a static magnetic field are treated as illustrative examples.

1. INTRODUCTION

QUANTUM electrodynamics is usually considered for processes *in vacuo*. The quantization is applied to the electromagnetic field equations for a vacuum, and the interaction with matter is introduced through the currents associated with individual particles. Three steps are involved in applying quantum electrodynamics to a plasma in a self-consistent way. Firstly, quantum electrodynamics should be used to calculate the responses of the plasma to an electromagnetic disturbance. Secondly, the current describing the linear response should be included in the electromagnetic field equations before these are quantized. Thirdly, the other currents describing the response of the plasma should be included along with the single-particle currents in treating the interaction with matter.

The purpose of this paper is to discuss the first and third stages in the above generalization. In Section 2 quantum electrodynamics is used to calculate the responses of a collisionless electron gas following a method introduced by TSYTOVICH (1961). In Section 3 the conventional diagrammatic technique of quantum electrodynamics is generalized to include the effects of the non-linear responses of the medium. The method is related to one used by GAILITIS and TSYTOVICH (1965). In this way a theory is developed which is a generalization both of the classical theory of a collisionless plasma and of quantum electrodynamics *in vacuo*. Some illustrative examples are given in Section 4.

The second stage is not discussed in detail here. It is assumed that the radiation field may be regarded as a collection of wave quanta, called photons for simplicity, in various wave modes (labelled σ, σ', etc.) with wave properties determined as in the classical case, and with occupation numbers $N^{\sigma}(\mathbf{k})$, $N^{\sigma'}(\mathbf{k})$, etc. Similarly the propagator for the electromagnetic field, i.e. the photon propagator, is assumed to be the same as in the classical case, but written in appropriate 4-vector notation (e.g. MELROSE, 1973). Thus the quantization of the field equations in the presence of a medium is by-passed here. [This quantization has been discussed by JAUCH and WATSON (1948*a*,*b*); BREVIK and LAUTRUP (1970) for a non-dispersive medium described phenomenologically, and by WATSON and JAUCH (1949) for a dispersive medium. These discussions are not directly relevant to the general case of a spatially dispersive medium.]

Explicit expressions for the linear response of an electron gas in a static magnetic field and for the non-linear response of an electron gas in the absence of a static magnetic field are discussed in Appendices B and C respectively. The general expression found for the polarization tensor for an electron gas in a static magnetic field is essentially the same as that found by SVETOZAROVA and TSYTOVICH (1962). This result includes relativistic effects and is more general than results obtained by many subsequent authors, e.g. by KELLY (1964); GREEN *et al.* (1969); HARRIS (1969); CANUTO and VENTURA (1972). The method of derivation is summarized in Appendix A, and the general result is written down in Appendix B where an error in Svetozarova and Tsytovich's expression for the interesting case of the long-wavelength limit is corrected. The general expression found in Appendix C for the non-linear response tensor is of formal interest in that it implies the existence of a generalization of FURRY's (1937) theorem; the limiting case of a degenerate electron gas may be of interest in connection with non-linear processes in solid state plasmas

2. RESPONSE TENSORS

TSYTOVICH (1961) calculated the polarization tensor for an electron gas using the following technique: in the formula for the vacuum polarization tensor the electron propagator is replaced by a propagator averaged over the distribution of electrons. Tsytovich's result reduces to well known results for a collisionless plasma in appropriate limits, e.g. it reduces to SILIN's (1960) result in the classical (but relativistic) limit, and it leads to LINDHARD's (1954) result in the completely degenerate non-relativistic limit. Tsytovich's result also includes the vacuum polarization tensor.

Tsytovich's method can be used to calculate the non-linear response tensors for an electron gas. To avoid confusion let the lowest order non-linear response be called the 'quadratic' response, the next highest order non-linear response be called the 'cubic' response, and so on. Following Tsytovich, the linear response tensor can be written down in terms of the amplitude for the 'bubble' diagram Fig. 1. Likewise

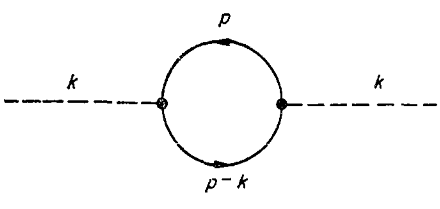

FIG. 1.—The 'bubble' diagram.

the quadratic and cubic response tensors can be written down in terms of the amplitudes for the 'triangle' diagrams Fig. 2 and the 'box' diagrams Fig. 3. There are two topologically distinct 'triangle' diagrams, and six distinct 'box' diagrams. The required tensor is found by averaging the amplitudes for these diagrams. [Actually, the average is to be taken if the quadratic (or cubic, etc.) response is that to a single disturbance, but it may be convenient not to average if the bi-linear (or tri-linear, etc.) response to two (or more) distinct disturbances is required.]

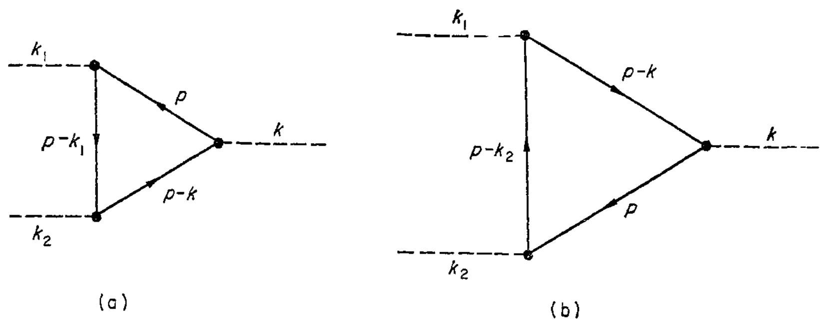

FIG. 2.—The 'triangle' diagrams. For processes *in vacuo* the amplitudes corresponding to diagrams (a) and (b) are equal but opposite by Furry's Theorem.

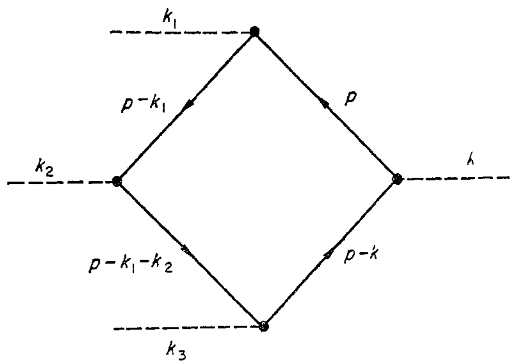

FIG. 3.—A 'box' diagram. There are six distinct such diagrams corresponding to the six orderings of the final proton lines.

The tensors

In the following the notation for the polarization tensor is that used by MELROSE (1973) and the notation for the Dirac algebra is that used by BERESTETSKII *et al.* (1971), with the exceptions that here the electronic charge is $-e$ and 'Sp' denotes the trace over Dirac matrices. The units are such that $\hbar = c = 1$.

The linear polarization 4-tensor is given by

$$\alpha^{\mu\nu}(k) = -ie^2 Sp\int \frac{d^4p}{(2\pi)^4}\gamma^\mu \bar{G}(p)\gamma^\nu \bar{G}(p-k). \tag{1}$$

The averaged propagator $\bar{G}$ is given by

$$\begin{aligned}\bar{G}(P) &= g^+(P)\frac{\not{p}^+ + m}{2\varepsilon} + g^-(P)\frac{\not{p}^- + m}{2\varepsilon},\\ g^\pm(P) &= \frac{1-n^\pm(\mathbf{p})}{\pm E - \varepsilon + i0} + \frac{n^\pm(\mathbf{p})}{\pm E - \varepsilon - i0},\end{aligned} \tag{2}$$

where $n^+(\mathbf{p})$ and $n^-(\mathbf{p})$ are the occupation numbers for electrons and positrons respectively, and with

$$\begin{aligned}P^\mu &:= [E, \mathbf{p}], \qquad p^\mu := [\varepsilon, \mathbf{p}],\\ \varepsilon &:= \varepsilon(\mathbf{p}) := (|\mathbf{p}|^2 + m^2)^{1/2},\\ \not{p}^\pm &:= \pm\gamma^0\varepsilon - \boldsymbol{\gamma}\cdot\mathbf{p}.\end{aligned} \tag{3}$$

As explained by TSYTOVICH (1961) only terms arising from the principal value for one of the propagators and from the semi-residue for the other are to be retained in (1). This gives only the hermitian part of $\alpha^{\mu\nu}$. The anti-hermitian part may be obtained using the Kramers–Kronig relations. [The terms which come from taking both principal values simultaneously vanish identically, while the terms which arise from both semi-residues taken simultaneously do not give the correct anti-hermitian part of $\alpha^{\mu\nu}$ because (2) corresponds to the Feynman propagator which does not take causal effects into account.]

The quadratic non-linear response tensor is given by

$$\alpha^{\mu\nu\rho}(k, k_1, k_2) = \tfrac{1}{2}[\alpha_1^{\mu\nu\rho}(k, k_1, k_2) + \alpha_2^{\mu\nu\rho}(k, k_1, k_2)] \tag{4}$$

with

$$\alpha_1^{\mu\nu\rho}(k, k_1, k_2) = ie^3 Sp \int \frac{d^4p}{(2\pi)^4} \gamma^\mu \bar{G}(p) \gamma^\nu \bar{G}(p - k_1) \gamma^\rho \bar{G}(p - k), \tag{5}$$

and

$$\alpha_2^{\mu\nu\rho}(k, k_1, k_2) = \alpha_1^{\mu\rho\nu}(k, k_2, k_1). \tag{6}$$

In (5) $k = k_1 + k_2$ is to be understood. An explicit expression for this tensor is derived and discussed in Appendix C.

The cubic non-linear response tensor is given by

$$\alpha^{\mu\nu\rho\tau}(k, k_1, k_2, k_3) = \tfrac{1}{6}[\alpha_1^{\mu\nu\rho\tau}(k, k_1, k_2, k_3) + \ldots + \alpha_6^{\mu\nu\rho\tau}(k, k_1, k_2, k_3)], \tag{7}$$

with

$$\alpha_1^{\mu\nu\rho\tau}(k, k_1, k_2, k_3) = -ie^4 Sp \int \frac{d^4p}{(2\pi)^4} \gamma^\mu \bar{G}(p) \gamma^\nu \bar{G}(p - k_1) \times \gamma^\rho \bar{G}(p - k_1 - k_2) \gamma^\tau \bar{G}(p - k), \tag{8}$$

and where the remaining five terms on the RHS of (7) are obtained from (8) by making all permutations among (ν, k_1), (ρ, k_2) and (τ, k_3). In (8) $k = k_1 + k_2 + k_3$ is to be understood.

Symmetry properties

Of the symmetry properties pointed out by MELROSE (1972), namely

$$\alpha^{\mu\nu\rho}(k, k_1, k_2) = \alpha^{\mu\rho\nu}(k, k_2, k_1) = \alpha^{\nu\mu\rho}(-k_1, -k_2, k), \tag{9}$$

the former is imposed in (4) and the latter can be proven as follows. Starting from (5) and using the invariance of the trace under cyclic permutations of the matrices, one obtains

$$\alpha_1^{\mu\nu\rho}(k, k_1, k_2) = \alpha_2^{\nu\mu\rho}(-k_1, -k, k_2), \tag{10}$$

which suffices to establish the result. The corresponding result for $\alpha^{\mu\nu\rho\tau}$ can be established similarly.

[It is interesting that the symmetry property is satisfied for the whole of the tensor rather than for only the non-resonant part. This may be attributed to the fact that, due to the use of the Feynman propagator, no distinction is drawn between 'disturbances' and 'responses', i.e. between causes and effects, in (5). The resonant parts of (5) are non-physical.]

The case $\mathbf{B} \neq 0$

The above applies only in the absence of an ambient magnetic field. In the presence of a static field, $\mathbf{B}$ say, the electron propagator between $\mathbf{r}, t$ and $\mathbf{r}', t'$ depends on $\mathbf{r}$ and $\mathbf{r}'$ separately rather than only on their difference $\mathbf{r} - \mathbf{r}'$, as would be implied by the use of Fourier transformed propagators in (1), (5) and (8). Fourier transforming in time only, (1) would be replaced by

$$\alpha^{\mu\nu}(\mathbf{r} - \mathbf{r}'; \omega) = -ie^2 Sp \int \frac{\mathrm{d}E}{2\pi} \gamma^\mu \bar{G}(\mathbf{r}, \mathbf{r}', E) \gamma^\nu \bar{G}(\mathbf{r}', \mathbf{r}, E - \omega), \tag{11}$$

where the dependence of $\alpha^{\mu\nu}$ on only $\mathbf{r} - \mathbf{r}'$ (rather than on both $\mathbf{r}$ and $\mathbf{r}'$) follows from translational invariance in a homogeneous medium. Likewise (5) would be replaced by

$$\alpha_1^{\mu\nu\rho}(\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2; \omega, \omega_1, \omega_2) = ie^3 Sp \int \frac{\mathrm{d}E}{2\pi} \gamma^\mu \bar{G}(\mathbf{r}, \mathbf{r}_1, E) \times \gamma^\nu \bar{G}(\mathbf{r}_1, \mathbf{r}_2, E - \omega_1) \gamma^\rho \bar{G}(\mathbf{r}_2, \mathbf{r}, E - \omega). \tag{12}$$

Homogeneity requires that $\alpha_1^{\mu\nu\rho}$ depend only on the differences between the coordinates, e.g. on $\mathbf{r} - \mathbf{r}_1$ and $\mathbf{r} - \mathbf{r}_2$. This ensures that on taking the spatial Fourier transform, with $\mathbf{k}$, $\mathbf{k}_1$ and $\mathbf{k}_2$ the Fourier components for the $\mathbf{r}$, $\mathbf{r}_1$ and $\mathbf{r}_2$ dependences respectively, the identity $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ must be satisfied. Likewise $\omega = \omega_1 + \omega_2$ is to be understood in (12).

3. EXTENSION OF THE DIAGRAMMATIC TECHNIQUE

When treating quantum electrodynamics *in vacuo*, the interaction Hamiltonian is of the form

$$H(t) = \int \mathrm{d}^3\mathbf{r}[j^\mu(\mathbf{r}, t) A_\mu(\mathbf{r}, t)], \tag{13}$$

where

$$j^\mu(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x) \tag{14}$$

is the current associated with a single particle (assumed to be an electron). On including the effects of an ambient medium, the linear current associated with the response of the medium is incorporated into the photon propagator, but the nonlinear responses, described by

$$j^\mu(x) = \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2 \ldots \mathrm{d}^4y_m \alpha^{\mu\nu_1\nu_2\ldots\nu_m}(x, y_1, y_2, \ldots, y_m) \times A_{\nu_1}(y_1) A_{\nu_2}(y_2) \ldots A_{\nu_m}(y_m), \tag{15}$$

with $m = 2, 3, \ldots$, need to be included in the interaction Hamiltonian.

It is straightforward to include the additional currents in the formal expansion of the S-matrix. However, the usual diagrammatic technique, wherein a one-to-one correspondence is set up between terms in this expansion and Feynman diagrams, needs to be generalized. It is convenient to do this by introducing new topological structures, here called multiple photon vertices. The rules for writing down the amplitudes for each diagram, e.g. as given by BERESTETSKII *et al.* (1971) also need to be generalized.

Multiple-photon vertices

Suppose those terms, in the expansion of S-matrix, would involve the current (15) are associated with a diagrammatic element of the form shown in Fig. 4. For the current (15) the tensor has $m+1$ indices and the corresponding diagrammatic element has $m+1$ photon lines joined to a blob. Each such diagrammatic element with m lines will be called an m-photon vertex.

Because of the asymmetry between the first written index plus argument and the remaining indices plus arguments in the tensors α, it is convenient to replace them by the more symmetric tensors

$$a^{\mu_1\mu_2\ldots\mu_m}(k_1, k_2, \ldots, k_m) := \alpha^{\mu_1\mu_2\ldots\mu_m}(-k_1, k_2, \ldots, k_m), \tag{16}$$

where $\alpha^{\mu_1\cdots\mu_m}$ is the $(m-1)$th order non-linear response tensor. That is the $(m-1)$th order non-linear current is given by

$$j^{\mu_1}(k_1) = \int \frac{d^4k_2}{(2\pi)^4} \cdots \frac{d^4k_m}{(2\pi)^4} (2\pi)^4 \, \delta^4(k_1 + \ldots + k_m)$$

$$\times \, a^{\mu_1\mu_2\ldots\mu_m}(k_1, k_2, \ldots, k_m) A_{\mu_2}(k_2) \ldots A_{\mu_m}(k_m). \tag{17}$$

The symmetry properties discussed by MELROSE (1972) imply that the non-resonant part of $a^{\mu_1\cdots\mu_m}$ is completely symmetric under all permutations of $(\mu_1, k_1) \ldots (\mu_m, k_m)$.

Rules for generalized diagrams

The standard rules for drawing the diagrams for the momentum-state representation are as follows for the case of processes *in vacuo* (e.g. BERESTETSKII *et al.*, 1971).

A given scattering process is specified by the number and kind of particles (including photons) in the initial and final states; the diagram for a specific scattering process has external lines equal in number and kind to the particles in the scattering process. The initial state is on the right and the final state is on the left of the diagram. Electrons are represented by solid lines which are labelled with the 4-momentum of the electron and with arrows pointing from right to left. Positrons are represented by solid lines labelled with minus the 4-momentum and with arrows pointing from left to right. Each solid line must be continuous with the arrow having a constant direction along it. Photons are represented by dashed lines labelled with the 4-momentum of the photon. Photon and electron–positron lines join at vertices. The link with the S-matrix expansion is in that each nth order term in this expansion corresponds to a diagram with n vertices.

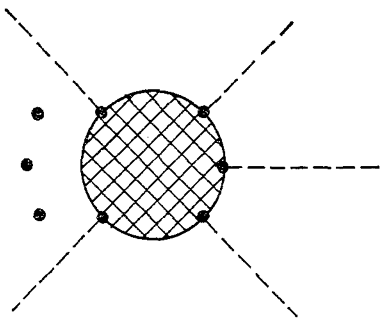

FIG. 4.—An m-photon vertex.

Let an *ordinary* diagram or diagrammatic element be defined as one which includes no multiple-photon vertices.

A generalization of the above rules to include the m-photon vertices is as follows:

A generalized nth order diagram consists of connected structures of ordinary diagrams (of order n_1, n_2, etc. say) and of multiple-photon vertices (m_1-photon, m_2-photon, etc. say) such that one has

$$n = n_1 + n_2 + \ldots + (m_1 - 2) + (m_2 - 2) + \ldots. \tag{18}$$

That is, an m-photon vertex is to be regarded as a diagrammatic substructure of $(m - 2)$th order.

To lowest order, which includes emission or absorption of a single photon by a single particle and the crossed processes of decay of a photon into an electron–positron pair or annihilation of a pair into a single photon, no multiple-photon vertices contribute. To next order, which includes scattering of one photon into another by a particle, double emission or absorption (i.e. of two photons by a single particle) and the annihilation of a pair into two photons, the 3-photon vertex does contribute, see Fig. 5. For example, in treating the scattering of a photon by an electron there are two diagrams (Figs. 5a and b) corresponding to the Compton effect and one (Fig. 5c) which involves the non-linear response of the medium. The amplitudes for each of these are to be added. Of the same order as the scattering process is the decay of a photon into two photons or the opposite coalescence process (generation of a second harmonic, of interest in non-linear optics, is a special case of such coalescence); these processes are described by the diagram Fig. 6.

However to next order, e.g. for the double Compton effect, there are $3! = 6$ ordinary diagrams, but there are ten additional diagrams involving the non-linear response of the medium, see Fig. 7.

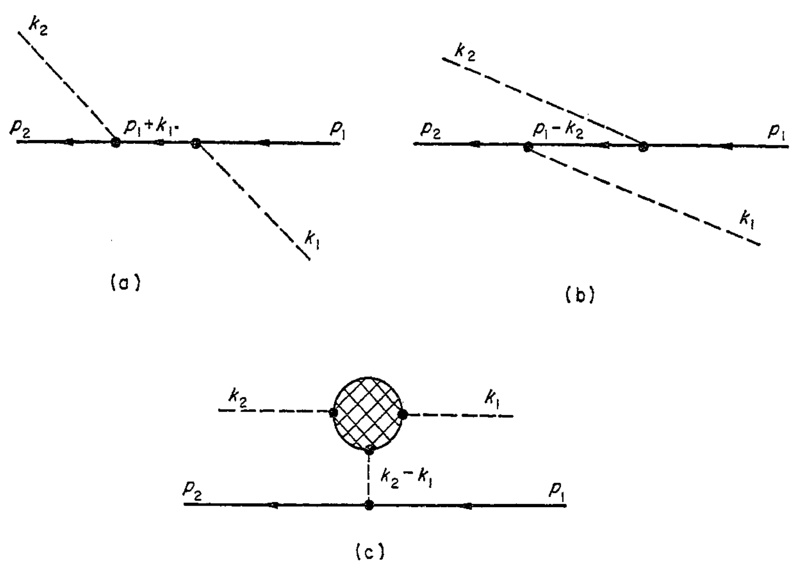

FIG. 5.—The diagrams for photon scattering. The two diagrams (a) and (b) are the usual ones for Compton scattering, while (c) describes non-linear scattering.

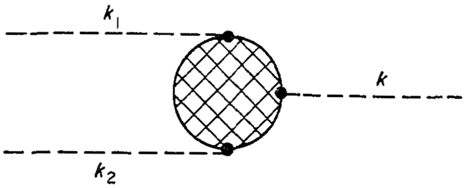

FIG. 6. The diagram for the decay of a photon into two photons.

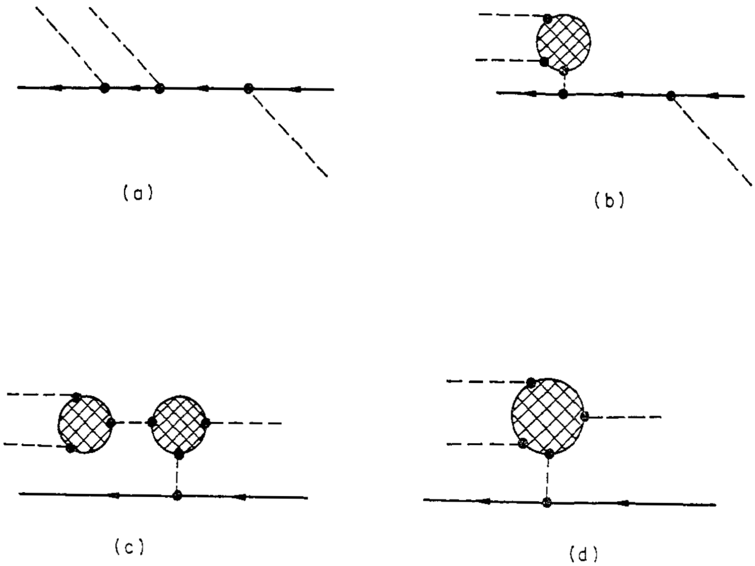

FIG. 7.—The diagrams for double Compton scattering. There are six diagrams of type (a), six of type (b), three of type (c) and one of type (d).

The transition probability

Let S_{fi} be the S-matrix element between an initial state i, with total 4-momentum P_i, and a final state f, with total 4-momentum P_f. The transition probability (for a system in a volume V over a time T) can be written as the modulus squared of the S-matrix element S_{fi} between the states. Appealing to conservation of 4-momentum one can write

$$S_{fi} =: i(2\pi)^4\, \delta^4(P_i - P_f)T_{fi}. \tag{19}$$

The probability per unit time of a transition is then given by

$$w_{i\to f} = V(2\pi)^2\, \delta^4(P_i - P_f)\, |T_{fi}|^2. \tag{20}$$

In evaluating T_{fi} it is convenient to separate certain factors relating to the normalization. Each particle (electron, photon, etc.) is normalized to one in an elemental range, e.g. in $d^3\mathbf{p}/(2\pi)^3$ or $d^3\mathbf{k}/(2\pi)^3$, in the volume V. In the absence of an ambient magnetic field electrons (positrons) may be described by a 4-spinor $u(\mathbf{p})$ ($v(\mathbf{p})$) normalized by

$$\bar{u}(\mathbf{p})\gamma^0 u(\mathbf{p}) = 2\varepsilon(\mathbf{p}) \qquad (= \bar{v}(\mathbf{p})\gamma^0 v(\mathbf{p})). \tag{21}$$

The factor for each initial electron or positron is chosen to be $(2\varepsilon V)^{1/2}$ where $\varepsilon = \varepsilon(\mathbf{p})$ is the energy of the particle.

For photons in a mode σ with dispersion relation $\omega = \omega^\sigma(\mathbf{k})$ and polarization vector $\mathbf{e}^\sigma(\mathbf{k})$, the vector potential may be written as

$$\mathbf{A}^\sigma(\mathbf{r}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} [a^\sigma(\mathbf{k})\mathbf{e}^\sigma(\mathbf{k})e^{(i\mathbf{k}\cdot\mathbf{r}-\omega^\sigma(\mathbf{k})t)} + \text{complex conjugate}]. \tag{22}$$

After calculating the electrical energy density, the total energy density, in the waves in the range $d^3\mathbf{k}/(2\pi)^3$, is given by

$$W_T{}^\sigma(\mathbf{k}) = \frac{|\omega^\sigma(\mathbf{k})a^\sigma(\mathbf{k})|^2}{4V[W_E/W_T]^\sigma(\mathbf{k})}, \tag{23}$$

where $[W_E/W_T]^\sigma(\mathbf{k})$ is the ratio of electric to total energy in the waves, see e.g. Appendix I of MELROSE and SY (1972). It follows that the appropriate normalization factor for photons is

$$|a^\sigma(\mathbf{k})|^2 = \frac{4\pi}{\omega^\sigma(\mathbf{k})}\left(\frac{W_E}{W_T}\right)^\sigma(\mathbf{k}). \tag{24}$$

In vacuo one has $[W_E/W_T] = \frac{1}{2}$. The factor for each photon, which would be $(2\omega V)^{1/2}$ *in vacuo*, is $\{\omega^\sigma(\mathbf{k})V[W_T/W_E]^\sigma(\mathbf{k})\}^{1/2}$ in a medium.

It is convenient to write

$$T_{fi} =: \frac{M_{fi}}{\Pi(2\varepsilon V)^{1/2}\Pi(\omega^\sigma[W_T/W_E]^\sigma V)^{1/2}}, \tag{25}$$

where the products are over all electrons (or positrons) in the initial state and over all photons in the initial state.

The probability per unit time for a particular process is

$$dw_{i\to f} = \frac{V(2\pi)^4\,\delta^4(P_i - P_f)\,|M_{fi}|^2\,D_f}{\Pi(2\,|\varepsilon|\,V)\Pi(|\omega^\sigma|\,[W_T/W_E]^\sigma V)}, \tag{26}$$

where the density of final states is the product

$$D_f = \Pi\,\frac{d^3\mathbf{p}}{(2\pi)^3 2|\varepsilon|}\,\Pi\left(\frac{W_E}{W_T}\right)^\sigma \frac{d^3\mathbf{k}}{(2\pi)^3\,|\omega^\sigma|} \tag{27}$$

for each electron (or positron) in the final state and for each photon in the final state. When M_{fi} is a product of Dirac matrices taking the trace (*Sp*) in evaluating $|M_{fi}|^2$ is to be understood in (26).

Thus the problem is reduced to evaluating the scattering amplitude M_{fi}.

Rules for evaluating the scattering amplitude

The rules for writing the scattering amplitude M_{fi} (in the absence of an ambient magnetic field) are as follows:

(1) Incoming electron, positron and photon lines are associated with u, $\bar{v}$ and $\sqrt{4\pi}\,e_\mu$ respectively (e_μ is the polarization 4-vector), and outgoing lines with $\bar{u}$, v and $\sqrt{4\pi}\,e_\mu{}^*$ respectively.

(2) Internal electron lines are associated with iG and internal photon lines with $-iD_{\mu\nu}$ (see (41) below). In all cases the argument is the 4-momentum associated with that line. With each vertex is associated a 4-vector $-i\gamma^\mu$. The 4-vector indices on the photon propagator are those associated with the two vertices between which it propagates.

(3) 4-Momentum is conserved at each vertex. The integral over $d^4p/(2\pi)^4$ is to be carried out for any undetermined 4-momentum around an internal loop.

(4) The amplitudes u, $\bar{u}$, v or $\bar{v}$, the matrices associated with the vertices and the propagators G are written according to matrix multiplication along any solid line in the reverse direction to the arrow.

(5) Each nth order diagram (in the absence of m-photon vertices) contributes to iM_{fi} with a factor $(-e)^n$. An additional factor -1 is included for each closed electron loop. An additional factor -1 is included for each line which joins a positron in the initial state to a positron in the final state. When identical fermions are present in the initial or final states the overall phase factor is arbitrary but the total amplitude must change sign on interchanging two identical fermions.

To these the following rule is added:

(6) With each m-photon vertex is associated a vertex function

$$-ia^{\mu_1 \cdots \mu_m}(k_1, \ldots, k_m).$$

The power of $-e$ in iM_{fi} is the product of the factors for the ordinary diagrammatic components in the generalized diagram, i.e. m-photon vertices contribute no power of $-e$.

4. ILLUSTRATIVE EXAMPLES

In this section the probabilities for single emission (the Cerenkov effect) and for scattering of photons (which is the Compton effect *in vacuo*) are written down; these probabilities include those of the crossed processes. The probability for coalescence or decay of photons, which latter process is called photon splitting in a magnetic field *in vacuo*, is then discussed. Finally, emission of a photon in the presence of an ambient magnetic field is discussed.

The Cerenkov effect

The Cerenkov effect, described by the diagram Fig. 8, is the simplest example of the use of the above formalism. The rules for writing down the scattering amplitude give

$$M_{fi} = \sqrt{4\pi}\, e e_\mu{}^* \bar{u}' \gamma^\mu u, \tag{28}$$

with $u = u(\mathbf{p})$ and $\bar{u}' = \bar{u}(\mathbf{p}')$. The modulus can be written in the form

$$|M_{fi}|^2 = 4\pi\, e^2 e_\mu{}^* e_\nu Sp[\rho'^{(e)} \gamma^\mu \rho^{(e)} \gamma^\nu], \tag{29}$$

where

$$\rho'^{(e)} = u'\bar{u}', \qquad \rho^{(e)} = u\bar{u} \tag{30}$$

are the polarization density matrices for final and initial electrons respectively. For unpolarized electrons one has

$$\rho^{(e)} = \tfrac{1}{2}(\not{p} + m). \tag{31}$$

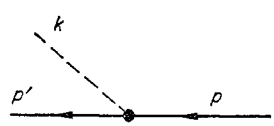

FIG. 8.—The diagram for the Cerenkov effect.

Averaging over the initial and summing over the final states of polarization (the sum is twice the average) one obtains

$$|M_{fi}|^2 = 8\pi^2 e^2 e_\mu{}^* e_\nu \tfrac{1}{4} Sp[(\not{p}' + m)\gamma^\mu(\not{p} + m)\gamma^\nu]. \tag{32}$$

Evaluating the trace in the rest frame of the medium and in the gauge $\phi = 0$ proceeds as follows: using

$$\begin{gathered} e^\mu = [0, \mathbf{e}^\sigma], \qquad k^\mu = [\omega^\sigma, \mathbf{k}], \\ p' = p - k, \qquad p^2 = m^2, \qquad p'^2 = m^2, \\ pk = \tfrac{1}{2}k^2 = \tfrac{1}{2}((\omega^\sigma)^2 - |\mathbf{k}|^2), \\ e^\mu e_\mu{}^* = -1, \end{gathered} \tag{33}$$

one finds

$$\begin{aligned} &\tfrac{1}{4}(e_\mu{}^* e_\nu) Sp[(\not{p} - \not{k} + m)\gamma^\mu(\not{p} + m)\gamma^\nu] \\ &\qquad = e_\mu{}^* e_\nu[(p^\mu - k^\mu)p^\nu + p^\mu(p^\nu - k^\nu) + g^{\mu\nu}\{m^2 - p(p-k)\}] \\ &\qquad = 2\{|(\mathbf{p} - \tfrac{1}{2}\mathbf{k}) . \mathbf{e}^\sigma|^2 - \tfrac{1}{4}(\omega^\sigma)^2 + \tfrac{1}{4}|\mathbf{k} \times \mathbf{e}^\sigma|^2\}. \end{aligned} \tag{34}$$

The final expression for the probability, namely

$$dw = \frac{|M_{fi}|^2}{4\,|\varepsilon\varepsilon'\omega^\sigma|}\left(\frac{W_E}{W_T}\right)^\sigma (2\pi)^4\,\delta^4(p' - p + k)\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3}, \tag{35}$$

after integrating over $\mathbf{p}'$, leads to the following expression for the probability per unit time that a photon in the range $\mathrm{d}^3\mathbf{k}/(2\pi)^3$ be emitted:

$$w^\sigma(\mathbf{p}, \mathbf{k}) = \frac{8\pi^2 e^2}{|\varepsilon\varepsilon'\omega^\sigma|}\left(\frac{W_E}{W_T}\right)^\sigma |(\mathbf{p} - \tfrac{1}{2}\mathbf{k}).\mathbf{e}^\sigma|^2 - \tfrac{1}{4}(\omega^\sigma)^2 + \tfrac{1}{4}|\mathbf{k} \times \mathbf{e}^\sigma|^2\}\,\delta(\varepsilon' - \varepsilon + \omega^\sigma), \tag{36}$$

with

$$\varepsilon' := [|\mathbf{p} - \mathbf{k}|^2 + m^2]^{1/2}.$$

In the classical limit this reduces to the well-known result

$$w^\sigma(\mathbf{p}, \mathbf{k}) = \frac{8\pi^2 e^2}{\hbar\omega^\sigma}\left(\frac{W_E}{W_T}\right)^\sigma |\mathbf{e}^\sigma . \mathbf{v}|^2\,\delta(\omega^\sigma - \mathbf{k} . \mathbf{v}), \tag{37}$$

where ordinary units are re-introduced.

The probability for absorption of a photon may be obtained from (36) by the replacement $\mathbf{k} \to -\mathbf{k}$ with the following understood:

$$\begin{gathered} \omega^\sigma(-\mathbf{k}) = -\omega^\sigma(\mathbf{k}), \qquad \mathbf{e}^\sigma(-\mathbf{k}) = \mathbf{e}^{\sigma^*}(\mathbf{k}), \\ \left(\frac{W_E}{W_T}\right)^\sigma(-\mathbf{k}) = \left(\frac{W_E}{W_T}\right)^\sigma(\mathbf{k}). \end{gathered} \tag{38}$$

The result (36) is a generalization of earlier results. A relativistic quantum treatment of the Cerenkov effect in a non-dispersive medium was given by GINZBURG (1940). The effect for a dispersive medium was discussed by WATSON and JAUCH (1949), and for an isotropic spatially dispersive medium by TSYTOVICH (1961).

Scattering of photons

The total amplitude for scattering of photons includes the two for the Compton effect (Fig. 5(a and b) plus one for the additional diagram (Fig. 5c) which includes a 3-photon vertex. The sum of the amplitudes for the diagrams in Figs. 5(a) and 5(b) gives

$$M_{fi}^{(cs)} = -4\pi e^2 \bar{u}_2 \{\not{e}_2{}^* G(p_1 + k_1)\not{e}_1 + \not{e}_1 G(p_1 - k_2)\not{e}_2{}^*\} u_1, \tag{39}$$

with $u_1 = u(\mathbf{p}_1)$, $\not{e}_1 = \gamma^\mu e_\mu{}^{\sigma_1}(\mathbf{k}_1)$ etc. The amplitude for the diagram Fig. 5(c) is

$$M_{fi}^{(nl)} = -4\pi e \bar{u}_2 e_{1\mu} e_{2\nu}{}^* a^{\mu\nu\rho}(-k_2, k_1, k_2 - k_1) D_{\rho\tau}(k_2 - k_1)\gamma^\tau u_1. \tag{40}$$

The photon propagator in an appropriate form was written down by MELROSE (1973), namely

$$D_{\mu\nu}(k) = -g_{\mu\rho}\frac{4\pi\omega^2 G_\alpha{}^* G^\beta \lambda^\alpha{}_\beta{}^\rho{}_\nu(k)}{\Lambda(k)(k^\alpha G_\alpha{}^*)^2}, \tag{41}$$

where $A^\mu(k)G_\mu{}^* = 0$ is the gauge condition, and where the other tensors were defined by MELROSE (1973).

The probability per unit time of the scattering of one photon in the mode σ in the range $d^3\mathbf{k}_1/(2\pi)^3$ into a photon in the mode σ' in the range $d^3\mathbf{k}_2/(2\pi)^3$ is given by

$$w^{\sigma\sigma'}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2) = \frac{\pi\,|M_{fi}|^2}{2\varepsilon_1\varepsilon_2\omega^\sigma\omega^{\sigma'}}\left(\frac{W_E}{W_T}\right)^\sigma\left(\frac{W_E}{W_T}\right)^{\sigma'}\delta(\varepsilon_1 + \omega^\sigma - \varepsilon_2 - \omega^{\sigma'}), \tag{42}$$

with $\omega^\sigma \equiv \omega^\sigma(\mathbf{k}_1)$, $\omega^{\sigma'} \equiv \omega^{\sigma'}(\mathbf{k}_2)$, $\varepsilon_1 = (|\mathbf{p}_1|^2 + m^2)^{1/2}$, $\varepsilon_2 = (|\mathbf{p}_1 + \mathbf{k}_1 - \mathbf{k}_2|^2 + m^2)^{1/2}$, and with

$$M_{fi} = M_{fi}^{(cs)} + M_{fi}^{(nl)}. \tag{43}$$

By way of illustration, consider scattering by an electron at rest in a (classical) thermal plasma. Ignoring quantum corrections (39) in the gauge $\phi = 0$ reduces to

$$M_{fi}^{(cs)} = -4\pi e^2 \frac{\mathbf{e}^\sigma \cdot \mathbf{e}^{\sigma'*}}{m\omega^\sigma}\bar{u}_2\gamma^0 u_1, \tag{44}$$

with $\omega^\sigma(\mathbf{k}_1) = \omega^{\sigma'}(\mathbf{k}_2)$ in this limit. Now $\omega^\sigma = \omega^{\sigma'}$ implies that $D_{\rho\tau}$ in (40) is to be evaluated at zero frequency. In this case it is convenient to use the Coulomb gauge (i.e. $G^\mu = [0, \mathbf{k}]$ in (41)) when one finds

$$D_{00}(k)|_{\omega=0} = \frac{4\pi}{|\mathbf{k}|^2\varepsilon^l(0, \mathbf{k})}, \qquad D_{i0} = D_{0i} = D_{ij} = 0, \tag{45}$$

where the zero frequency limit of the longitudinal part of the dielectric tensor for a thermal classical electron gas is

$$\varepsilon^l(0, \mathbf{k}) = \frac{1}{|\mathbf{k}|^2\lambda_D{}^2}, \tag{46}$$

where λ_D is the Debye length. In the same limit the relevant components of the tensor $a^{\mu\nu\rho}$ are

$$a^{ij0}(k, k_1, k_2)|_{\omega_2=0} = -\frac{e}{m\lambda_D{}^2}\delta^{ij}. \tag{47}$$

Thus one finds

$$M_{fi}^{(nl)} = -M_{fi}^{(cs)} \tag{48}$$

in this limit. That is Compton scattering and non-linear scattering interfere destructively for an electron at rest in a thermal plasma, as is well known.

GAILITIS and TSYTOVICH (1964) discussed quantum effects in the scattering of electron plasma waves into transverse waves using a formalism related to the above.

Photon splitting

The simplest example of the use of the generalized diagrams introduced above is in describing the coalescence of two photons into one or the decay of one photon into two. The diagram for this process is Fig. 6, and the amplitude is

$$M_{fi} = (4\pi)^{3/2} e_\mu e_\nu'^{*} e_\rho''^{*} \alpha^{\mu\nu\rho}(k, k', k''), \tag{49}$$

where $a^{\mu\nu\rho}$ has been replaced by $\alpha^{\mu\nu\rho}$ using (16). The probability per unit time for the coalescence process for initial photons in modes σ', σ'' in the ranges $d^3\mathbf{k}'/(2\pi)^3$, $d^3\mathbf{k}''/(2\pi)^3$ is given by (in the rest frame of the plasma in the gauge $\phi = 0$)

$$w_\sigma^{\sigma'\sigma''}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') = \frac{8(2\pi)^7}{\omega^\sigma \omega^{\sigma'} \omega^{\sigma''}} \left(\frac{W_E}{W_T}\right)^\sigma \left(\frac{W_E}{W_T}\right)^{\sigma'} \left(\frac{W_E}{W_T}\right)^{\sigma''} \times |e_i^\sigma e_j^{\sigma'*} e_l^{\sigma''*} \alpha_{ijl}(k, k', k'')|^2 \, \delta^4(k - k' - k''), \tag{50}$$

where 3-vector notation is used. This result is formally identical to the familiar classical result, e.g. TSYTOVICH (1970).

In principle the above method could be used to treat photon splitting in a strong magnetic field. Existing treatments of the problem, e.g. by BIALYNICKA-BIRULA and BIALYNICKA-BIRULA (1970), who pointed out errors in earlier results, and by ADLER (1971) involve a perturbation expansion in the effects of the ambient magnetic field. The present method treats the effects of the ambient magnetic field exactly. As explained by BIALYNICKA-BIRULA and BIALYNICKA-BIRULA (1970), it is necessary to include the effects of the ambient magnetic field on the properties of the photons when treating the photon-splitting problem. Thus to use (50) one would require not only knowledge of α_{ijl} for a vacuum for $B \neq 0$, but also the characteristic properties of the waves (i.e. $\omega^\sigma(\mathbf{k})$, $\mathbf{e}^\sigma(\mathbf{k})$ and $[W_E/W_T]^\sigma(\mathbf{k})$) found by solving the wave equation with the vacuum polarization tensor for $B \neq 0$ included.

An advantage of the present method when applied to this photon-splitting problem would be that the method allows for the inclusion of co-operative effects associated with an ambient electron gas. An ambient medium may well play a significant role in the suggested application in connection with neutron stars.

The case $\mathbf{B} \neq 0$

To treat processes in the presence of an ambient magnetic field requires modification to the formalism described in Section 3. It is not possible to use momentum space directly. However, after writing down the amplitude in coordinate space and constructing $|M_{fi}|^2$ the vertex functions $-i\gamma^\mu$ appear in combinations of the form

$$\Gamma^\mu_{qq'}(\mathbf{r}) = \overline{\psi}_q(\mathbf{r})\gamma^\mu\psi_{q'}(\mathbf{r}),$$

whose Fourier transform is given by equation (B.3) of Appendix B. Using this fact it is possible to write down the emission probability directly by analogy with (35).

Thus the probability per unit time of emission of a photon in the mode σ in the range $d^3\mathbf{k}/(2\pi)^3$ by an electron with initial (final) quantum numbers $q(q')$ is given by

$$w^{\sigma}_{qq'}(\mathbf{k}) = 8\pi^2 e^2 \left(\frac{W_E}{W_T}\right)^{\sigma} |\mathbf{e}^{\sigma *} \cdot \mathbf{\Gamma}_{qq'}(\mathbf{k})|^2 \, \delta(\varepsilon_q - \varepsilon_{q'} - \omega^{\sigma}), \tag{51}$$

with $\mathbf{\Gamma}_{qq'}(\mathbf{k})$ is given by equation (B.3) of Appendix B. In this case the factor $4\varepsilon_q\varepsilon_{q'}$, whose counterpart appears explicitly in the denominator in (35) and (36), is incorporated into the definition of the functions $\mathbf{\Gamma}$ in (51).

The probability of absorption is obtained from (51) by replacing $\mathbf{k}$ by $-\mathbf{k}$ using (38), and interchanging q and q'. The requirement from detailed balancing that the probability of absorption from q' to q be equal to the probability of emission from q to q' implies that

$$\mathbf{\Gamma}_{qq'}(-\mathbf{k}) = \mathbf{\Gamma}^{*}_{q'q}(\mathbf{k}) \tag{52}$$

be satisfied to within a phase factor. The required identity follows from the results of Appendix B.

The probability (51) was derived in a slightly different form and in a different way by TSYTOVICH (1962). The probability for emission *in vacuo* is well known in the context of the quantum theory of synchrotron radiation, see e.g. the book by SOKOLOV and TERNOV (1968) and references cited therein.

5. CONCLUSION AND DISCUSSION

The general conclusion of this paper is that it is possible to formulate the theory of electrodynamics in such a way that the theory reduces to that for a classical collisionless plasma in the classical limit and to the conventional theory of quantum electrodynamics *in vacuo*.

Although the generalized theory is of formal interest, possible practical applications are restricted. This is because the regime where relativistic quantum effects are significant is typically far removed from the regime where the co-operative effects characteristic of a plasma predominate. The relatively extreme conditions required for intrinsically quantum plasma effects to be important may well obtain in the atmospheres of neutron stars or, perhaps, of white dwarf stars, see e.g. the review by CANUTO and CHIU (1971).

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APPENDIX A

CALCULATION OF THE TENSORS FOR $\mathbf{B} \neq 0$

The solution of Dirac's equation for an electron in a static magnetic field leads to energy eigenvalues

$$p^0 = \epsilon\varepsilon(p_z, n, s), \tag{A.1}$$

$$\varepsilon(p_z, n, s) = [m^2 + p_z^2 + (2n + 1 + s)eB]^{1/2}, \tag{A.2}$$

with $\epsilon = \pm 1$, $s = \pm 1$ and $n = 0, 1, 2, \ldots$. With the choice $A = (0, Bx, 0)$ of gauge, and in the Schrödinger representation and the standard representation of the Dirac matrices, the wavefunctions $\psi^\epsilon_{n,s}(x, p_y, p_z)$, which are normalized according to

$$eB\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}p_y}{2\pi}\frac{\mathrm{d}p_z}{2\pi}\,\bar{\psi}^\epsilon_{n,s}(x, p_y, p_z)\gamma^0\psi^\epsilon_{n,s}(x, p_y, p_z) = 1, \tag{A.3}$$

can be written in the form (e.g. JOHNSON and LIPPMANN, 1949)

$$\psi^\epsilon_{n,s}(x, p_y, p_z) = \frac{A_n(x, p_y, p_z, p^0)\phi_s^\epsilon}{[\sqrt{eB}\,2p^0(p^0 + m)]^{1/2}} \tag{A.4}$$

with p^0 given by (A.1) and with

$$A_n(x, p_y, p_z, p^0) = \begin{bmatrix} (p^0 + m)v_n & 0 & -p_z v_n & ip_n v_{n-1} \\ 0 & (p^0 + m)v_n & -ip_{n+1}v_{n+1} & p_z v_n \\ p_z v_n & -ip_n v_{n-1} & (-p^0 + m)v_n & 0 \\ ip_{n+1}v_{n+1} & -p_z v_n & 0 & (-p^0 + m)v_n \end{bmatrix}, \tag{A.5}$$

$$\phi_1^{1} = \phi_1^{-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \phi_{-1}^{1} = \phi_{-1}^{-1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{A.6}$$

Here the following notation is used:

$$v_n := v_n(\xi) := \frac{e^{-\xi^2/2}H_n(\xi)}{[\sqrt{\pi}\,2^n n!]^{1/2}}, \quad \xi = \sqrt{eB}\left(x + \frac{p_y}{eB}\right). \tag{A.7}$$

and

$$p_n := [2n\,eB]^{1/2}, \tag{A.8}$$

where H_n is a Hermite polynomial.

The propagator in (11) and (12) was given by SVETOZAROVA and TSYTOVICH (1962):

$$\bar{G}(\mathbf{r}, \mathbf{r}', E) = \sum_{q>0} \psi_q(\mathbf{r})\bar{\psi}_q(\mathbf{r}')\left\{\frac{1 - n_q^+}{E - \varepsilon_q + i0} + \frac{n_q^+}{E - \varepsilon_q - i0}\right\}$$
$$- \sum_{q<0} \psi_q(\mathbf{r})\bar{\psi}_q(\mathbf{r}')\left\{\frac{1 - n_q^-}{-E - \varepsilon_q + i0} + \frac{n_q^-}{-E - \varepsilon_q - i0}\right\}, \tag{A.9}$$

where q denotes the set of quantum numbers ϵ, s, n, p_z with $q > 0$ ($q < 0$) corresponding to $\epsilon = 1$ ($\epsilon = -1$) and where n_q^+ and n_q^- are the occupation numbers for electrons and positrons respectively. The propagator (A.9) is the Feynman one. It is possible but tedious to reduce the form (A.9) for $n_q\pm = 0$, after inverting the Fourier transform w.r.t. time, to forms given by, e.g. SCHWINGER (1951); KÄLLEN (1958); KAITNA and URBAN (1964).

The traces can be reduced as follows:

$$Sp[\gamma^{\mu_1}\bar{G}(\mathbf{r}_1, \mathbf{r}_2, E_1)\gamma^{\mu_2}\bar{G}(\mathbf{r}_2, \mathbf{r}_3, E_2) \ldots \gamma^{\mu_N}\bar{G}(\mathbf{r}_N, \mathbf{r}_1, E_N)]$$

$$= \sum_{q_1,\ldots,q_N} Sp[\Gamma^{\mu_1}_{q_Nq_1}(\mathbf{r}_1)M(s_1)\,\Gamma^{\mu_2}_{q_1q_2}(\mathbf{r}_2)M(s_2)\ldots\Gamma^{\mu_N}_{q_{N-1}q_N}(\mathbf{r}_N)M(s_N)], \tag{A.10}$$

with

$$\Gamma^{\mu}_{qq'}(\mathbf{r}) = \bar{\psi}_q(\mathbf{r})\gamma^{\mu}\psi_q(\mathbf{r}), \tag{A.11}$$

and with $M(s) = \phi_s^{\epsilon}\phi_s^{-\epsilon}$ a diagonal 4×4 matrix which is (1, 0, 0, 0) for $s = 1$ and (0, 1, 0, 0) for $s = -1$.

The final step in evaluating the tensors is Fourier transforming in space; this reduces to evaluating

$$\Gamma^{\mu}_{qq'}(\mathbf{k}) = \int \mathrm{d}^3\mathbf{r}\, \mathrm{e}^{i\mathbf{k}.\mathbf{r}}\, \Gamma^{\mu}_{qq'}(\mathbf{r}). \tag{A.12}$$

The following integral is required

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{ixy}v_n(x)v_m(x) = i^{n-m}\, J^m_{n-m}(\tfrac{1}{2}y^2) \;= i^{m-n}J^n_{m-n}(\tfrac{1}{2}y^2), \tag{A.13}$$

with

$$J^n_{\nu}(x) : = \left(\frac{n!}{(n+\nu)!}\right)^{1/2} \mathrm{e}^{-1/2x}x^{1/2\nu}\, L^{\nu}_n(x) = (-)^{\nu}\, J^{n+\nu}_{-\nu}(x), \tag{A.14}$$

where L^{ν}_n is a generalized Laguerre polynomial, e.g. in the notation of GRADSTEYN and RYZHIK (1965).

APPENDIX B

Explicit evaluation of the tensor $\alpha^{\mu\nu}$ using the method of Appendix A was carried out by SVETOZAROVA and TSYTOVICH (1962). The result for the 3-tensor $\alpha_{ij}(\mathbf{k}, \omega)$ whose components are minus the $\mu = i$, $\mu = j$ components of $\alpha^{\mu\nu}$, and which is related to the dielectric tensor by

$$\varepsilon_{ij}(\mathbf{k}, \omega) = \delta_{ij} + \frac{4\pi c}{\omega^2}\, \alpha_{ij}(\mathbf{k}, \omega),$$

reduces to

$$\alpha_{ij}(\mathbf{k}, \omega) = \frac{e^3B}{2\pi} \sum_{n,\nu,s,s'} \int \frac{\mathrm{d}p_z}{2\pi}\Bigg(\frac{n_q^+ - n_{q'}^+}{\omega - \varepsilon_q + \varepsilon_{q'}}\, (\mathbf{\Gamma}^{++}_{qq'})_i(\mathbf{\Gamma}^{++}_{qq'})_j^*$$

$$- \frac{n_q^- - n_{q'}^-}{\omega + \varepsilon_q - \varepsilon_{q'}}\, (\mathbf{\Gamma}^{--}_{qq'})_i(\mathbf{\Gamma}^{--}_{qq'})_j^* + \frac{n_q^+ + n_{q'}^- - 1}{\omega - \varepsilon_q - \varepsilon_{q'}}\, (\mathbf{\Gamma}^{+-}_{qq'})_i(\mathbf{\Gamma}^{+-}_{qq'})_j^*$$

$$- \frac{n_q^- + n_{q'}^+ - 1}{\omega + \varepsilon_q + \varepsilon_{q'}}\, (\mathbf{\Gamma}^{-+}_{qq'})_i(\mathbf{\Gamma}^{-+}_{qq'})_j^*\Bigg), \tag{B.1}$$

where * denotes complex conjugation, and with $n' = n + \nu$ and $p_{z'} = p_z - k_z$. For coordinate axes corresponding to

$$\mathbf{k} = (k_{\perp}, 0, k_z), \qquad \mathbf{B} = (0, 0, B), \tag{B.2}$$

explicit evaluation of (A.12) gives (apart from an irrelevant phase factor)

$$\mathbf{\Gamma}^{\epsilon\epsilon'}_{qq'} = \frac{1}{\sqrt{4\epsilon\epsilon' r\varepsilon_q\varepsilon_{q'}}}\Bigg(\frac{1+ss'}{2}\,\{rp_{\perp}'\,J^n_{\nu+s} + p_{\perp}J^{n+s}_{\nu-s},\; is(rp_{\perp}'J^n_{\nu+s} - p_{\perp}J^{n+s}_{\nu-s}),\; (rp_z' + p_z)J^n_{\nu}\}$$

$$+ \frac{1-ss'}{2}\; \{-s(rp_z' - p_z)J^n_{\nu},\; -i(rp_z' - p_z)J^n_{\nu},\; s(rp_{\perp}'J^n_{\nu-s} - p_{\perp}J^{n+s}_{\nu-s}\Bigg), \tag{B.3}$$

with

$$r := \frac{\epsilon\varepsilon_q + m}{\epsilon'\varepsilon_{q'} + m}, \quad p_\perp := [eB(2n + 1 + s)]^{1/2},$$
$$p_\perp' := [eB(2n + 2\nu + 1 + s')]^{1/2} \tag{B.4}$$

and where the argument of the functions J_ν^n, etc. is $k_\perp^2/2eB$.

The anti-hermitian part may be obtained from the hermitian part (B.1) by replacing ω by $\omega + i0$ and retaining only the resulting semi-residues.

Long-wavelength limit

In the long-wavelength limit, i.e. $k_\perp \to 0$ and $k_z \to 0$, one has $p_z' = p_z$ and $J_\nu^n = \delta_{\nu 0}$; (B.1) reduces to

$$\alpha_{11}(\mathbf{k}, \omega) = \alpha_{22}(\mathbf{k}, \omega) = \sum_{n=0}^{\infty} \frac{e^3B}{2\pi} \int \frac{dp_z}{2\pi} \left\{ (f_{n+1}^s - f_n^s) \frac{\varepsilon_{n+1} - \varepsilon_n}{\omega^2 - (\varepsilon_{n+1} - \varepsilon_n)^2} \right.$$
$$\left. \times \left(1 - \frac{m^2 + p_z^2}{\varepsilon_n\varepsilon_{n+1}}\right) + (f_{n+1}^s + f_n^s) \frac{\varepsilon_{n+1} + \varepsilon_n}{\omega^2 - (\varepsilon_{n+1} + \varepsilon_n)^2} \left(1 + \frac{m^2 + p_z^2}{\varepsilon_n\varepsilon_{n+1}}\right) \right\},$$

$$\alpha_{12}(\mathbf{k}, \omega) = -\alpha_{21}(\mathbf{k}, \omega) = \sum_{n=0}^{\infty} -\frac{ie^3B}{2\pi} \int \frac{dp_z}{2\pi} \left\{ (f_{n+1}^D - f_n^D) \frac{\omega}{\omega^2 - (\varepsilon_{n+1} - \varepsilon_n)^2} \right.$$
$$\left. \times \left(1 - \frac{m^2 + p_z^2}{\varepsilon_n\varepsilon_{n+1}}\right) + (f_{n+1}^D - f_n^D) \frac{\omega}{\omega^2 - (\varepsilon_{n+1} + \varepsilon_n)^2} \left(1 + \frac{m^2 + p_z^2}{\varepsilon_n\varepsilon_{n+1}}\right) \right\},$$

$$\alpha_{33}(\mathbf{k}, \omega) = \sum_{n=0}^{\infty} \frac{4e^3B}{2\pi} \int \frac{dp_z}{2\pi} \left\{ f_{n+1}^s \frac{\varepsilon_{n+1}}{\omega^2 - 4\varepsilon_{n+1}^2} \left(1 - \frac{p_z^2}{\varepsilon_{n+1}^2}\right) \right.$$
$$\left. + f_n^s \frac{\varepsilon_n}{\omega^2 - 4\varepsilon_n^2} \left(1 - \frac{p_z^2}{\varepsilon_n^2}\right) \right\}, \tag{B.5}$$

$$\alpha_{13}(\mathbf{k}, \omega) = \alpha_{31}(\mathbf{k}, \omega) = \alpha_{23}(\mathbf{k}, \omega) = \alpha_{32}(\mathbf{k}, \omega) = 0,$$

where it is assumed that $n_q\pm$ are functions only of

$$\varepsilon_n := [m^2 + p_z^2 + 2neB]^{1/2}, \tag{B.6}$$

with

$$f_n^s := n^+(\varepsilon_n) + n^-(\varepsilon_n), \qquad f_n^D := n^+(\varepsilon_n) - n^-(\varepsilon_n). \tag{B.7}$$

Svetozarova and Tsytovich's result differs from (B.5) in the 12 and 21 components. (The 12, 21, 23 and 32 components must vanish for an electron gas with the quantum numbers of the vacuum, i.e. for $n_q^+ = n_q^-$; their incorrect result does not do so.)

Classical limit

The classical limit is achieved by expanding in ω/ε, k_z/p_z and ν/n and retaining the lowest order non-trivial terms after setting $n = \infty$. The relevant expansion for large n is given by

$$J_\nu^n(z^2/2n) = \left(\frac{(n+\nu)!}{n!n^\nu}\right)^{1/2} \sum_{s=0}^{\infty} c_s(z/2n)^s J_{\nu+s}(z),$$
$$c_0 = 1, \qquad c_1 = -\tfrac{1}{2}(\nu + 1), \qquad c_2 = \tfrac{1}{8}(\nu + 1)(\nu + 2), \tag{B.8}$$
$$(s + 1)c_{s+1} = -\tfrac{1}{2}(\nu + 1)c_s + \tfrac{1}{4}(\nu + s)c_{s-1} - \tfrac{1}{2}nc_{s-2},$$

where $J_\nu(z)$ is a Bessel function. To lowest order, after replacing ν by $-\nu$, the following classical expression is rederived

$$\alpha_{ij}(\mathbf{k}, \omega) = \sum_{\nu=-\infty}^{\infty} e^2 \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \int_0^{\infty} \frac{dp_\perp p_\perp}{(2\pi)^2} \left(\frac{V_i(\nu, \mathbf{k})\, V_j^*(\nu, \mathbf{k})}{\omega - \nu\Omega - k_z v_z} \left\{ \frac{\omega - k_z v_z}{v_\perp} \frac{\partial}{\partial p_\perp} \right. \right.$$
$$\left. \left. + k_z \frac{\partial}{\partial p_z} \right\} f(\mathbf{p}) - \frac{p_z}{p_\perp} \left\{ v_z \frac{\partial}{\partial p_\perp} - v_\perp \frac{\partial}{\partial p_z} \right\} f(\mathbf{p}) \right), \tag{B.9}$$

with

$$V_i(\nu, \mathbf{k}) = \left(v_\perp \frac{\nu}{z} J_\nu(z), iv_\perp J_\nu'(z), v_z J_\nu(z)\right),$$
$$z = \frac{k_\perp v_\perp}{\Omega}, \quad \Omega = \frac{eB}{\varepsilon}, \quad v_\perp = \frac{p_\perp}{\varepsilon}, \quad v_z = \frac{p_z}{\varepsilon}, \tag{B.10}$$

and where n_q^- has been set equal to zero and n_q^+ to $2f(\mathbf{p})$.

APPENDIX C

THE TENSOR α_{ijl} FOR $\mathbf{B} = 0$

The components of the 3-tensor $\alpha_{ijl}(\mathbf{k}, \omega; \mathbf{k}_1, \omega_1; \mathbf{k}_2, \omega_2)$ are given by (plus) the $\mu = i$, $\nu = j$, $\rho = l$ components of (4). The following results were derived by Mr. P. J. Blamey (unpublished).

Writing

$$(f^{\mu\nu\rho})^{\epsilon_1\epsilon_2\epsilon_3}(p_1, p_2, p_3) := \frac{Sp[\gamma^\mu(\not{p}_1 + m)\gamma^\nu(\not{p}_2 + m)\gamma^\rho(\not{p}_3 + m)]}{8\epsilon_1\epsilon_2\epsilon_3\varepsilon_1\varepsilon_1\varepsilon_1}$$
$$\epsilon_i := \frac{p_i^0}{|p_i^0|}, \qquad \varepsilon_i = |p_i^0| = [m^2 + |\mathbf{p}_i|^2]^{1/2}, \tag{C.1}$$

the 3-tensor components reduce to

$$\begin{aligned} f_{ijl}^{\epsilon_1\epsilon_2\epsilon_3}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \frac{1}{2\epsilon_1\epsilon_2\epsilon_3\varepsilon_1\varepsilon_2\varepsilon_3} [&p_{1i}p_{2j}p_{3l} + p_{3i}p_{1j}p_{2l} \\ &- p_{2i}p_{3j}p_{1l} + p_{2i}p_{1j}p_{3l} + p_{3i}p_{2j}p_{1l} + p_{1i}p_{3j}p_{2l} \\ &+ (m^2 + \mathbf{p}_1\cdot\mathbf{p}_2 - \epsilon_1\epsilon_2\varepsilon_1\varepsilon_2)(-p_{1i}\delta_{jl} - p_{1j}\delta_{il} + p_{1l}\delta_{ij}) \\ &+ (m^2 + \mathbf{p}_1\cdot\mathbf{p}_2 - \epsilon_1\epsilon_2\varepsilon_1\varepsilon_2)(-p_{3i}\delta_{jl} + p_{3j}\delta_{il} - p_{3l}\delta_{ij}) \\ &+ (m^2 + \mathbf{p}_1\cdot\mathbf{p}_3 - \epsilon_1\epsilon_3\varepsilon_1\varepsilon_3)(p_{2i}\delta_{jl} - p_{2j}\delta_{il} - p_{2l}\delta_{ij})] \\ &= -f_{ijl}^{-\epsilon_1-\epsilon_2-\epsilon_3}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = f_{lij}^{\epsilon_3\epsilon_1\epsilon_2}(\mathbf{p}_3, \mathbf{p}_1, \mathbf{p}_2) \\ &= -f_{ijl}^{\epsilon_1\epsilon_2\epsilon_3}(-\mathbf{p}_1, -\mathbf{p}_2, -\mathbf{p}_3). \end{aligned} \tag{C.2}$$

It is convenient to write

$$\begin{aligned} a_{ijl}^{(1)}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2) &= f_{ijl}^{+++}(\mathbf{p}, \mathbf{p} - \mathbf{k}_1, \mathbf{p} - \mathbf{k}_2), \\ a_{ijl}^{(2)}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2) &= f_{ijl}^{++-}(\mathbf{p}, \mathbf{p} - \mathbf{k}_1, \mathbf{p} - \mathbf{k}_2), \\ a_{ijl}^{(3)}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2) &= f_{ijl}^{+-+}(\mathbf{p}, \mathbf{p} - \mathbf{k}_1, \mathbf{p} - \mathbf{k}_2), \\ a_{ijl}^{(4)}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2) &= f_{ijl}^{-++}(\mathbf{p}, \mathbf{p} - \mathbf{k}_1, \mathbf{p} - \mathbf{k}_2). \end{aligned} \tag{C.3}$$

Explicit evaluation of (5) gives

$$[\alpha_1^{\mu\nu\rho}(k, k_1, k_2)]_{\mu=i,\nu=j,\rho=l} = e^3 \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{4} a_{ijl}^{(r)}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}) \times \{A_+^{(r)}(\mathbf{p}, k_1, k) + A_-^{(r)}(\mathbf{p}, k_1, k)\} \tag{C.4}$$

with

$$\begin{aligned} A_\pm^{(1)}(\mathbf{p}, k_1, k) = &-\frac{n^\pm(\mathbf{p})}{(\omega \mp \varepsilon \pm \varepsilon_0)(\omega_1 \mp \varepsilon \pm \varepsilon_1)} + \frac{n^\pm(\mathbf{p} - \mathbf{k}_1)}{(\omega_1 \mp \varepsilon \pm \varepsilon_1)(\omega - \omega_1 \mp \varepsilon_1 \pm \varepsilon_0)} \\ &- \frac{n^\pm(\mathbf{p} - \mathbf{k})}{(\omega \mp \varepsilon \pm \varepsilon_1)(\omega - \omega_1 \mp \varepsilon_1 \pm \varepsilon_0)}; \end{aligned}$$

$$\begin{aligned} A_\pm^{(2)}(\mathbf{p}, k_1, k) = &-\frac{n^\pm(\mathbf{p})}{(\omega_1 \mp \varepsilon \pm \varepsilon_1)(\omega \mp \varepsilon \mp \varepsilon_0)} + \frac{n^\pm(\mathbf{p} - \mathbf{k}_1)}{(\omega_1 \mp \varepsilon \pm \varepsilon_1)(\omega - \omega_1 \mp \varepsilon_1 \mp \varepsilon_0)} \\ &- \frac{1 - n^\mp(\mathbf{p} - \mathbf{k}))}{(\omega \mp \varepsilon \mp \varepsilon_0)(\omega - \omega_1 \mp \varepsilon_1 \mp \varepsilon_0)} \end{aligned}$$

$$A^{(3)}_{\pm}(\mathbf{p}, k_1, k) = -\frac{n^{\pm}(\mathbf{p})}{(\omega \mp \varepsilon \pm \varepsilon_0)(\omega_1 \mp \varepsilon \mp \varepsilon_1)} + \frac{1 - n^{\mp}(\mathbf{p} - \mathbf{k}_1)}{(\omega_1 \mp \varepsilon \mp \varepsilon_1)(\omega - \omega_1 \pm \varepsilon_1 \pm \varepsilon_0)}$$
$$- \frac{n^{\pm}(\mathbf{p} - \mathbf{k})}{(\omega \mp \varepsilon \pm \varepsilon_1)(\omega - \omega_1 \pm \varepsilon_1 \pm \varepsilon_1)},$$

$$A^{(4)}_{\pm}(\mathbf{p}, k_1, k) = \frac{1 - n^{\mp}(\mathbf{p})}{(\omega \pm \varepsilon \pm \varepsilon_0)(\omega_1 \pm \varepsilon \pm \varepsilon_1)} - \frac{n^{\pm}(\mathbf{p} - \mathbf{k}_1)}{(\omega_1 \pm \varepsilon \pm \varepsilon_1)(\omega - \omega_1 \pm \varepsilon_0 \mp \varepsilon_1)}$$
$$+ \frac{n^{\pm}(\mathbf{p} - \mathbf{k})}{(\omega \pm \varepsilon \pm \varepsilon_0)(\omega - \omega_1 \pm \varepsilon_0 \mp \varepsilon_1)}, \tag{C.5}$$

and with

$$\varepsilon := \varepsilon(\mathbf{p}), \qquad \varepsilon_0 := \varepsilon(\mathbf{p} - \mathbf{k}), \qquad \varepsilon_1 := \varepsilon(\mathbf{p} - \mathbf{k}_1).$$

The other term $\alpha_2^{\mu\nu\rho}$ may be obtained by making the interchanges indicated in (6). However, because the average of these two terms, see (4), must vanish for a vacuum (i.e. for $n^+ = n^- = 0$) by FURRY's (1937) theorem, it is convenient to write $\alpha_2^{\mu\nu\rho}$ in such a form that Furry's theorem is trivially satisfied. Such a form is

$$[\alpha_2^{\mu\nu\rho}(k, k_1, k_2)]_{\mu=i,\nu=j,\rho=l} = e^3 \int \frac{d^3p}{(2\pi)^3} \sum_{r=1}^{4} a^{(r)}_{ijl}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2)$$
$$\times \{-A^{(r)}_{+}(-\mathbf{p}, -k_1, -k) - A^{(r)}_{-}(-\mathbf{p}, -k_1, -k)\}. \tag{C.6}$$

The average (4) then gives

$$\alpha_{ijl}(k, k_1, k_2) = \frac{e^3}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{r=1}^{4} a^{(r)}_{ijl}(\mathbf{p}, \mathbf{k}_1, \mathbf{k})\ \{\bar{A}^{(r)}_{+}(\mathbf{p}, k_1, k) + \bar{A}^{(r)}_{-}(\mathbf{p}, k_1, k)\}, \tag{C.7}$$

with

$$\bar{A}^{(1)}_{\pm}(\mathbf{p}, k_1, k) = -\frac{(n^{\pm}(\mathbf{p}) - n^{\mp}(\mathbf{p}))}{(\omega \mp \varepsilon \pm \varepsilon_0)(\omega_1 \mp \varepsilon \pm \varepsilon_1)} + \frac{(n^{\pm}(\mathbf{p} - \mathbf{k}_1) - n^{\mp}(-\mathbf{p} + \mathbf{k}_1))}{(\omega_1 \mp \varepsilon \pm \varepsilon_1)(\omega - \omega_1 \mp \varepsilon_1 \pm \varepsilon_0)}$$
$$- \frac{(n^{\pm}(\mathbf{p} - \mathbf{k}) - n^{\mp}(-\mathbf{p} + \mathbf{k}))}{(\omega \mp \varepsilon \pm \varepsilon_0)(\omega - \omega_1 \mp \varepsilon_1 \pm \varepsilon_0)},$$

$$\bar{A}^{(2)}_{\pm}(\mathbf{p}, k_1, k) = -\frac{(n^{\pm}(\mathbf{p}) - n^{\mp}(-\mathbf{p}))}{(\omega_1 \mp \varepsilon \pm \varepsilon_1)(\omega \mp \varepsilon \mp \varepsilon_0)} + \frac{(n^{\pm}(\mathbf{p} - \mathbf{k}_1) - n^{\mp}(-\mathbf{p} + \mathbf{k}_1))}{(\omega_1 \mp \varepsilon \pm \varepsilon_1)(\omega - \omega_1 \mp \varepsilon_1 \mp \varepsilon_0)}$$
$$+ \frac{(n^{\mp}(\mathbf{p} - \mathbf{k}) - n^{\pm}(-\mathbf{p} + \mathbf{k}))}{(\omega \mp \varepsilon \mp \varepsilon_1)(\omega - \omega_1 \mp \varepsilon_1 \mp \varepsilon_0)},$$

$$\bar{A}^{(3)}_{\pm}(\mathbf{p}, k_1, k) = -\frac{(n^{\pm}(\mathbf{p}) - n^{\mp}(-\mathbf{p}))}{(\omega \mp \varepsilon \pm \varepsilon_0)(\omega_1 \mp \varepsilon \mp \varepsilon_1)} - \frac{(n^{\mp}(\mathbf{p} - \mathbf{k}_1) - n^{\pm}(-\mathbf{p} + \mathbf{k}_1))}{(\omega_1 \mp \varepsilon \mp \varepsilon_1)(\omega - \omega_1 \pm \varepsilon_1 + \varepsilon_0)}$$
$$- \frac{(n^{\pm}(\mathbf{p} - \mathbf{k}) - n^{\mp}(-\mathbf{p} + \mathbf{k}))}{(\omega \mp \varepsilon \pm \varepsilon_1)(\omega - \omega_1 \pm \varepsilon_1 \pm \varepsilon_0)},$$

$$\bar{A}^{(4)}_{\pm}(\mathbf{p}, k_1, k) = -\frac{(n^{\mp}(\mathbf{p}) - n^{\pm}(-\mathbf{p}))}{(\omega \pm \varepsilon \pm \varepsilon_0)(\omega_1 \pm \varepsilon \pm \varepsilon_1)} - \frac{n^{\pm}(\mathbf{p} - \mathbf{k}_1) - n^{\mp}(-\mathbf{p} + \mathbf{k}_1))}{(\omega_1 \pm \varepsilon \pm \varepsilon_1)(\omega - \omega_1 \pm \varepsilon_0 \mp \varepsilon_1)}$$
$$+ \frac{n^{\pm}(\mathbf{p} - \mathbf{k}) - n^{\mp}(-\mathbf{p} + \mathbf{k}))}{(\omega \pm \varepsilon \pm \varepsilon_0)(\omega - \omega_1 \pm \varepsilon_0 \mp \varepsilon_1)}. \tag{C.8}$$

Not only does α_{ijl} vanish for a vacuum, but also for an electron gas whose quantum numbers are those of the vacuum i.e. for $n^+(\mathbf{p}) = n^-(-\mathbf{p})$. This is an obvious generalization of Furry's theorem.

In the non-relativistic limit with

$$\varepsilon, \varepsilon_0, \varepsilon_1 \gg |\omega|, |\omega_1|, |\omega - \omega_1| \gg |\varepsilon - \varepsilon_0|, |\varepsilon - \varepsilon_1|, |\varepsilon_0 - \varepsilon_1|, \tag{C.9}$$

the expansion (C.7) reduces to

$$\alpha_{ijl}(k, k_1, k_2) = -\frac{1}{2}\frac{n_e e^3}{m^2}\left(\frac{k_i}{\omega}\delta_{jl} + \frac{k_{1j}}{\omega_1}\delta_{il} + \frac{k_{2l}}{\omega_2}\delta_{ij}\right), \tag{C.10}$$

with

$$n_e := 2\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,[n^+(\mathbf{p}) - n^-(-\mathbf{p})], \tag{C.11}$$

$$|\omega|/|\mathbf{k}|,\ |\omega_1|/|\mathbf{k}_1|,\ |\omega_2|/|\mathbf{k}_2| \gg [2\varepsilon_F/m]^{1/2}, \tag{C.12}$$

where ε_F is the Fermi energy.

Blamey also showed that in the classical limit (C.7) reduces to the result found using the Vlasov equations.