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Gyromagnetic Absorption at the Fundamental

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Abstract

Using a conventional procedure to treat gyromagnetic absorption in the centre of the line at the fundamental, it is found that there is a small range of angles ($\sim 5°$ in the solar corona) where the damping is weak. The procedure used by Gershman (1960) is shown to be nonphysical in that the time-irreversible properties of the waves are calculated using the time-reversible response of the plasma, and vice versa.

1. Introduction

In an earlier paper (Melrose 1973) I stated that I believed the treatment by Gershman (1960) of gyromagnetic absorption at the fundamental to be nonphysical. My reasons are given here. The result in question is Gershman's equation (1.23), which refers to absorption in 'the centre of the line' (see Section 3 below) at the fundamental of the gyrofrequency Ω_e. Gershman's result has been quoted in discussions of the escape of electromagnetic waves from the solar corona by Ginzburg (1964, equation (12.39)), Zheleznyakov (1970, equation (26.119)) and Fung and Yip (1966, equation (3.13)). For this application the gyrofrequency must exceed the plasma frequency ω_p; $\Omega_e > \omega_p$ is assumed in the following discussion.

In deriving his result, Gershman (1960) followed a procedure used by Sitenko and Stepanov (1956, equations (34) and (36)), who made an error which was corrected by Gershman; Sitenko and Stepanov explained the method more clearly than did Gershman. Before indicating my objections in detail to Gershman's treatment, it is necessary to summarize the theory of weakly damped waves.

2. Weakly Damped Waves

A separation of the dielectric tensor into hermitian (h) and anti-hermitian (a) parts, i.e.

$$\varepsilon_{ij}(\boldsymbol{k},\omega) = \varepsilon^{\mathrm{h}}_{ij}(\boldsymbol{k},\omega) + \varepsilon^{\mathrm{a}}_{ij}(\boldsymbol{k},\omega)\,, \tag{1}$$

is equivalent to a separation of the linear response into time-reversible and time-irreversible parts respectively. Consequently, in the wave equation, e.g. in the form (Melrose 1968)

$$\Lambda_{ij}(\boldsymbol{k},\omega)\,E_j(\boldsymbol{k},\omega) = -(4\pi\mathrm{i}/\omega)\,j^{\mathrm{ext}}_i(\boldsymbol{k},\omega)\,, \tag{2}$$

with

$$\Lambda_{ij}(\boldsymbol{k},\omega) = (k^2c^2/\omega^2)(\kappa_i\kappa_j - \delta_{ij}) + \varepsilon_{ij}(\boldsymbol{k},\omega)\,, \tag{3}$$

$$\boldsymbol{k} = k\boldsymbol{\kappa}\,, \qquad k = |\boldsymbol{k}|\,,$$

there is both an explicit source term, described by $\boldsymbol{j}^{\mathrm{ext}}$, and an implicit source term associated with the time-irreversible response of the medium. The properties of waves are found by solving

$$\Lambda^{\mathrm{h}}_{ij}(\boldsymbol{k},\omega)\,E_j(\boldsymbol{k},\omega) = -\varepsilon^{\mathrm{a}}_{ij}(\boldsymbol{k},\omega)\,E_j(\boldsymbol{k},\omega)\,, \tag{4}$$

firstly neglecting the right-hand member to find the time-reversible wave properties, and secondly regarding the right-hand member as a source term in calculating the damping.

The time-reversible wave properties are (i) the dispersion relation, $\omega = \omega^{\sigma}(\boldsymbol{k})$ for waves in the mode σ say, which is any particular solution of

$$\Lambda^{\mathrm{R}}(\boldsymbol{k},\omega) = \det[\Lambda^{\mathrm{h}}_{ij}(\boldsymbol{k},\omega)] = 0\,, \tag{5}$$

(ii) the (unimodular) polarization vector $\boldsymbol{e}^{\sigma}(\boldsymbol{k})$, which can be constructed from

$$\lambda^{\mathrm{h}}_{ij}(\boldsymbol{k},\omega^{\sigma}(\boldsymbol{k})) = \lambda^{\mathrm{h}}_{ss}(\boldsymbol{k},\omega^{\sigma}(\boldsymbol{k}))\,e^{\sigma}_i(\boldsymbol{k})\,e^{\sigma *}_j(\boldsymbol{k})\,, \tag{6}$$

where the asterisk denotes complex conjugation and where $\lambda^{\mathrm{h}}_{ij}(\boldsymbol{k},\omega)$ is defined by

$$\Lambda^{\mathrm{h}}_{ij}(\boldsymbol{k},\omega)\,\lambda^{\mathrm{h}}_{jl}(\boldsymbol{k},\omega) = \Lambda^{\mathrm{R}}(\boldsymbol{k},\omega)\,\delta_{il}\,, \tag{7}$$

and (iii) the energetics in the waves, for which it suffices to specify the ratio of electrical to total energy as a function of $\boldsymbol{k}$:

$$[W_{\mathrm{E}}/W_{\mathrm{T}}]^{\sigma}(\boldsymbol{k}) = \left(\frac{\lambda^{\mathrm{h}}_{ss}(\boldsymbol{k},\omega)}{\omega\,\partial\{\Lambda^{\mathrm{R}}(\boldsymbol{k},\omega)\}/\partial\omega}\right)_{\omega=\omega^{\sigma}(\boldsymbol{k})}. \tag{8}$$

In describing the damping, which may be temporal or spatial or any combination thereof, one basic parameter suffices; this may be chosen as the absorption coefficient γ, which is defined such that if the damping were purely temporal the energy in waves would vary as $\exp(-\gamma t)$. This parameter may be calculated by regarding the right-hand member of (4) as a source term and calculating the work done by it using standard techniques:

$$\gamma^{\sigma}(\boldsymbol{k}) = -2\mathrm{i}\omega^{\sigma}(\boldsymbol{k})\,[W_{\mathrm{E}}/W_{\mathrm{T}}]^{\sigma}(\boldsymbol{k})\,e^{\sigma *}_i(\boldsymbol{k})\,e^{\sigma}_j(\boldsymbol{k})\,\varepsilon^{\mathrm{a}}_{ij}(\boldsymbol{k},\omega^{\sigma}(\boldsymbol{k}))\,. \tag{9}$$

Alternatively, one can calculate γ by solving

$$\Lambda(\boldsymbol{k},\omega-\tfrac{1}{2}\mathrm{i}\gamma) = 0\,, \tag{10}$$

with

$$\begin{aligned}\Lambda(\boldsymbol{k},\omega) &= \det[\Lambda^{\mathrm{h}}_{ij}(\boldsymbol{k},\omega)+\varepsilon^{\mathrm{a}}_{ij}(\boldsymbol{k},\omega)]\\ &= \Lambda^{\mathrm{R}}(\boldsymbol{k},\omega)+\lambda^{\mathrm{h}}_{ij}(\boldsymbol{k},\omega)\,\varepsilon^{\mathrm{a}}_{ji}(\boldsymbol{k},\omega)+\Lambda^{\mathrm{h}}_{ij}(\boldsymbol{k},\omega)\,E_{ji}(\boldsymbol{k},\omega)+\det[\varepsilon^{\mathrm{a}}_{ij}(\boldsymbol{k},\omega)]\,,\end{aligned} \tag{11}$$

where E_{ij} is defined by

$$\varepsilon^{\mathrm{a}}_{ij}(\boldsymbol{k},\omega)\,E_{jl}(\boldsymbol{k},\omega) = \delta_{il}\det[\varepsilon^{\mathrm{a}}_{ij}(\boldsymbol{k},\omega)]\,. \tag{12}$$

One is to solve equation (10) by expanding both in γ/ω and in the components of $\varepsilon^{\mathrm{a}}_{ij}$. The zeroth order term is just the form (5), while the first order terms, when $\omega = \omega^{\sigma}(\boldsymbol{k})$ is inserted, reproduce equation (9).

3. Dielectric Tensor for $Y \approx 1$

In discussing the wave properties for $\omega \approx \Omega_e$ it is necessary to have an explicit expression for the dielectric tensor under this condition. In terms of the magnetoionic parameters $X = \omega_p^2/\omega^2$, which is assumed to be less than unity, and $Y = \Omega_e/\omega \approx 1$, the dielectric tensor for a thermal plasma can be approximated by

$$\varepsilon_{ij}^{\mathrm{h}} \approx a_{ij} - \left(\frac{X}{1-Y}\right)\phi(y)\, b_{ij}, \qquad \varepsilon_{ij}^{\mathrm{a}} \approx \mathrm{i}(\tfrac{1}{2}\pi)^{\frac{1}{2}}\left(\frac{X}{n\beta_{\mathrm{th}}\,|\cos\theta|}\right)\exp(-y^2)\, b_{ij}. \tag{13a, b}$$

In equations (13), θ is the angle between $\boldsymbol{k}$ and the background magnetic field $\boldsymbol{B}$; $\phi(y)$ is the function

$$\phi(y) = 2y\exp(-y^2)\int_0^y \exp t^2\,\mathrm{d}t \tag{14}$$

$$= 2y^2 + \dots, \qquad \text{for} \qquad y^2 \ll 1, \tag{15a}$$

$$= 1 + \tfrac{1}{2}y^{-2} + \dots, \qquad y^2 \gg 1; \tag{15b}$$

β_{th} is the ratio of the thermal speed to the velocity of light (that is, $mc^2\beta_{\mathrm{th}}^2$ is the temperature in energy units); and the remaining quantities are given by

$$n = kc/\omega, \qquad y^2 = (1-Y)^2/2n^2\beta_{\mathrm{th}}^2\cos^2\theta, \tag{16}$$

$$a_{ij} = \begin{bmatrix} 1-\dfrac{X}{2(1+Y)} & -\dfrac{\mathrm{i}X}{2(1+Y)} & \dfrac{X\tan\theta}{2Y} \\[2ex] \dfrac{\mathrm{i}X}{2(1+Y)} & 1-\dfrac{X}{2(1+Y)} & \dfrac{\mathrm{i}X\tan\theta}{2Y} \\[2ex] \dfrac{X\tan\theta}{2Y} & -\dfrac{\mathrm{i}X\tan\theta}{2Y} & 1-X \end{bmatrix}, \tag{17}$$

and

$$b_{ij} = \begin{bmatrix} \tfrac{1}{2} & -\tfrac{1}{2}\mathrm{i} & \tfrac{1}{2}\left(\dfrac{1-Y}{Y}\right)\tan\theta \\[2ex] \tfrac{1}{2}\mathrm{i} & \tfrac{1}{2} & \tfrac{1}{2}\mathrm{i}\left(\dfrac{1-Y}{Y}\right)\tan\theta \\[2ex] \tfrac{1}{2}\left(\dfrac{1-Y}{Y}\right)\tan\theta & -\tfrac{1}{2}\mathrm{i}\left(\dfrac{1-Y}{Y}\right)\tan\theta & \tfrac{1}{2}\left(\dfrac{1-Y}{Y}\right)^2\tan^2\theta \end{bmatrix}, \tag{18}$$

where the coordinate axes are such that $\boldsymbol{B}$ is along the 3-axis and $\boldsymbol{k}$ is in the 1–3 plane.

For $y^2 \gg 1$ the dielectric tensor (13a) reduces to that used in magnetoionic theory, while for $y^2 \ll 1$ the second term in (13a) is negligible. The 'centre of the line' may be defined as the region $y^2 \lesssim 1$. (The error made by Sitenko and Stepanov (1956, equation (33)) was in the form of ε_{ij} for $y^2 \ll 1$.) In the case of gyromagnetic absorption at the fundamental, unlike absorption at higher harmonics, the form of $\varepsilon_{ij}^{\mathrm{h}}$ and, consequently, the time-reversible properties of the waves change rapidly as functions of frequency near $y^2 = 1$.

4. Procedure used by Sitenko and Stepanov and by Gershman

The procedure used by Sitenko and Stepanov (1956) and by Gershman (1960) for treating absorption in the centre of the line at the fundamental is based on the observation that term by term $\varepsilon_{ij}^{\mathrm{a}}$, given by equation (13b), is of order β_{th}^{-1} times $\varepsilon_{ij}^{\mathrm{h}}$, given by (13a). Consequently, these authors solved equation (10) by, firstly, solving

$$\lambda_{ij}^{\mathrm{h}}(\boldsymbol{k},\omega)\,\varepsilon_{ij}^{\mathrm{a}}(\boldsymbol{k},\omega) = 0 \tag{19}$$

for the time-reversible wave properties (only the dispersion equation was found) and, secondly, finding γ by equating to zero the sum of the first-order terms in expansions in γ/ω and in the components of Λ_{ij} (compared with those of $\varepsilon_{ij}^{\mathrm{a}}$). They actually solved for the spatial damping decrement but, as noted by Gershman, the spatial and temporal decrements are related.

The above procedure is nonphysical in that the time-reversible properties of the waves would depend in a fundamental way on the time-irreversible response of the medium, while the time-irreversible properties of the waves would be essentially independent of the time-irreversible response of the medium. (There is an additional objection to the procedure. By implication equation (11), with the forms (13) inserted, is to be expanded in powers of β_{th}. However, when this is done consistently the final term $\det[\varepsilon_{ij}^{\mathrm{a}}]$ is of the same order as the second term, which is the term retained in equation (19), and likewise the third term is of the same order as the first term. The retention of only the first two terms in equation (11) is then inconsistent with the procedure adopted.)

5. Region of Weak Damping

It can be argued on general grounds that damping of the ordinary mode at the fundamental for $\Omega_{\mathrm{e}} > \omega_{\mathrm{p}}$ must be zero at $\sin\theta = 0$. (This is not the case with Gershman's (1960) result, but, as noted by Sitenko and Stepanov (1956), the procedure used is invalid for sufficiently small $\sin\theta$.) Thus there must be a range of angles in which the damping is weak and in which the procedure summarized in Section 2 is valid. For $y^2 \ll 1$, this procedure gives the following result for the dispersion relation for the ordinary mode:

$$n^2 = (B-F)/2A\,, \qquad F^2 = B^2-4AC\,, \tag{20a, b}$$

with

$$A = 1-X+\tfrac{7}{4}X\sin^2\theta\,, \tag{21a}$$

$$B = 2(1-X)(1-\tfrac{1}{4}X)-\tfrac{1}{4}X^2\tan^2\theta - X^2\sin^2\theta+\tfrac{7}{4}X\sin^2\theta\,, \tag{21b}$$

$$C = (1-X)(1-\tfrac{1}{2}X)-\tfrac{1}{2}(1-\tfrac{1}{2}X)X^2\tan^2\theta\,. \tag{21c}$$

A remarkable feature of this result is that for

$$\tan^2\theta > 2(1-X)/X^2\,, \tag{22}$$

the waves are evanescent. Except for $X \approx 1$ this occurs only in a small range of angles about $\theta = \frac{1}{2}\pi$. (The approximation breaks down for θ sufficiently close to $\frac{1}{2}\pi$ because $y^2 \ll 1$ becomes impossible; the waves are undamped at $\theta = \frac{1}{2}\pi$.)

In deriving an expression for γ (for $y^2 \ll 1$) from equation (9), it is convenient to use (6) in writing

$$e_i^{\sigma *} e_j^{\sigma} \varepsilon_{ij}^{\mathrm{a}} = \lambda_{ij}^{\mathrm{h}} \varepsilon_{ji}^{\mathrm{a}} / \lambda_{ss}^{\mathrm{h}},$$

and to leave the resulting expression as a function of n^2:

$$\frac{\gamma}{\omega} = \frac{(\frac{1}{2}\pi)^{\frac{1}{2}} X}{n\beta_{\mathrm{th}} |\cos\theta|} \frac{W_{\mathrm{E}}}{W_{\mathrm{T}}} \left(\frac{n^4 \sin^2\theta - n^2\{2(1-X)+\sin^2\theta\} + 2(1-X)(1-\frac{1}{2}X)}{n^4 - n^2(4-\frac{5}{2}X + \frac{7}{4}X\sin^2\theta) + 3 - 3X + \frac{1}{2}X^2(1-\tan^2\theta)} \right). \tag{23}$$

It is understood that n^2 is given by equation (20a). It can be shown that whenever an expansion in powers of θ is justified the lowest order term gives

$$W_{\mathrm{E}}/W_{\mathrm{T}} = \tfrac{1}{2}n, \tag{24}$$

with $n \approx (1-\frac{1}{2}X)^{\frac{1}{2}}$. The result (23) is valid only if the waves are weakly damped in the sense that γ is less than the line width, i.e. only for

$$\gamma \lesssim \omega n \beta_{\mathrm{th}} |\cos\theta| . \tag{25}$$

For small values of θ and moderate values of X, e.g. for $X \sim \frac{1}{2}$, the leading term in equation (23) is of order

$$\gamma/\omega \sim \theta^4/\beta_{\mathrm{th}} .$$

Consequently, it follows that the waves are weakly damped only for $\theta \lesssim \beta_{\mathrm{th}}^{\frac{1}{4}}$. In the solar corona ($\beta_{\mathrm{th}} \sim 10^{-2}$), this window, through which waves in the ordinary mode can pass from the region $\omega_{\mathrm{p}} < \omega < \Omega_{\mathrm{e}}$ to $\omega > \Omega_{\mathrm{e}}$, extends over a range of angles of less than about 5°. Other than in this window, the waves are so strongly damped that the conventional theory of weakly damped waves is inapplicable.

References

Fung, P. C. W., and Yip, W. K. (1966). *Aust. J. Phys.* **19**, 759.

Gershman, B. N. (1960). *Zh. éksp. teor. Fiz.* **38**, 912; English translation in *Soviet Phys. JETP* **11**, 657.

Ginzburg, V. L. (1964). 'The Propagation of Electromagnetic Waves in Plasma' (Pergamon: Oxford).

Melrose, D. B. (1968). *Astrophys. Space Sci.* **2**, 171.

Melrose, D. B. (1973). *Aust. J. Phys.* **26**, 229.

Sitenko, A. G., and Stepanov, K. N. (1956). *Zh. éksp. teor. Fiz.* **31**, 642; English translation (1957) in *Soviet Phys. JETP* **4**, 512.

Zheleznìakov, V. V. (1970). 'Radio Emission of the Sun and Planets' (Pergamon: Oxford).

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