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Vacuum Polarization and Photon Propagation in a Magnetic Field.

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Summary. — The vacuum polarization tensor in the presence of a static homogeneous magnetic field is calculated exactly as a function of both the magnetic field and the wave vector, and is regularized explicitly by using Shabad's diagonalization with respect to tensor indices. The wave properties for the two electromagnetic modes in the birefringent vacuum are calculated exactly using a technique from plasma physics. The strong-field limit is considered explicitly and it is shown that the two modes reduce to forms equivalent to the magnetoacoustic and shear Alfvén modes in a plasma with Alfvén speed much greater than the speed of light.

1. - Introduction.

Interest in the vacuum polarization tensor in the presence of a static homogeneous magnetic field has been connected with interest in the properties of photons in the birefringent vacuum. TOLL (1) used the Kramers-Kronig relations to calculate the refractive indices from the known absorption coefficient for photo-pair production. More recently, approaches have involved the use of the Heisenberg-Euler effective Lagrangian (2,3) and Schwinger's proper-time

(1) J. S. TOLL: Thesis (unpublished), Princeton University (1952).
(2) J. MCKENNA and P. M. PLATZMAN: *Phys. Rev.*, **129**, 2354 (1963); J. J. KLEIN and B. P. NIGAM: *Phys. Rev.*, **135**, B 1279 (1964); E. BREZIN and C. ITZYKSON: *Phys. Rev. D*, **3**, 618 (1970); Z. BIALYNICKA-BIRULA and I. BIALYNICKI-BIRULA: *Phys. Rev. D*, **2**, 2341 (1970).
(3) S. L. ADLER: *Ann. of Phys.*, **67**, 599 (1971).

technique [3,4]. However only approximate expressions have been presented for the vacuum polarization tensor and for the wave properties. Here we present exact expressions for the regularized polarization tensor for a magnetized vacuum and for the wave properties of the two natural electromagnetic modes, and we correct an earlier treatment of the strong-field limit. We also develop a simple method for regularizing the vacuum polarization tensor explicitly.

In sect. 2 we write down the electron propagator in a magnetic field in the Géhénian representation and we use it in sect. 3 to calculate the vacuum polarization tensor. The regularization procedure is developed in sect. 4 and explicit expressions for the exact regularized vacuum polarization tensor are exhibited. The properties of the two natural modes are derived in sect. 5 by means of a method familiar in plasma physics. The strong-field limit is taken in sect. 6 where it is pointed out that Constantinescu's [5] approximate expression is unacceptable, and where it is shown that for very strong fields the wave properties reduce to those of the magnetoacoustic mode and shear Alfvén mode in a very diffuse strongly magnetized plasma (specifically, one with Alfvén speed much greater than the speed of light).

Our notation is that of BERESTESKII et *al.* [6] (but with $-e$ for the electronic charge and Sp for the trace over Dirac matrices) and unrationalized Gaussian units with $\hbar = c = 1$ are used. The symbols $:=$ and $=:$ define the quantities on the left and right respectively, and $A^\mu = (A^0, \mathbf{A})$ relates a 4-vector to its time and space components and $\mathbf{A} = (A_1, A_2, A_3)$ relates the 3-vector to its Cartesian components.

2. – The electron propagator.

A convenient form of the electron propagator for present purposes is that derived by GÉHÉNIAU [7] and also by SCHWINGER [8] and KÄLLEN [9]. With the 3-axis along the static magnetic field $\boldsymbol{B}$ the propagator from $x'^\mu = (t', \mathbf{r}')$ to $x^\mu = (t, \mathbf{r})$ may be written as

$$G(x, x'; \boldsymbol{B}) = \varphi(x, x')\Delta(x - x') \tag{1}$$

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[4] A. MINGUZZI: *Nuovo Cimento*, **4**, 476 (1956); **6**, 501 (1957); **9**, 145 (1958); **19**, 847 (1961).

[5] D. H. CONSTANTINESCU: *Nucl. Phys.*, **36** B, 121 (1972); *Lett. Nuovo Cimento*, **5**, 766 (1972).

[6] V. B. BERESTETSKII, E. M. LIFSHITZ and L. P. PITAEVSKII: *Relativistic Quantum Theory, Part One* (Oxford, 1971).

[7] J. GÉHÉNIAU: *Physica (Utrecht)*, **16**, 822 (1950); J. GÉHÉNIAU and M. DEMEUR: *Physica (Utrecht)*, **17**, 71 (1951).

[8] J. SCHWINGER: *Phys. Rev.*, **82**, 664 (1951).

[9] A. O. G. KÄLLEN: *Handbuch der Physik*, Vol. **5**, Part 1 (Berlin, 1958) (reprinted in translation as *Quantum Electrodynamics* (New York, N.Y., 1972)).

with

$$\Delta(x) = -\left(i(\gamma\cdot\partial) + e\gamma\cdot b(x) + m\right)\int_0^\infty \frac{\mathrm{d}\lambda}{8\pi^2}\,\frac{\mathbf{1} - i\boldsymbol{\Sigma}\,\mathrm{tg}\,(eB/2\lambda)}{(2\lambda/eB)\,\mathrm{tg}\,(eB/2\lambda)}\cdot\exp\left[\frac{ieB(x^2+y^2)}{4\,\mathrm{tg}\,(eB/2\lambda)} + \frac{i\lambda}{2}(z^2 - t^2) - \frac{im^2}{2\lambda}\right]. \tag{2}$$

In (2)

$$b^\mu(x) := (0, \tfrac{1}{2}\boldsymbol{B}\times\boldsymbol{r}) \tag{3}$$

is unrelated to the choice of gauge, $\mathbf{1}$ is the unit matrix, and

$$\boldsymbol{\Sigma} := i\gamma^1\gamma^2 = \mathrm{diag}(+1, -1, +1, -1)\,. \tag{4}$$

The other quantity in (1) is

$$\varphi(x, x') := \exp\left[-ie\int_x^{x'} \mathrm{d}x_\mu A^\mu(x)\right], \tag{5}$$

where the path of integration is a straight line from x to x'. This is the only gauge-dependent term in $\boldsymbol{G}(x, x'; B)$. For the choice of gauge

$$A = (0, Bx, 0) \tag{6}$$

(5) becomes

$$\varphi(x, x') = \exp\left[-\tfrac{1}{2}ieB(x + x')(y - y')\right]. \tag{7}$$

Using the standard representation of the γ-matrices one has

$$i(\gamma\cdot\partial) + e\gamma\cdot b(x) + m = \tag{8}$$

$$= \begin{vmatrix} i\dfrac{\partial}{\partial t} + m & 0 & i\dfrac{\partial}{\partial z} & i\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right) + iR_- \\ 0 & i\dfrac{\partial}{\partial t} + m & i\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) - iR_+ & -i\dfrac{\partial}{\partial z} \\ -i\dfrac{\partial}{\partial z} & -i\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right) - iR_- & -i\dfrac{\partial}{\partial t} + m & 0 \\ -i\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) + iR_+ & i\dfrac{\partial}{\partial z} & 0 & -i\dfrac{\partial}{\partial t} + m \end{vmatrix}$$

with $R_\pm := \frac{1}{2}eB(x \pm iy)$, and hence

$$\text{(9)} \qquad \Delta(x) = -\frac{eB}{16\pi^2}\int_0^\infty \frac{d\lambda}{\lambda} \exp\left[-\frac{im^2}{2\lambda}\right] \cdot \exp\left[\frac{1}{4} ieB(x^2+y^2)\,\text{ctg}\frac{eB}{2\lambda} + \frac{1}{2} i\lambda(z^2-t^2)\right] \boldsymbol{B}(\lambda; x)$$

with

$$\text{(10)} \qquad \boldsymbol{B}(\lambda; x) = \begin{vmatrix} (\lambda t + m)C_- & 0 & -\lambda z C_- & -R_- C_+ C_- \\ 0 & (\lambda t + m)C_+ & -R_+ C_+ C_- & \lambda z C_+ \\ \lambda z C_- & R_- C_+ C_- & (-\lambda t + m)C_- & 0 \\ R_+ C_+ C_- & -\lambda z C_+ & 0 & (-\lambda t + m)C_+ \end{vmatrix}$$

and with $C_\pm := \text{ctg}(eB/2\lambda) \pm i$. (Convergence of the integral is ensured by allowing the path of integration in the λ-plane to make a small positive angle with the real axis, *i.e.* by replacing λ by $(1 + id)\lambda$ with $d > 0$.)

3. – The unregularized vacuum polarization tensor.

The vacuum polarization tensor is given by

$$\text{(11)} \qquad \alpha^{\mu\nu}(x - x') = -ie^2\,\text{Sp}\,[\gamma^\mu \boldsymbol{G}(x, x'; B)\gamma^\nu \boldsymbol{G}(x', x; B)]\,.$$

The advantage of the representation (1) is that then

$$\text{(12)} \qquad \alpha^{\mu\nu}(x - x') = -ie^2\,\text{Sp}\,[\gamma^\mu \Delta(x - x')\gamma^\nu \Delta(x' - x)]$$

is manifestly independent of the choice of gauge for the magnetostatic field (and manifestly dependent only on $x - x'$, rather than on x and x' independently, as required by translational invariance). Choosing $x' = 0$ and inserting (9) with (10) gives

$$\text{(13)} \qquad \alpha^{\mu\nu}(x) = -\frac{ie^4B^2}{(4\pi)^4}\int_0^\infty \frac{d\lambda}{\lambda}\int_0^\infty \frac{d\lambda'}{\lambda'} \exp\left[-\frac{im^2}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda'}\right)\right] \cdot \exp\left[\frac{i}{2}(\lambda+\lambda')(z^2-t^2) + \frac{ieB}{4}(x^2+y^2)\left(\text{ctg}\frac{eB}{2\lambda}+\text{ctg}\frac{eB}{2\lambda'}\right)\right] \cdot \text{Sp}\,[\gamma^\mu \boldsymbol{B}(\lambda; x)\gamma^\nu \boldsymbol{B}(\lambda'; -x)]\,.$$

After changing the variables of integration to

$$\text{(14)} \qquad \alpha := \frac{eB}{2}\left(\frac{1}{\lambda}+\frac{1}{\lambda'}\right) \qquad \text{and} \qquad \beta := \frac{eB}{2}\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right),$$

the range of β-integration becomes $[-a, a]$ and one may omit the terms odd in β and take twice the β-integral over $[0, a]$ of the terms even in β. The final step is to Fourier transform to find

$$\alpha^{\mu\nu}(k) := \int d^4x\, \alpha^{\mu\nu}(x) \exp[ikx]\,. \tag{15}$$

The relevant integrals have been evaluated by BOGOLIUBOV and SHIRKOV [10]. With $k^\mu = (w, \boldsymbol{k})$ and

$$\boldsymbol{k} = (k_\perp, 0, k_\parallel) \tag{16}$$

one obtains

$$\alpha^{\mu\nu}(k) = \frac{e^2 m^2}{(2\pi)^2}\int_0^\infty \frac{d\alpha}{\alpha}\int_0^\alpha d\beta\, D^{\mu\nu}(k;\alpha,\beta)\exp\left[-\frac{i\alpha}{L}+\frac{i(\alpha^2-\beta^2)}{4\alpha L}\frac{k^2}{m^2}\right] \tag{17}$$

with $L := B/B_c$, where $B_c := m^2/e$ is the «critical field», with

$$D^{\mu\nu}(k; a, \beta) = d^{\mu\nu}(k; a, \beta)\exp\left[\frac{i}{2L}\left[\frac{\alpha^2-\beta^2}{2\alpha}-\frac{\cos\beta-\cos a}{\sin a}\right]\frac{k_\perp^2}{m^2}\right], \tag{18}$$

and with

$$\left\{\begin{aligned}
d^{00} &= \operatorname{ctg}\alpha\left[-\frac{(\omega^2+k_\parallel^2)(\alpha^2-\beta^2)}{4m^2\alpha^2}-\frac{iL}{\sin a\cos a}+1\right]-\frac{k_\perp^2(\cos\beta-\cos a)}{2m^2\sin^3 a},\\
d^{11} &= \frac{\cos\beta}{\sin a}\left[-\frac{(\omega^2-k_\parallel^2)(\alpha^2-\beta^2)}{4m^2\alpha^2}+\frac{iL}{a}-1\right]-\frac{k_\perp^2(\cos\beta-\cos a)}{2m^2\sin^3 a},\\
d^{22} &= \frac{\cos\beta}{\sin a}\left[-\frac{(\omega^2-k_\parallel^2)(\alpha^2-\beta^2)}{4m^2a^2}+\frac{iL}{\alpha}-1\right]+\frac{k_\perp^2(\cos\beta-\cos a)}{2m^2\sin^3 a},\\
d^{33} &= \operatorname{ctg} a\left[-\frac{(\omega^2+k_\parallel^2)(\alpha^2-\beta^2)}{4m^2\alpha^2}+\frac{iL}{\sin a\cos a}-1\right]+\frac{k_\perp^2(\cos\beta-\cos a)}{2m^2\sin^3\alpha},\\
d^{01} &= -\frac{\omega k_\perp\cos\beta}{2m^2\sin a}\left(1-\frac{\beta\operatorname{tg}\beta}{\alpha\operatorname{tg}\alpha}\right),\\
dB &= -\frac{\omega k_\parallel(\alpha^2-\beta^2)\operatorname{ctg} a}{2m^2a^2}\\
d^{13} &= -\frac{k_\perp k_\parallel\cos\beta}{2m^2\sin\alpha}\left(1-\frac{\beta\operatorname{tg}\beta}{\alpha\operatorname{tg}\alpha}\right),\\
d^{02} &= d^{12} = d^{23} = 0\,,
\end{aligned}\right. \tag{19}$$

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[10] N. N. BOGOLIUBOV and D. V. SHIRKOV: *Introduction to the Theory of Quantized Fields*, subsect. *14.1* (New York, N. Y., *1959*).

and where the remaining components follow from

$$d^{\mu\nu}(k;\alpha,\beta)=d^{\nu\mu}(k;\alpha,\beta)=d^{\mu\nu}(-k;\alpha,\beta)=d^{\mu\nu}(k;\alpha,-\beta)\,. \tag{20}$$

In the weak-field limit (18) with (19) gives

$$D_0^{\mu\nu}(k;\alpha,\beta):=\lim_{L\to 0}D^{\mu\nu}(k;\alpha,\beta)=\\ =\frac{1}{\alpha}\left[g^{\mu\nu}\left\{1-\frac{iL}{\alpha}+\frac{\alpha^2-\beta^2}{4\alpha^2}\,\frac{k^2}{m^2}\right\}-\frac{\alpha^2-\beta^2}{2\alpha^2}\,\frac{k^\mu k^\nu}{m^2}\right], \tag{21}$$

and L may be incorporated into the variables of integration ($\alpha/L\to$ a, $\beta/L\to\beta$) in (17).

4. - The regularized vacuum polarization tensor.

The polarization tensor (17) is divergent and it must be regularized. Also (17) does not satisfy the requirements of gauge invariance and charge continuity, *i.e.* one does not have $\alpha^{\mu\nu}k_\nu=0$ and $k_\mu\alpha^{\mu\nu}=0$. It is desirable to write the regularized vacuum polarization tensor in a form which is manifestly convergent and which manifestly satisfies the requirements of gauge invariance and charge continuity.

Formally the regularization may be achieved as follows. The presence of a magnetostatic (or other electromagnetic) field does not introduce any additional divergent terms (the divergences are from the limit a $\to 0$ in (17)) and consequently the divergences may be removed by subtracting from (17) the zero-field limit of (17). The regularized form of (17) may then be found by adding to the result the well-known expression for the regularized zero-field vacuum polarization tensor. This procedure gives

$$\operatorname{reg} \mathrm{a}^{\mathrm{Pv}}(\mathrm{k})=\alpha^{\mu\nu}(k)-\alpha_0^{\mu\nu}(k)+\operatorname{reg}\alpha_0^{\mu\nu}(k) \tag{22}$$

with $\alpha^{\mu\nu}$ given by (17) with (18) and (19), $\alpha_0^{\mu\nu}$ given by (17) with (21), and with

$$\operatorname{reg}\alpha_0^{\mu\nu}(k)=A(k)\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right), \tag{23}$$

$$A(k)=-\frac{e^2k^2}{(2\pi)^2}\left(\frac{1}{9}-\frac{(1-\psi\operatorname{ctg}\mathrm{p})(4m^2+2k^2)}{3\mathrm{k}^2}\right], \tag{24}$$

and $\sin^2\psi:=k^2/4m^2$.

However (22) is neither manifestly convergent nor manifestly gauge invariant. To obtain an explicitly convergent and gauge-invariant result we note firstly that it must be possible to write

$$\operatorname{reg}\alpha^{\mu\nu}(k) = \sum F_i f_i^{\mu\nu}, \tag{25}$$

where the F_i are functions of invariants and the sum is over all $f_i^{\mu\nu}$ which can be constructed from k^μ and $F^{\mu\nu}$ (the Maxwell 4-tensor constructed from B) using $g^{\mu\nu}$ and $\varepsilon^{\mu\nu\varrho\sigma}$. Now SHABAD [11] has shown that $\operatorname{reg}\alpha^{\mu\nu}(k)$ has the following four eigenvectors:

$$\begin{cases} b_1^\mu = F^{\mu\varrho}k_\varrho\,, \qquad b_2^\mu = F^{*\varrho\mu}k_\varrho\,, \\ b_3^\mu = k^2 F^{\mu\varrho}F_{\varrho\sigma}k^\sigma - k^\mu k^\varrho F_{\varrho\sigma}F^{\sigma\tau}k_\tau\,, \\ b_4^\mu = k^\mu\,, \end{cases} \tag{26}$$

with $F^{*\mu\nu} := -\frac{1}{2}\varepsilon^{\mu\nu\varrho\sigma}F_{\varrho\sigma}$. Furthermore gauge invariance implies that the eigenvalue corresponding to b_4^μ is zero. Thus with

$$f_i^{\mu\nu} := \frac{b_i^\mu b_i^\nu}{(b_i)^2} \tag{27}$$

(25) becomes

$$\operatorname{reg}\alpha^{\mu\nu}(k) = \sum_{i=1}^{3} F_i f_i^{\mu\nu}\,. \tag{28}$$

Furthermore, explicit evaluation shows

$$g^{\mu\nu} = \sum_{i=1}^{4} f_i^{\mu\nu}\,, \tag{29}$$

and it is convenient to introduce

$$f_0^{\mu\nu} := g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} = \sum_{i=1}^{3} f_i^{\mu\nu}\,, \tag{30}$$

and to replace *(28)* by

$$\operatorname{reg}\alpha^{\mu\nu}(k) = \sum_{i=0}^{2} G_i f_i^{\mu\nu}\,. \tag{31}$$

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[11] A. E. SHABAD: *Ann. of Phys.*, **90**, 166 (1975).

Comparison of (31) and (22) allows one to identify the G_i. There results

$$G_i=\frac{e^2}{(2\pi)^2}\cdot \tag{32}$$

$$\cdot\int_0^\infty\frac{d\alpha}{\alpha}\int_0^\alpha d\beta\, g_i\exp\left[-\frac{i\alpha}{L}+\frac{i(\alpha^2-\beta^2)}{4\alpha L}\frac{k^2}{m^2}+\frac{i}{2L}\left[\frac{\alpha^2-\beta^2}{2\alpha}-\frac{\cos\beta-\cos\alpha}{\sin\alpha}\right]\frac{k_\perp^2}{m^2}\right]$$

with

$$\left\{\begin{aligned}
g_0&=k^2\frac{\cos\beta}{2\sin\alpha}\left(1-\frac{\beta\operatorname{tg}\beta}{\alpha\operatorname{tg}\alpha}\right),\\
g_1&=-k_\perp^2\left\{\frac{\cos\beta-\cos\alpha}{\sin^3\alpha}-\frac{\cos\beta}{2\sin\alpha}\left(1-\frac{\beta\operatorname{tg}\beta}{\alpha\operatorname{tg}\alpha}\right)\right\},\\
g_2&=(\omega^2-k_\parallel^2)\left\{\frac{\alpha^2-\beta^2}{2\alpha^2}\operatorname{ctg}\alpha-\frac{\cos\beta}{2\sin\alpha}\left(1-\frac{\beta\operatorname{tg}\beta}{\alpha\operatorname{tg}\alpha}\right)\right\}.
\end{aligned}\right. \tag{33}$$

Note that contributions from $\alpha_0^{\mu\nu}$ and $\operatorname{reg}\alpha_0^{\mu\nu}$ cancel. Both contribute only to G_0 and the relevant identity is

$$A(k)=\frac{e^2k^2}{(2\pi)^2}\int_0^\infty\frac{d\alpha}{\alpha}\int_0^\alpha d\beta\,\alpha^2-\beta^2\exp\left[-i\alpha+\frac{i(\alpha^2-\beta^2)}{4\alpha}\frac{k^2}{m^2}\right] \tag{34}$$

with $A(k)$ given by (24). (Formally the integral in (34) diverges, and one integrates around the pole in the usual way to find a finite result.)

It is worth emphasizing that the regularization procedure (22) has not been used explicitly here, and that the essential step is the decomposition (28) which SHABAD ([11]) referred to as diagonalization with respect to tensor indices. The decomposition could be applied directly to $\alpha^{\mu\nu}$, without the formal intermediate step (22), with the result that $\alpha^{\mu\nu}$ would be regularized. Thus the expansion in eigenvectors, as proposed by SHABAD ([11]), is in effect an alternative regularization procedure. In fact if we had applied this procedure directly to the unregularized form of $\alpha^{\mu\nu}$ and then taken the zero-field limit we would have rederived the regularized zero-field vacuum polarization tensor, *i.e.* (23) with (24).

An alternative form of G_i may be obtained by the transformation $\alpha\to -ieB\alpha$, $\beta\to -ieB\beta$:

$$G_i=\frac{e^3B}{(2\pi)^2}\int_0^\infty\frac{d\alpha}{\alpha}\int_0^\alpha d\beta\,\gamma_i\exp[-\alpha m^2]\cdot \tag{35}$$

$$\cdot\exp\left[\frac{\alpha^2-\beta^2}{4\alpha}(k^2+k_\perp^2)+\frac{\cosh eB\beta-\cosh eB\alpha}{2eB\sinh eB\alpha}k_\perp^2\right]$$

with

$$(36)\qquad \begin{cases} \gamma_0 = k^2 \dfrac{\cosh eB\beta}{2\sinh eB\alpha}\left(1-\dfrac{\beta \operatorname{tgh} eB\beta}{\alpha \operatorname{tgh} eB\alpha}\right), \\[2ex] \gamma_1 = +k_\perp^2\left\{\dfrac{\cosh eB\beta}{2\sinh eB\alpha}\left(1-\dfrac{\beta \operatorname{tgh} eB\beta}{\alpha \operatorname{tgh} eB\alpha}\right)+\dfrac{\cosh eB\beta-\cosh eB\alpha}{\sinh^3 eB\alpha}\right\}, \\[2ex] \gamma_2 = (\omega^2-k_\parallel^2)\left\{\dfrac{\alpha^2-\beta^2}{2\alpha^2}\operatorname{ctgh} eB\alpha-\dfrac{\cosh eb\beta}{2\sinh eB\alpha}\left(1-\dfrac{\beta \operatorname{tgh} eB\beta}{\alpha \operatorname{tgh} eB\alpha}\right)\right\}. \end{cases}$$

The replacement of the upper limit of α-integration by infinity in (35) is achieved by rotating through $\pi/2$ in the α-plane and this is possible provided no poles are encountered. A sufficient condition for the absence of poles is that pair production be impossible, *e.g.* $\omega < m[1+(1+2B/B_c)^{\frac{1}{2}}]$ [3].

5. – Dispersion relations and polarization vectors.

Maxwell's equations imply the wave equation

$$(37)\qquad \{k^2 g^{\mu\nu} - k^\mu k^\nu + 4\pi \operatorname{reg} \alpha^{\mu\nu}(k)\} A_\mu(k) = 0 \,.$$

SHABAD [11] suggested that the dispersion relations for the electromagnetic modes of the birefringent vacuum are given by $k^2 = -4\pi F_i$, where the F_i are the eigenvalues of $\operatorname{reg}\alpha^{\mu\nu}(k)$, cf. (28). (It follows from (27) and (30) that the b_i^μ are eigenvectors of $k^2 g^{\mu\nu} - k^\mu k^\nu$ with eigenvalues k^2.) However, this is not the case. There are only two modes and all that one can conclude is that the dispersion relation for each is of the form $k^2 = -4\pi \sum_{i=1}^{3} a_i F_i$ with $\sum_{i=1}^{3} a_i = 1$ and with the a_i not necessarily independent of k^2. Hence Shabad's procedure is not useful in general.

We use the following procedure familiar in plasma physics for calculating the wave properties. Firstly, choose the radiation gauge $A^\mu(k) = (0, \boldsymbol{A}(k))$ so that (37) reduces to three simultaneous equations for the three components of $\boldsymbol{A}(k)$. Secondly, set the determinant of the coefficients equal to zero to find the dispersion equation. Each distinct solution of the dispersion equation is the dispersion relation for a specific wave mode. Finally, when any dispersion relation is satisfied one may solve for the polarization vector $\boldsymbol{e}$ which is defined to be a unimodular vector along the solution for $\boldsymbol{A}$. Although this procedure involves an explicit choice of gauge (and of inertial frame) it can be shown that equivalent results are obtained for any other choice of gauge (or frame) [12].

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[12] D. B. MELROSE: *Plasma Phys.*, 15, 99 (1973).

If we introduce the refractive index μ and the wave angle 8 via

$$\mu^2 := \frac{|\boldsymbol{k}|^2}{\omega^2}, \qquad \theta := \operatorname{arctg}(k_\perp/k_\parallel), \tag{38}$$

and write, cf. (32) and (33),

$$\chi_0 := \frac{4\pi G_0}{k^2}, \qquad \chi_1 := \frac{4\pi G_1}{k_\perp^2}, \qquad \chi_2 := \frac{4\pi G_2}{\omega^2 - k_\parallel^2}, \tag{39}$$

the three simultaneous equations become

$$\begin{pmatrix} (1-\mu^2\cos^2\theta)(1+\chi_0) & 0 & \mu^2\sin 8\cos\theta(1+\chi_0) \\ 0 & (1-\mu^2)(1+\chi_0)+\mu^2\sin^2\theta\chi_1 & 0 \\ \mu^2\sin\theta\cos\theta(1+\chi_0) & 0 & (1-\mathrm{p}^a\sin^2\theta)(1+\chi_0)+\chi_2 \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = 0 . \tag{40}$$

The solutions are

$$\mu_1^2 = \frac{1+\chi_0}{1+\chi_0 + \sin^2\theta\chi_1}, \qquad \underline{e}_1 = (0, 1, 0), \tag{41a}$$

$$\mu_2^2 = \frac{1+\chi_0+\chi_2}{1+\chi_0+\cos^2\theta\chi_2}, \qquad \underline{e}_2 = \frac{((1+\chi_0+\chi_2)\cos\theta, 0, -(1+\chi_0)\sin\theta)}{[(1+\chi_0)^2+\cos^2\theta\chi_2(\chi_2+2+2\chi_0)]^{\frac{1}{2}}}. \tag{41b}$$

However, these solutions for μ^2 are implicit because the χ_i remain functions of μ^2.

Our results reproduce those of ADLER [3] by assuming $\mu^2 \approx 1$ in (41*a*), (41*b*) and by evaluating G_1 and G_2 for $k^a = 0$ (*i.e.* $k_\perp = \omega \sin 8$, $k_\parallel = \omega \cos 8$). One obtains, denoting the case $k^a = 0$ by an asterisk,

$$\mu_1 \approx 1 + \frac{2\pi}{\omega^2} G_1^*, \tag{42a}$$

$$\mu_2 \approx 1 + \frac{2\pi}{\omega^2} G_2^* \tag{42b}$$

with

$$G_i^* = \frac{e^3 B}{(2\pi)^2} \int_0^\infty \frac{d\alpha}{\alpha} \int_0^a d\beta\, \gamma_i^* \exp[-\alpha m^2] \cdot \exp\left[\omega^2 \sin^2\theta \left[\frac{\alpha^2-\beta^2}{4\alpha} + \frac{\cosh eB\beta - \cosh eB\alpha}{2eB \sinh eB\alpha}\right]\right] \tag{43}$$

and with

$$
(44)\qquad \begin{cases} \gamma_0^* = 0\,, \\[2mm] \gamma_1^* = +\,\omega^2 \sin^2\theta \left\{ \dfrac{\cosh eB\beta}{2 \sinh eB\alpha} \left(1 - \dfrac{\beta \operatorname{tgh} eB\beta}{\alpha \operatorname{tgh} eB\alpha}\right) + \dfrac{\cosh eB\beta - \cosh eB\alpha}{\sinh^3 eB\alpha} \right\}, \\[2mm] \gamma_2^* = \omega^2 \sin^2\theta \left\{ \dfrac{\alpha^2 - \beta^2}{2\alpha^2} \operatorname{ctgh} eB\alpha - \dfrac{\cosh eB\beta}{2 \sinh eB\alpha} \left(1 - \dfrac{\beta \operatorname{tgh} eB\beta}{\alpha \operatorname{tgh} eB\alpha}\right) \right\}. \end{cases}
$$

As explained by ADLER, the result for μ_2 applies only for $\omega < 2m$ while that for μ_1 applies only for $\omega < m[1 + (1 + 2B/B_c)^{\frac{1}{2}}]$, these being the thresholds for pair production by the differently polarized photons.

On setting $k^2 = 0$ in (17) one obtains the results of CONSTANTINESCU [5], whose F and G are calculated from

$$(45a)\qquad F = -\frac{4\pi}{m^2}\,\alpha^{22}\,,$$

$$(45b)\qquad G = -\frac{4\pi}{m^2}\,(\alpha^{33} - \operatorname{cosec}^2\theta\,\alpha^{22} + \operatorname{ctg}^2\theta\,\alpha^{11} - 2 \operatorname{ctg}\theta\,\alpha^{13})\,.$$

If we use (31) these may be written in the simpler form

$$(46a)\qquad F = +\frac{4\pi}{m^2}\,G_1^*\,,$$

$$(46b)\qquad G = \frac{4\pi}{m^2}\operatorname{cosec}^2\theta\,(G_1^* + G_2^*)\,,$$

where F and G are now both manifestly renormalized.

6. – The strong-field limit.

In the presence of a strong magnetic field $B \gg B_c$, the dominant contribution in (35) gives G_0 and G_1 of order $\exp[-B/B_c]$ and

$$
(47)\qquad G_2 \approx \frac{e^3 B}{8\pi^2}(\omega^2 - k_\parallel^2)\exp[-k_\perp^2/2eB]\int_0^\infty \frac{d\alpha}{\alpha}\int_0^\alpha d\beta\,\exp[-\alpha m^2]\,\frac{\alpha^2 - \beta^2}{\alpha^2}\cdot\exp\left[\frac{(\alpha^2 - \beta^2)(\omega^2 - k_\parallel^2)}{4\alpha}\right]
$$

A change of integration variable to $\eta := \beta/\alpha$ allows the order of integration to be reversed. The α-integral is then trivial and the β-integral follows. One obtains

$$G_2 = \frac{e^3B}{2\pi^2} \exp\left[-k_\perp^2/2eB\right] \cdot \left\{\frac{4m^a}{[4m^2-(\omega^2-k_\parallel^2)]^{\frac12}(\omega^2-k_\parallel^2)^{\frac12}} \operatorname{arctg}\left[\frac{(\omega^2-k_\parallel^2)^{\frac12}}{(4m^2-(\omega^2-k_\parallel^2))^{\frac12}}\right]-1\right\}. \tag{48}$$

The refractive indices of the two modes are given by

$$\mu_1^2 = 1 + O(\exp[-B/B_c])\,, \tag{49a}$$

$$\mu_2^2 = \frac{1+\chi_2}{1+\chi_2\cos^2\theta} \tag{49b}$$

with $\chi_2 = 4\pi G_2/(\omega^2-k_\parallel^2)$.

For $\omega^2 - k_\parallel^2 \ll 4m^2$ and for $k_\perp^2 \ll 2eB$ $(=2m^2B/B_c)$ (48) gives

$$\chi_2 \approx \frac{e^2}{3\pi}\frac{B}{B_c}\,, \tag{50}$$

and (49*b*) becomes

$$\mu_2^2 \approx \begin{cases} 1 + \dfrac{e^2}{3\pi}\dfrac{B}{B_c}\sin^2\theta & (\chi_2 \ll 1)\,,\\[2ex] \dfrac{1}{\cos^2\theta} & (\chi_2 \gg 1)\,. \end{cases} \tag{51}$$

This result in the extreme limit $B \gg (3x/e^2)B_c$ $(\approx 2.5\cdot 10^{17}\,\mathrm{G})$ has a simple interpretation by analogy with waves in plasmas. The refractive indices $\mu_1^2 \approx 1$ and $\mu_2^2 \approx 1/\cos^2\theta$ correspond to those of the magnetoacoustic mode and the (shear) Alfvén mode in the limit of Alfvén speed much greater than the speed of light [13]. The polarization vectors (41*a*) and (41*b*) for $\chi_2 \gg 1$ also coincide with those of the magnetoacoustic and Alfvbn modes respectively. In particular, for the Alfvbn mode (2-mode) the photons propagate along the magnetic-field lines irrespective of the wave angle θ.

The foregoing results do not agree with those of CONSTANTINESCU [5], whose approximation to the vacuum polarization tensor in the strong-field limit does not satisfy gauge invariance. Also our result shows no evidence of a longitudinal photonlike resonance for $e^2B/B_c \approx 1$, as suggested by COVER and KALMAN [14].

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[13] T. H. STIX: *The Theory of Plasma Waves* (New York, N. Y., 1962), p. 33.
[14] R. A. COVER and G. KALMAN: *Phys. Rev. Lett.*, **33**, 1113 (1974).

● RIASSUNTO (•)

Si calcola il tensore di polarizzazione del vuoto in presenza di un carnpo magnetico omogeneo statico esattamente in funzione sia del campo magnetico che del vettore d'onda, *e* si regolarizza esplicitamente usando la diagonalizzazione di Shabad rispetto agli indici tensoriali. Si calcolano esattamente le proprietà d'onda per i due modi elettromagnetici nel vuoto birifrangente usando una tecnica della fisica del plasma. Si considera esplicitamente il limite del campo forte e si mostra che i due modi si riducono a forme equivalenti al modo magnetoacustico ed ai modi di Alfvén di taglio in un plasma con la velocitii di Alfvén molto maggiore della velocità della luce.

(•) *Traduzione a cura della Redazione.*

Поляризация вакуума и распространение фотона в магнитном поле.

Резюме (•). — Точно вычисляется тензор поляризации вакуума в присутствии статического однородного магнитного поля как функция магнитного поля и волного вектора. В явном виде производится регуляризация тензора поляризации вакуума, используя диагонализацию Шабада по отношению к тензорным индексам. Определяются волновые свойства двух электромагнитных мод в двоякопреломляющем вакууме, используя технику физики плазмы. В явном виде рассматривается предел сильного поля. Показывается, что две элетромагнитные моды сводятся к выражениям, эквивалентным магнитноакустической и сдвиговой альфвеновской модам в плазме, когда скорость Альфвена много больше, чем скорость света.

(•) *Переведено редакцией.*

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