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LETTER TO THE EDITOR

Generalized Kramers–Kronig formula for spatially dispersive media

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Abstract. For a spatially dispersive medium the requirements of causality and special relativity are used to develop a generalized dispersion formula (a generalization of the Kramers–Kronig formula). An error is pointed out in earlier versions of this formula. The vacuum polarization tensor is shown to satisfy the generalized dispersion formula explicitly.

Causality (no effect precedes its cause) implies that any response function is analytic in the upper half of the complex frequency plane (Toll 1952, 1956, Hilgevoord 1962). For spatially dispersive media (when the response depends on $\boldsymbol{k}$ as well as ω) causality and special relativity lead to the requirement that the response be non-zero only inside the forward light cone. The associated restriction on the properties of the response function will be called the generalized dispersion formula. Here we point out an error in existing discussions of the generalized dispersion formula (Leontovich 1961, Silin and Rukhadze 1961, appendix). In addition we show explicitly that the vacuum polarization tensor satisfies the generalized dispersion formula.

Considering the generalized dispersion formula, the linear response of a medium (e.g. a plasma or the vacuum) to an electromagnetic disturbance may be described in terms of the polarization tensor $\alpha^{\mu\nu}(\omega, \boldsymbol{k})$ (cf Melrose 1973) which is the Fourier transform of a tensor operator $\tilde{\alpha}^{\mu\nu}(\tau, \boldsymbol{\xi})$. (Our notation is that of Berestetskii *et al* 1971 and unrationalized Gaussian units with $\hbar = c = 1$ are used. The symbols := and =: define the quantities on the left and right, respectively.) The causal condition may be expressed by

$$\tilde{\alpha}^{\mu\nu}(\tau, \boldsymbol{\xi}) = \theta(\tau)\tilde{\alpha}^{\mu\nu}(\tau, \boldsymbol{\xi}), \tag{1}$$

where $\theta(\tau)$ is the unit step function. On Fourier transforming, (1) leads to the usual Kramers–Kronig dispersion formula. Special relativity requires that the response function vanish for τ and $\boldsymbol{\xi}$ outside the forward light cone. This condition may be expressed by

$$\tilde{\alpha}^{\mu\nu}(\tau, \boldsymbol{\xi}) = \theta(\tau - \boldsymbol{\beta} \cdot \boldsymbol{\xi})\tilde{\alpha}^{\mu\nu}(\tau, \boldsymbol{\xi}), \tag{2}$$

where $\boldsymbol{\beta}$ is any vector for which the surface $\tau = \boldsymbol{\beta} \cdot \boldsymbol{\xi}$ lies entirely within the space-like region, i.e. outside both the forward and backward light cones. When all such surfaces with $\beta^2 \leqslant 1$ are taken into account, the θ-function in (2) excludes all space–time except the forward light cone, as required.

The Fourier transform of $\theta(\tau-\boldsymbol{\beta}\cdot\boldsymbol{\xi})$ is

$$\chi(\omega,\boldsymbol{k}):=\int \mathrm{d}^3\boldsymbol{\xi}\int \mathrm{d}\tau\,\exp[\mathrm{i}(\omega\tau-\boldsymbol{k}\cdot\boldsymbol{\xi})]\theta(\tau-\boldsymbol{\beta}\cdot\boldsymbol{\xi})=\frac{(2\pi)^3\,\mathrm{i}\delta^3(\boldsymbol{k}-\omega\boldsymbol{\beta})}{\omega+\mathrm{i}0}, \tag{3}$$

where the small positive imaginary part indicates how one is to integrate around the pole at $\omega=0$. Using the convolution theorem, the Fourier transform of (2) becomes

$$\begin{aligned}\alpha^{\mu\nu}(\omega,\boldsymbol{k})&=\int\frac{\mathrm{d}\zeta}{2\pi}\int\frac{\mathrm{d}^3\boldsymbol{\eta}}{(2\pi)^3}\chi(\omega-\zeta,\boldsymbol{k}-\boldsymbol{\eta})\alpha^{\mu\nu}(\zeta,\boldsymbol{\eta})\\&=\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{(\omega-\zeta+\mathrm{i}0)}\alpha^{\mu\nu}(\zeta,\boldsymbol{k}+\boldsymbol{\beta}(\zeta-\omega)).\end{aligned} \tag{4}$$

The Plemelj formula (Montgomery and Tidman 1964, § 5.3) allows one to rewrite (4) as

$$\alpha^{\mu\nu}(\omega,\boldsymbol{k})=\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{(\zeta-\omega)}\alpha^{\mu\nu}(\zeta,\boldsymbol{k}+\boldsymbol{\beta}(\zeta-\omega)), \tag{5}$$

where the principal value of the integral is to be taken.

Equation (5) is our generalized dispersion formula. The usual Kramers–Kronig relation follows from (5) by choosing $\boldsymbol{\beta}=\boldsymbol{0}$. Separating (5) into Hermitian (H) and anti-Hermitian (A) parts gives

$$\alpha^{\mu\nu(\mathrm{H,A})}(\omega,\boldsymbol{k})=\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\zeta}{(\zeta-\omega)}\alpha^{\mu\nu(\mathrm{A,H})}(\zeta,\boldsymbol{k}+\boldsymbol{\beta}(\zeta-\omega)). \tag{6}$$

Alternatively, (5) may be separated into real and imaginary parts. Equation (6) implies two relations between the dissipative and non-dissipative parts of the response tensor, but these relations are not independent due to the skew symmetry of the Hilbert transform.

A result similar to (5) was first obtained by Leontovich (1961). He assumed $\boldsymbol{\beta}$ parallel to $\boldsymbol{k}$ and this restriction was relaxed by Silin and Rukhadze (1961, appendix). All these authors applied a Lorentz transformation to the usual Kramers–Kronig formula to obtain their results. However, they assumed incorrectly that a quantity effectively equal to $\alpha^{\mu\nu}(\omega,\boldsymbol{k})/\omega$ (Leontovich) or $\alpha^{\mu\nu}(\omega,\boldsymbol{k})/\omega^2$ (Silin and Rukhadze) transforms as a 4-tensor. Melrose (1973) has shown that $\alpha^{\mu\nu}(\omega,\boldsymbol{k})$ itself transforms as a 4-tensor.

As an application of the generalized dispersion relation we have considered the vacuum polarization tensor. In this case a separation into Hermitian and anti-Hermitian parts is effectively the same as a separation into real and (i times the) imaginary parts. The imaginary part of the tensor is (Feynman 1949)

$$\begin{aligned}\operatorname{Im}\alpha^{\mu\nu}(\omega,\boldsymbol{k})&=\frac{e^2\operatorname{sgn}\omega}{6\pi}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{(\omega^2-|\boldsymbol{k}|^2)}\right)[m^2+\tfrac{1}{2}(\omega^2-|\boldsymbol{k}|^2)]\\&\quad\times\left(1-\frac{4m^2}{(\omega^2-|\boldsymbol{k}|^2)}\right)^{1/2}\theta(\omega^2-|\boldsymbol{k}|^2-4m^2).\end{aligned} \tag{7}$$

A direct application of the dispersion relation to find the real part of the tensor leads to a divergent integral, but after a double subtraction (Lifshitz and Pitaevskii 1974, § 108) the renormalized, gauge invariant real part of the tensor is obtained. Both the real and imaginary parts of the tensor are proportional to $[g^{\mu\nu}-k^\mu k^\nu/(\omega^2-|\boldsymbol{k}|^2)]$ and it is necessary to apply the dispersion relation only to the scalar function $\alpha(\omega, \boldsymbol{k}) := \frac{1}{3}\alpha^{\mu}{}_{\mu}(\omega, \boldsymbol{k})/(\omega^2 - |\boldsymbol{k}|^2)^2$.

By inspection of (7) one may write

$$\operatorname{Im}\alpha(\omega, \boldsymbol{k}) =: F(\omega^2 - |\boldsymbol{k}|^2)\operatorname{sgn}\omega\,\theta(\omega^2 - |\boldsymbol{k}|^2 - 4m^2), \tag{8}$$

where F is a function of its argument. Substituting this into the real part of (5) gives

$$\operatorname{Re}\alpha(\omega, \boldsymbol{k}) = \frac{1}{\pi}\int_0^\infty d\zeta \left(\frac{F(A\zeta^2 + B\zeta + C)\theta(A\zeta^2 + B\zeta + C - 4m^2)}{(\zeta - \omega)} + \frac{F(A\zeta^2 - B\zeta + C)\theta(A\zeta^2 - B\zeta + C - 4m^2)}{(\zeta + \omega)}\right), \tag{9}$$

with $A := 1 - \beta^2$, $B := 2(\beta^2\omega - \boldsymbol{k}\cdot\boldsymbol{\beta})$, $C := -(\boldsymbol{k} - \omega\boldsymbol{\beta})^2$. For arbitrary $\beta^2 < 1$, the θ-functions in (9) imply

$$\operatorname{Re}\alpha(\omega, \boldsymbol{k}) = \frac{1}{\pi}\left(\int_{\zeta_-}^\infty \frac{d\zeta}{(\zeta - \omega)} F(A\zeta^2 + B\zeta + C) + \int_{\zeta_+}^\infty \frac{d\zeta}{(\zeta + \omega)} F(A\zeta^2 - B\zeta + C)\right), \tag{10}$$

with $\zeta_\pm := \{\pm B + [B^2 + 4A(4m^2 - C)]^{1/2}\}/2A$. Setting $x := A\zeta^2 + B\zeta + C$ and $x := A\zeta^2 - B\zeta + C$ in the first and second integrals, respectively, of (10) gives

$$\operatorname{Re}\alpha(\omega, \boldsymbol{k}) = \frac{1}{\pi}\int_{4m^2}^\infty \frac{dx\, F(x)}{x - (\omega^2 - |\boldsymbol{k}|^2)}. \tag{11}$$

For the special case $\beta^2 = 1$ equation (9) becomes

$$\operatorname{Re}\alpha(\omega, \boldsymbol{k}) = \frac{1}{\pi}\int_0^\infty d\zeta \left(\frac{F(B'\zeta + C)\theta(B'\zeta + C - 4m^2)}{(\zeta - \omega)} + \frac{F(-B'\zeta + C)\theta(-B'\zeta + C - 4m^2)}{(\zeta + \omega)}\right), \tag{12}$$

with $B' := 2(\omega - \boldsymbol{k}\cdot\boldsymbol{\beta})$. For $\omega > \boldsymbol{k}\cdot\boldsymbol{\beta}$ ($\omega < \boldsymbol{k}\cdot\boldsymbol{\beta}$ may be shown to give the same result), the θ-functions in (12) imply

$$\operatorname{Re}\alpha(\omega, \boldsymbol{k}) = \int_{\zeta_0}^\infty \frac{d\zeta}{(\zeta - \omega)} F(B'\zeta + C), \tag{13}$$

with $\zeta_0 := (4m^2 - C)/B'$. A change of variable to $x := B'\zeta + C$ allows (13) to be rewritten in the form (11).

It has therefore been shown that (9) reduces to (11) for arbitrary $\beta^2 \leq 1$. Expression (11) is independent of $\boldsymbol{\beta}$ and has been evaluated by Lifshitz and Pitaevskii (1974, § 110). The vacuum polarization tensor therefore satisfies the generalized dispersion formula for $\boldsymbol{\beta}$ an arbitrary vector of magnitude less than or equal to unity.

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