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LEGENDRE EXPANSION OF THE QUASI-LINEAR EQUATIONS FOR ANISOTROPIC PARTICLES AND LANGMUIR WAVES

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ABSTRACT

The quasi-linear diffusion and friction coefficients for axisymmetric electron distributions interacting with Langmuir waves are evaluated explicitly by expanding the distribution of waves in Legendre polynomials. The quasi-linear equations are then reduced to a form in which both the distributions of waves and of particles are simultaneously expanded in Legendre polynomials, and all coefficients are evaluated explicitly. It is argued that such expansions are likely to be justified in practice and that the results obtained should prove useful in discussing quasi-linear relaxation under various conditions in three dimensions rather than one dimension. New results are anticipated for the problem of the propagation of electron streams causing type III solar radio bursts.

The influence of the magnetic field on the Langmuir waves is neglected, i.e., $(\omega_c/\omega_p)^2 \ll 1$ is assumed.

Subject headings: functions: numerical methods — plasmas — Sun: radio radiation — turbulence

I. INTRODUCTION

In an earlier paper (Melrose and Stenhouse 1977) the emission and absorption of Langmuir waves by an axisymmetric distribution of particles was discussed. In this paper that discussion is extended to the quasi-linear relaxation of the particles due to the waves. The extension is in two parts. First, in § III, the quasi-linear diffusion coefficient are evaluated explicitly in terms of the Legendre expansion coefficients of the wave distribution. This is analogous to the explicit evaluation of the emission and absorption coefficients by Melrose and Stenhouse (1977) in terms of the Legendre expansion coefficients of the particle distribution. Second, in § IV, both the equation describing emission and absorption of the waves and the equation describing the effect on the distribution of particles (referred to as "quasi-linear relaxation") are evaluated explicitly, after expanding both distributions in their Legendre series.

This extension is motivated by the desire to simplify the analytic description of the quasi-linear relaxation of anisotropic (e.g., streaming) particles. At the outset it should be stated that for the expansion of either distribution in Legendre polynomials to lead to significant simplification, it must be possible to represent the gross features of the angular distribution with the first few terms from the Legendre series only. Indeed, attempts to justify the use of one-dimensional equations have led to a widespread belief that very narrow angular distributions of Langmuir waves develop due to streaming instabilities, and very narrow distributions are unfavorable for expansion in Legendre polynomials. It will be argued below that in practice very narrow distributions are not to be expected and, incidentally, this implies that quasi–one-dimensional treatments are inadequate. It is argued in § VI that this may have important implications for our understanding of the physics of electron beams of astrophysical interest, in particular beams causing type III solar radio bursts.

The present status of the theory of quasi-linear relaxation (of anisotropic particles due to Langmuir waves) may be summarized as follows. Quasi-linear relaxation of a "one-dimensional" stream of electrons was discussed first by Drummond and Pines (1962) and Vedenov, Velikov, and Sagdeev (1962). The asymptotic state of the particles is a plateau distribution, and two-thirds of the initial kinetic energy in the particles passes to the waves (Shapiro 1963). The evolution of the relaxation was discussed analytically by Ivanov and Rudakov (1966); Grognard (1975) pointed out the inconsistency of the neglect of spontaneous emission, but his numerical solutions showed essentially the same gross features as the earlier analytic discussion. Quasi-linear relaxation in three dimensions was discussed by Sizonenko and Stepanov (1965) and Bernstein and Engelmann (1966); but they differed on what the asymptotic state would be. Numerical calculations by Morse and Nielson (1969) tended to support Bernstein

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and Engelmann's conclusion that the unstable region of $\boldsymbol{k}$-space shrinks to zero; see also the discussion by Davidson (1972, § 9.3). More recently Ivanov, Soboleva, and Yushmanov (1975) studied the dynamical evolution of a collection of particles in three dimensions numerically, and calculations of two-dimensional quasi-linear relaxation were reported by Appert, Tran, and Vaclavik (1976). These studies both indicate that there are two stages, the first being similar to the one-dimensional case and the second being qualitatively different from earlier predictions. Most important for present purposes, Ivanov, Soboleva, and Yushmanov (1975) and Appert, Tran, and Vaclavik (1976) found that the angular distribution of Langmuir waves did not become very narrow but actually tended to broaden, although they found a narrowing in $k = (k_\perp^2 + k_\parallel^2)^{1/2}$ as would be implied by shrinkage of the unstable region of $\boldsymbol{k}$-space to zero. The results of Ivanov *et al.* and Appert *et al.* provide preliminary justification for the expansion in Legendre polynomials here.

(Grognard [1977] has pointed out that Appert *et al.*'s [1976] results must be partly nonphysical due to their neglect of spontaneous emission. However, in view of the foregoing comments on the one-dimensional case, it is reasonable to expect the qualitative form of the quasi-linear relaxation to be affected little by this error.)

II. EQUATIONS

The standard quasi-linear equations for the electron momentum distribution $f(\boldsymbol{p}, \boldsymbol{r}, t)$ and Langmuir wave energy density $T(\boldsymbol{k}, \boldsymbol{r}, t)$, including spontaneous emission, are used (Melrose 1970; Tsytovich 1970):

$$\frac{d}{dt} f \equiv \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right) f = \frac{\partial}{\partial \boldsymbol{p}}\cdot\left(\mathbf{D}\cdot\frac{\partial}{\partial \boldsymbol{p}} + \boldsymbol{A}\right) f, \tag{1}$$

$$\frac{d}{dt} T \equiv \left(\frac{\partial}{\partial t} + \frac{\partial \omega_k}{\partial \boldsymbol{k}}\cdot\frac{\partial}{\partial \boldsymbol{r}} - \frac{\partial \omega_k}{\partial \boldsymbol{r}}\cdot\frac{\partial}{\partial \boldsymbol{k}}\right) T = \gamma_k T + \alpha_k\,. \tag{2}$$

The normalizations of f and T are

$$\int d^3\boldsymbol{p} f(\boldsymbol{p}, \boldsymbol{r}, t) = n_0(\boldsymbol{r}, t) = \text{local electron density}; \tag{3}$$

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} T(\boldsymbol{k}, \boldsymbol{r}, t) = W^l(\boldsymbol{r}, t) = \text{total energy density associated with Langmuir waves}; \tag{4}$$

$$\boldsymbol{p} = \gamma m \boldsymbol{v}; \qquad \gamma = (1 - v^2/c^2)^{-1/2}, \quad \text{the Lorentz factor};$$

$$-e, m = \text{electron charge and rest mass};$$

$$\omega_p = (4\pi n_0 e^2/m)^{1/2}; \qquad \text{the electron thermal speed is } v_e = (\kappa T_e/m)^{1/2};$$

$$\omega_k = (\omega_p^2 + 3k^2 v_e^2)^{1/2}; \qquad \text{Debye length is } \lambda_D = k_D^{-1} = v_e/\omega_p;$$

$$\frac{\partial}{\partial \boldsymbol{k}} \omega_k = (3v_e^2/\omega_k)\boldsymbol{k} \approx 3v_e(\boldsymbol{k}/k_D); \tag{5}$$

$$\frac{\partial}{\partial \boldsymbol{r}} \omega_k \approx \frac{\partial}{\partial \boldsymbol{r}} \omega_p = \frac{\omega_p}{2n_0}\frac{\partial n_0}{\partial \boldsymbol{r}}\,. \tag{6}$$

$\mathbf{D}$, $\boldsymbol{A}$, γ_k, and α_k are given by

$$\mathbf{D}(\boldsymbol{p}) = \frac{e^2}{2\pi}\int d^3\boldsymbol{k}\hat{k}\hat{k}T(\boldsymbol{k})\delta(\omega_p - \boldsymbol{k}\cdot\boldsymbol{v})\,, \tag{7}$$

$$\boldsymbol{A}(\boldsymbol{p}) = \frac{e^2\omega_p}{2\pi}\int d^3\boldsymbol{k}(\hat{k}/k)\delta(\omega_p - \boldsymbol{k}\cdot\boldsymbol{v}) = (e\omega_p/v)^2\hat{p}\log(v/v_e)\,, \tag{8}$$

$$\gamma_k = (2\pi e/k)^2\omega_p\int d^3\boldsymbol{p}\delta(\omega_p - \boldsymbol{k}\cdot\boldsymbol{v})\,\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{p}} f(\boldsymbol{p})\,, \tag{9}$$

$$\alpha_k = (2\pi e\omega_p/k)^2\int d^3\boldsymbol{p}\delta(\omega_p - \boldsymbol{k}\cdot\boldsymbol{v}) f(\boldsymbol{p})\,. \tag{10}$$

We make use of the unit vector notation

$$\hat{k} = \boldsymbol{k}/k; \qquad \hat{p} = \boldsymbol{p}/p, \qquad \text{etc.} \quad (\text{one has } \hat{p} = \hat{v})\,.$$

It follows from (7) that $\mathbf{D}$ has three real and positive eigenvalues, whence equation (1) is well posed for advancing

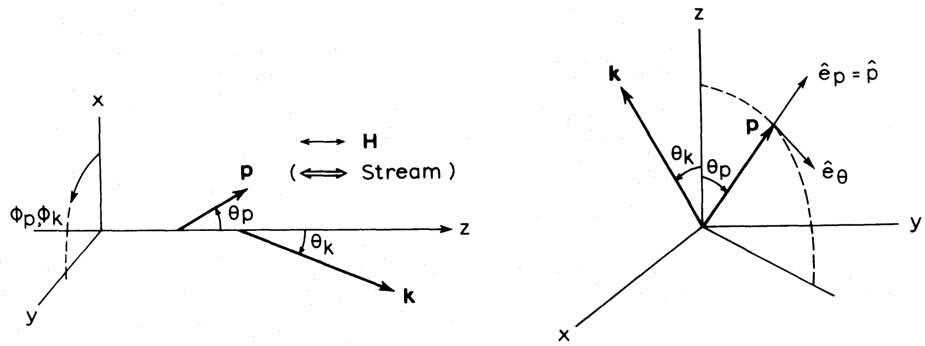

FIG. 1.—Definition of coordinate frames for $\boldsymbol{r}$, $\boldsymbol{k}$, and $\boldsymbol{p}$ ($\mu_p = \cos\theta_p$; $\mu_k = \cos\theta_k$). Definition of unit vectors $\hat{e}_p = \hat{p} = \boldsymbol{p}/p = \hat{v}$, and $\hat{e}_\theta$.

in time. This remains true if Coulomb interactions are included (Appendix A). Equation (1) is valid only for the high-velocity tail of the electron distribution, $|\boldsymbol{p}| > p_e = mv_e(1 - v_e^2/c^2)^{-1/2}$. ω_k has been approximated by ω_p at all places where actually ω_k occurs in (7)–(10), and the other arguments in $T(\boldsymbol{k})$ and $f(\boldsymbol{p})$ are left understood. The integration over $\boldsymbol{k}$ in (7) and (8) extends in practice over the sphere $|\boldsymbol{k}| \leqslant k_D$ in wave-vector space, on the grounds that Langmuir waves cease to exist for $k \geqslant k_D$ (in this way one arrives at the explicit expression for $\boldsymbol{A}$ in [8], by performing the integration over $\boldsymbol{k}$ in spherical coordinates k, θ_k, ϕ_k, such that $\theta_k = 0$ corresponds to the direction of $\boldsymbol{v}$). Henceforth, it is understood that $T(\boldsymbol{k})$ and $\{T_n(k)\}$ in (12) approach zero rapidly for $k \geqslant k_D$.

The effect of Coulomb interactions (collisions) may be included in $\mathbf{D}$ and $\boldsymbol{A}$; this is done for the nonrelativistic limit in Appendix A.

The Lorentz force term $(\boldsymbol{v} \times \boldsymbol{H})\cdot(\partial/\partial\boldsymbol{p})f(\boldsymbol{p})$ has been omitted in equation (1): It is identically zero when $f(\boldsymbol{p})$ is axisymmetric with respect to the local direction of $\boldsymbol{H}$. We assume a homogeneous field $\boldsymbol{H}$, and exclude both nonaxisymmetric *initial conditions*, and models in which small-scale ($\geqslant \lambda_D$) inhomogeneities play an essential role. Then $f(\boldsymbol{p})$ and $T(\boldsymbol{k})$ must be axisymmetric around $\boldsymbol{H}$. Note that the Lorentz term drops out in this case; it has not been set equal to zero by assuming $\boldsymbol{H} = 0$; $\boldsymbol{H}$ remains finite. On the other hand, in (1)–(10) the influence of $\boldsymbol{H}$ on the properties of the waves was neglected, requiring $(\omega_c/\omega_p)^2 \ll 1$, where ω_c is the electron cyclotron frequency.

When the high-velocity electrons are streaming (e.g., electron beams causing type III radio emission or hard X-ray emission in solar flares), there will be a reverse current (carried by the thermal electrons) which neutralizes the beam current. The beam then has no self-magnetic field, but there is an electric field $\boldsymbol{E}_r$ driving the reverse current. The corresponding force term $\boldsymbol{E}_r\cdot(\partial/\partial\boldsymbol{p})f$ operative on the beam electrons has been neglected in equation (1); see Appendix B for further details.

Henceforth, we choose the z-axis along the magnetic field lines (Fig. 1). Under the assumptions made above it is no further restriction to take all quantities independent of both perpendicular directions: $\partial/\partial x \equiv 0$; $\partial/\partial y \equiv 0$. There is also axisymmetry around the z-axis, or, adopting spherical coordinates (p, θ_p, ϕ_p), (k, θ_k, ϕ_k): $\partial/\partial\phi_p \equiv 0$, $\partial/\partial\phi_k \equiv 0$.

Both distributions can now be expanded in a Legendre series (see Appendix C):

$$f(p, \mu_p, z, t) = \sum_{n=0}^{\infty} f_n(p, z, t) P_n(\mu_p)\,, \tag{11}$$

$$T(k, \mu_k, z, t) = \sum_{n=0}^{\infty} T_n(k, z, t) P_n(\mu_k)\,. \tag{12}$$

It is emphasized that validity of (11) and (12) does not involve any approximation other than the assumed axial symmetry. The technical point that the expansions would not converge for sufficiently singular angular distributions is hypothetical. On the contrary, for any physically meaningful distribution, the Legendre expansion will converge *uniformly*. This property guarantees that the maximum of the absolute truncation error on $-1 \leq \mu \leq 1$ decreases steadily to zero upon increasing the order of the term at which the expansion is truncated. This renders Legendre expansion a very efficient approximation method (in contrast to, e.g., a Taylor expansion in μ).

Yet there remains the question of the usefulness of the expansion, which depends on the question at what term (11) and (12) can be truncated. This is briefly discussed in § V. In the next section, explicit expressions for $\mathbf{D}$, $\boldsymbol{A}$, γ_k, and α_k are derived and the equations for $\{f_n\}$ and $\{T_n\}$ are obtained in § IV.

III. EXPANSION OF $\mathbf{D}$, $\boldsymbol{A}$, γ_k, AND α_k

In principle the analysis is straightforward, and only a small part of it will be presented explicitly. Starting with equation (1), one obtains, in spherical coordinates ($\partial/\partial\phi_p = 0$):

$$\frac{df}{dt} = \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left\{D_{pp}\frac{\partial}{\partial p} - \frac{1}{p}D_{p\theta}w(\mu_p)\frac{\partial}{\partial\mu_p} + A_p\right\}f - \frac{1}{p}\frac{\partial}{\partial\mu_p}w(\mu_p)\left\{D_{\theta p}\frac{\partial}{\partial p} - \frac{1}{p}D_{\theta\theta}w(\mu_p)\frac{\partial}{\partial\mu_p}\right\}f. \tag{13}$$

We have converted $\partial/\partial\theta_p$ using $\partial/\partial\theta_p = -(1-\mu_p{}^2)^{1/2}\partial/\partial\mu_p$ and introduced the shorthand notation

$$w(\mu) = (1-\mu^2)^{1/2}. \tag{14}$$

(The index θ in $D_{\theta p}$, $D_{\theta\theta}$, etc., actually stands for θ_p.)

a) Coefficients for the Particle Diffusion Equation

The coefficients in (13) are given by (see Fig. 1):

$$D_{pp} = \hat{e}_p\cdot\mathbf{D}\cdot\hat{e}_p\,; \qquad D_{p\theta} = D_{\theta p} = \hat{e}_p\cdot\mathbf{D}\cdot\hat{e}_\theta\,; \qquad D_{\theta\theta} = \hat{e}_\theta\cdot\mathbf{D}\cdot\hat{e}_\theta\,;$$

$$A_p = \boldsymbol{A}\cdot\hat{e}_p\,; \qquad A_\theta = \boldsymbol{A}\cdot\hat{e}_\theta \propto \hat{e}_p\cdot\hat{e}_\theta = 0\,, \quad \text{cf. eq. (8)}. \tag{15}$$

Because A_θ vanishes identically it was already omitted from (13). We now proceed to obtain explicit expressions for the coefficients in (15) in terms of the expansion coefficients $T_n(k)$ of $T(\boldsymbol{k})$. For A_p one finds trivially

$$A_p = \frac{e^2\omega_p{}^2}{v^2}\log(v/v_e)\,; \qquad A_\theta = 0\,. \tag{16}$$

For the coefficients of $\mathbf{D}$, (7) is substituted in (15) and the integration over $\boldsymbol{k}$ must now be performed. For this we make use of the fact that one can choose any suitable coordinate frame: the integration is done in spherical coordinates k, μ'_k, ϕ'_k defined with respect to the frame x', y', z' as indicated in Figure 2. (For an alternative method we refer to Appendix D):

$$\begin{Bmatrix} D_{pp} \\ D_{p\theta} \\ D_{\theta\theta} \end{Bmatrix} = \frac{e^2}{2\pi}\int d^3\boldsymbol{k}\begin{Bmatrix} (\omega_p/kv)^2 \\ (\omega_p/kv)\sin\theta'_k\cos\phi'_k \\ \sin^2\theta'_k\cos^2\phi'_k \end{Bmatrix} T(\boldsymbol{k})\delta(\omega_p - \boldsymbol{k}\cdot\boldsymbol{v})\,. \tag{17}$$

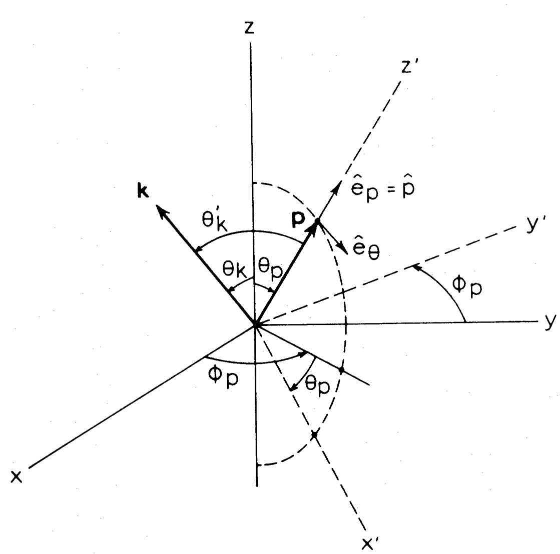

FIG. 2.—For the integration over $\boldsymbol{k}$ in (7), the x', y', z' frame attached to the (fixed) vector $\boldsymbol{p}$ is chosen; the y'-axis lies in the (x, y)-plane. The integration is done in spherical coordinates k, θ'_k, ϕ'_k defined with respect to $k_{x'}$, $k_{y'}$, $k_{z'}$ in the usual way ($k_{x'} = k\sin\theta'_k\cos\phi'_k$, etc.).

In (17), use was made of

1. $\hat{k}\cdot\hat{e}_p = \hat{k}\cdot\hat{v} = \boldsymbol{k}\cdot\boldsymbol{v}/kv = \omega_p/kv$, due to the δ-function.
2. $\hat{k}\cdot\hat{e}_\theta = \sin\theta'_k \cos\phi'_k$.

We now evaluate the integrals in (17) and take D_{pp} as an example. Since $\boldsymbol{k}\cdot\boldsymbol{v} = kv\mu'_k$, one finds

$$D_{pp} = \frac{e^2\omega_p^2}{2\pi v^2}\int_0^\infty dk \int_{-1}^{1} d\mu'_k \int_0^{2\pi} d\phi'_k T(\boldsymbol{k})\delta(\omega_p - kv\mu'_k)$$
$$= \frac{e^2\omega_p^2}{2\pi v^3}\sum_{s=0}^{\infty}\int_0^\infty \frac{dk}{k} T_s(k)\int_{-1}^{1} d\mu'_k \int_0^{2\pi} d\phi'_k P_s(\mu_k)\delta(\mu'_k - \omega_p/kv)\,. \tag{18}$$

The Legendre expansion (12) was substituted and the argument of the δ-function is rewritten [with $\delta(\lambda x - a) = |\lambda|^{-1}\delta(x - a/\lambda)$].

The crucial point is that the argument of P_s in (18) is μ_k rather than μ'_k: $T(\boldsymbol{k})$ can be expanded in a Legendre series (12) only with respect to the physical axis of symmetry, which is not the z'-axis. The quantity μ_k in (18) is to be regarded as a function of μ'_k. Now the addition theorem for spherical harmonics $Y_l^m(\theta, \phi)$ allows one to express $P_s(\mu_k)$ explicitly in terms of θ'_k and ϕ'_k (Jackson 1963, § 3.5; Brink and Satchler 1968, § 2.6; the Y_l^m are defined in Appendix C):

$$P_s(\mu_k) = \frac{4\pi}{2s+1}\sum_{m=-s}^{+s} Y_s^m(\theta_p, \pi)^* Y_s^m(\theta'_k, \phi'_k)\,. \tag{19}$$

The following steps are now made sequentially. Relation (19) is inserted in (18) and the ϕ'_k-integration is done via (Jackson 1963, relations 3.53 and 3.57):

$$\int_0^{2\pi} d\phi'_k Y_s^m(\theta'_k, \phi'_k) = 2\pi\delta_{m,0}Y_s^0(\theta'_k, \phi'_k) = 2\pi\delta_{m,0}\left(\frac{2s+1}{4\pi}\right)^{1/2} P_s(\mu'_k)\,. \tag{20}$$

Using this relation again, $Y_s^0(\theta_p, \pi)$ is expressed in terms of $P_s(\mu_p)$:

$$Y_s^0(\theta_p, \pi) = \left(\frac{2s+1}{4\pi}\right)^{1/2} P_s(\mu_p)\,. \tag{21}$$

Finally, the integration over μ'_k is done via the δ-function. We state the result for all three coefficients together; for $D_{p\theta}$ and $D_{\theta\theta}$ there is a slight complication due to the presence of factors $\sin\theta'_k$ and $\cos\phi'_k$, which is handled most easily by expressing the relevant combination in terms of Y_l^m (Jackson 1963, p. 66). As a result, the associated Legendre functions P_s^1 and P_s^2 (Appendix C) occur there:

$$\begin{Bmatrix} D_{pp} \\ D_{p\theta} \\ D_{\theta\theta} \end{Bmatrix} = \frac{e^2\omega_p^2}{v^3}\sum_{s=0}^{\infty}\int_{\omega_p/v}^{\infty}\frac{dk}{k}T_s(k)\times\begin{Bmatrix} P_s(\omega_p/kv)P_s(\mu_p) \\ -\left[\left(\frac{kv}{\omega_p}\right)^2 - 1\right]^{1/2}[s(s+1)]^{-1}P_s^1(\omega_p/kv)P_s^1(\mu_p) \\ \frac{1}{2}\left[\left(\frac{kv}{\omega_p}\right)^2 - 1\right]\left\{P_s\left(\frac{\omega_p}{kv}\right)P_s(\mu_p) + \frac{(s-2)!}{(s+2)!}P_s^2\left(\frac{\omega_p}{kv}\right)P_s^2(\mu_p)\right\} \end{Bmatrix}. \tag{22}$$

These are the desired expressions; the lower boundary of integration is now ω_p/v instead of zero, due to the fact that the δ-function is zero for $k < \omega_p/v$.

b) Isotropic Waves

The associated Legendre functions P_l^m are zero for $l < m$ (Appendix C). Hence, for isotropic Langmuir waves, having all $T_n(k) = 0$ except $T_0(k)$, one has $[P_0(x) \equiv 1]$:

$$D_{pp} = \frac{e^2\omega_p^2}{v^3}\int_{\omega_p/v}^{\infty}\frac{dk}{k}T_0(k)\,, \tag{23a}$$

$$D_{p\theta} = 0\,, \tag{23b}$$

$$D_0 \equiv D_{\theta\theta} = \frac{e^2}{2v}\int_{\omega_p/p}^{\infty} kdkT_0(k)[1 - (\omega_p/kv)^2]\,. \tag{23c}$$

It is interesting to note that $D_{\theta\theta}$ is nonzero for isotropic Langmuir waves. On inserting (23a, b, c) in (13), there is a term which describes "isotropic" pitch-angle scattering, namely,

$$\left(\frac{df}{dt}\right)_{sc} = \frac{1}{p^2} D_0(p) \frac{\partial}{\partial \mu_p} (1 - \mu_p^2) \frac{\partial f}{\partial \mu_p} \cdot \tag{24}$$

At first sight it might be surprising that isotropic Langmuir waves should lead to pitch-angle scattering. However, the wave-particle interaction must tend to make the particles isotropic (if they interact with a fixed isotropic distribution of waves), and a term of the form (24) must be present to allow this.

c) Coefficients for the Langmuir Wave Equation (2)

Explicit expressions for γ_k and α_k in terms of the expansion coefficients $\{f_n\}$ of f have been derived by Melrose and Stenhouse (1977). Here, we briefly outline how the integration over $\boldsymbol{p}$ in (9) and (10) is done along the lines expounded above. A frame x'', y'', z'' is attached to $\boldsymbol{k}$ (just reverse the rôles of $\boldsymbol{k}$ and $\boldsymbol{p}$ in Fig. 2), and the integration is performed in spherical coordinates p, θ''_p, ϕ''_p attached to the doubly primed frame. With $\boldsymbol{k}\cdot(\partial/\partial\boldsymbol{p}) = k(\partial/\partial p_{z''})$, one obtains for γ_k:

$$\gamma_k = \left(\frac{2\pi e}{k}\right)^2 \omega_p \int_0^\infty \frac{p^2}{v} dp \int_{-1}^1 d\mu''_p \int_0^{2\pi} d\phi''_p \, \delta(\mu''_p - \omega_p/kv) \frac{\partial}{\partial p_{z''}} f(\boldsymbol{p})$$

now

$$\frac{\partial}{\partial p_{z''}} = \mu''_p \frac{\partial}{\partial p} + \frac{1 - \mu''^2_p}{p} \frac{\partial}{\partial \mu''_p}$$

and

$$f(\boldsymbol{p}) = \sum_{s=0}^\infty f_s(p) \frac{4\pi}{2s+1} \sum_{m=-s}^{s} Y_s^m(\theta_k, \pi)^* Y_s^m(\theta''_p, \phi''_p)$$

(cf. [11] and [19]). After insertion of these relations, integrate over ϕ''_p with (20), use (21), and integrate over μ''_p via the δ-function, resulting in:

$$\gamma_k = \frac{(2\pi)^3 e^2 \omega_p}{k^2} \sum_{s=0}^\infty P_s(\mu_k) \int_{p_\phi}^\infty (p^2/v) dp \left\{ \frac{\omega_p}{kv} P_s\left(\frac{\omega_p}{kv}\right) \frac{\partial f_s}{\partial p} + \frac{1 - (\omega_p/kv)^2}{p} P'_s\left(\frac{\omega_p}{kv}\right) f_s(p) \right\}, \tag{25}$$

with

$$p_\phi = m v_\phi (1 - v_\phi^2/c^2)^{-1/2}; \qquad v_\phi = \omega_p/k; \qquad P'_s(x) = \frac{d}{dx} P_s(x) . \tag{26}$$

Following Melrose and Stenhouse (1977), $(\partial/\partial p) f_s$ is removed by partial integration, to retrieve their relations (29) and (36):

$$\begin{aligned}
\gamma_k &= \gamma_k^I + \gamma_k^R + \gamma_k^A , \\
\gamma_k^I &= -\frac{(2\pi)^3 e^2 \omega_p^2}{k^3} \frac{p_\phi^2}{v_\phi^2} \sum_{s=0}^\infty P_s(\mu_k) f_s(p_\phi) = -\frac{(2\pi)^3 e^2 \omega_p^2}{k^3} \frac{p_\phi^2}{v_\phi^2} f(p_\phi \hat{k}) , \\
\gamma_k^R &= -\frac{(2\pi)^3 e^2 \omega_p^2}{k^3} \sum_{s=0}^\infty P_s(\mu_k) \int_{p_\phi}^\infty dp \frac{2p}{c^2} P_s\left(\frac{\omega_p}{kv}\right) f_s(p) , \\
\gamma_k^A &= -\frac{(2\pi)^3 e^2 \omega_p^2}{k^3} \left[\left(\frac{\omega_p}{kc}\right)^2 - 1\right] \sum_{s=0}^\infty P_s(\mu_k) \int_{p_\phi}^\infty dp \frac{p}{v v_\phi} P'_s\left(\frac{\omega_p}{kv}\right) f_s(p) ;
\end{aligned} \tag{27}$$

and the coefficient of spontaneous emission is

$$\alpha_k = \frac{(2\pi)^3 e^2 \omega_p^2}{k^3} \sum_{s=0}^\infty P_s(\mu_k) \int_{p_\phi}^\infty \frac{p^2}{v} dp P_s\left(\frac{\omega_p}{kv}\right) f_s(p) . \tag{28}$$

Because $p_\phi \uparrow \infty$ as $k \downarrow (\omega_p/c)$, γ_k and α_k approach zero for $k \downarrow (\omega_p/c)$ and are identically zero for $k \leq (\omega_p/c)$. The nonrelativistic limit obtains by taking $c \uparrow \infty$. In that case $\gamma_k^R \downarrow 0$ (it is a purely relativistic term); otherwise there is little change: $p_\phi \to m v_\phi$ and $[(\omega_p/kc)^2 - 1] \to -1$. The reader is referred to Melrose and Stenhouse (1977) for a discussion of the physical meaning of the three terms contributing to γ_k.

IV. COMPLETE LEGENDRE EXPANSION

In this section we shall use the inner product notation for the integration over μ_p or μ_k (see Appendix C). Complete expansion is obtained by taking the inner product of equation (13) with $P_n(\mu_p)$ and of equation (2) with $P_n(\mu_k)$, using in addition expansions (11) and (12), together with the explicit expressions (22), (27), and (28). In this way (coupled) equations for the components $f_n(p, z, t)$ and $T_n(k, z, t)$ are derived, which in turn admit a condensed matrix notation. Starting with (1) and (13), left-hand side, one has

$$\left(P_n, \frac{df}{dt}\right) = \sum_l (P_n, P_l)\frac{\partial f_l}{\partial t} + v\sum_l (P_n, \mu_p P_l)\frac{\partial f_l}{\partial z}\,.$$

Now $\mu_p P_l(\mu_p)$ is reexpressed in terms of $P_{l\pm 1}$ using equation (C4), and all inner products are then trivial (cf. [C8]). The result is written as

$$\frac{2n+1}{2}\left(P_n, \frac{df}{dt}\right) = \left(\frac{\partial}{\partial t} + v\mathbf{H}\frac{\partial}{\partial z}\right)\boldsymbol{f}\Big|_{n\text{th comp.}}\,, \tag{29}$$

where (cf. [C4] and [C8])

$$H_{nl} = \frac{2n+1}{2}(P_n, xP_l) = \frac{n}{2n-1}\delta_{n,l+1} + \frac{n+1}{2n+3}\delta_{n,l-1}\,, \tag{30}$$

$$\mathbf{H} = \begin{pmatrix} 0 & 1/3 & 0 & 0 & 0\ldots \\ 1 & 0 & 2/5 & 0 & 0 \\ 0 & 2/3 & 0 & 3/7 & 0 \\ 0 & 0 & 3/5 & 0 & 4/9 \\ 0 & 0 & 0 & 4/7 & 0\ldots \\ \vdots & & & & \vdots \end{pmatrix} \quad \text{and} \quad \boldsymbol{f} = \begin{pmatrix} f_0(p, z, t) \\ f_1(p, z, t) \\ \vdots \end{pmatrix}\,. \tag{31}$$

Any square, even × even section $\{H_{nl}\}$, $0 \le n, l \le 2N+1$, of $\mathbf{H}$ is nonsingular, but all odd × odd sections, $0 \le n, l \le 2N$, have zero determinant.

The right-hand side of equation (13) is somewhat more complicated. After taking the inner product with P_n, substitution of the expansion (11) and appropriate partial integrations, one reexpresses all derivatives of Legendre polynomials in terms of associated Legendre functions (C2). One then obtains the desired equation for $\boldsymbol{f}$:

$$\left(\frac{\partial}{\partial t} + v\mathbf{H}\frac{\partial}{\partial z}\right)\boldsymbol{f} = \frac{1}{p^2}\left\{\frac{\partial}{\partial p}\mathbf{P}\frac{\partial}{\partial p} + \frac{\partial}{\partial p}\mathbf{Q} + \mathbf{R}\frac{\partial}{\partial p} + \mathbf{S}\right\}\boldsymbol{f} \tag{32}$$

with

$$P_{nl} = \frac{2n+1}{2}p^2(P_n, D_{pp}P_l)\,, \tag{33a}$$

$$Q_{nl} = \frac{2n+1}{2}p(P_n, D_{p\theta}P_l^1) + p^2 A_p \delta_{nl}\,, \tag{33b}$$

$$R_{nl} = -\frac{2n+1}{2}p(P_n^1, D_{p\theta}P_l)\,, \tag{33c}$$

$$S_{nl} = -\frac{2n+1}{2}(P_n^1, D_{\theta\theta}P_l^1)\,. \tag{33d}$$

Equation (2) is handled in the same way. Its left-hand side is written in terms of k, μ_k using

$$\frac{\partial \omega_k}{\partial z}\cdot\frac{\partial}{\partial k_z} = \frac{\omega_p}{L}\left(\mu_k\frac{\partial}{\partial k} + \frac{1-\mu_k^2}{k}\cdot\frac{\partial}{\partial \mu_k}\right)$$

with

$$L(z) = 2n_0(\partial n_0/\partial z)^{-1}\,. \tag{34}$$

With the aid of (C5), (C4), and (C8) one obtains the equation for the waves:

$$\left\{\frac{\partial}{\partial t} + \omega_p\mathbf{H}\left(\frac{3k}{k_D^2}\frac{\partial}{\partial z} - \frac{1}{L}\frac{\partial}{\partial k}\right) + \frac{\omega_p}{kL}\mathbf{J}\right\}\boldsymbol{T} = \boldsymbol{\Gamma}\boldsymbol{T} + \boldsymbol{\sigma} \tag{35}$$

with

$$J_{nl} = \frac{l(l+1)}{2l+1}(\delta_{n,l+1} - \delta_{n,l-1}) \tag{36}$$

$$\mathbf{J} = \begin{pmatrix} 0 & -2/3 & 0 & 0 & 0\ldots \\ 0 & 0 & -6/5 & 0 & 0 \\ 0 & 2/3 & 0 & -12/7 & 0 \\ 0 & 0 & 6/5 & 0 & -20/9 \\ 0 & 0 & 0 & 12/7 & 0\ldots \\ \vdots & & & & \vdots \end{pmatrix} \quad \text{and} \quad \boldsymbol{T} = \begin{pmatrix} T_0(k, z, t) \\ T_1(k, z, t) \\ \vdots \end{pmatrix}, \tag{37}$$

$$\Gamma_{nl} = \frac{2n+1}{2}(P_n, \gamma_k P_l); \qquad \sigma_n = \frac{2n+1}{2}(P_n, \alpha_k). \tag{38}$$

Because the first column of **J** is zero, **J** itself as well as any square section $\{J_{nl}\}$, $0 \leq n, l \leq N$, has zero determinant.

We close this section by obtaining explicit expressions for **P** through **S** in (33), **Γ** and **σ** in (38). Inspection of (22) and (27) shows that inner products of the type (P_n, P_sP_l), $(P_n, P_s{}^1P_l{}^1)$, and $(P_n{}^1, P_s{}^2P_l{}^1)$ must be evaluated. Starting with **Γ** and **σ**, one has

$$\Gamma_{nl} = -\frac{(2\pi)^3 e^2 \omega_p{}^2}{k^3} \sum_{s=|n-l|}^{n+l} \frac{2n+1}{2} A_{nsl} \tag{39}$$

$$\times \left\{\frac{p_\phi{}^2}{v_\phi{}^2} f_s(p_\phi) + \int_{p_\phi}^{\infty} dp \frac{2p}{c^2} P_s\left(\frac{\omega_p}{kv}\right) f_s(p) + \left[\left(\frac{\omega_p}{kc}\right)^2 - 1\right] \int_{p_\phi}^{\infty} dp \frac{p}{vv_\phi} P'_s\left(\frac{\omega_p}{kv}\right) f_s(p)\right\},$$

$$\sigma_n = \frac{(2\pi)^3 e^2 \omega_p{}^2}{k^3} \int_{p_\phi}^{\infty} \frac{p^2}{v} dp P_n\left(\frac{\omega_p}{kv}\right) f_n(p). \tag{40}$$

A_{nsl} is defined by

$$A_{nsl} = \int_{-1}^{1} P_n(x) P_s(x) P_l(x) dx = 2\begin{pmatrix} n & s & l \\ 0 & 0 & 0 \end{pmatrix}^2. \tag{41}$$

The quantity $\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ is a $3-j$ symbol (Appendix E). A_{nsl} is invariant under any index permutation, and $A_{nsl} \geq 0$. It is nonzero only when $n + s + l$ is even *and* when the maximum of n, s, and l is no greater than the sum of the other two. This "triangle property," in the form $|n - l| \leq s \leq n + l$, is taken into account in the sums in (39) and below. The explicit expression for A_{nsl} is given in Appendix E; for practical purposes the tabulation of Rotenberg *et al.* (1959) is very convenient.

Continuing with **P** through **S**, substitution of (22) in (33) gives

$$P_{nl} = e^2 \omega_p{}^2 \frac{p^2}{v^3} \sum_{s=|n-l|}^{n+l} \frac{2n+1}{2} A_{nsl} \int_{\omega_p/v}^{\infty} \frac{dk}{k} T_s(k) P_s(\omega_p/kv), \tag{42}$$

$$Q_{nl} = (e\omega_p p/v)^2 \log(v/v_e)\delta_{nl} - e^2\omega_p{}^2 \frac{p}{v^3} \sum_{s=|n-l|}^{n+l} \frac{2n+1}{2s(s+1)} \lambda^{(1)}{}_{nsl} A_{nsl} \int_{\omega_p/v}^{\infty} \frac{dk}{k}\left[\left(\frac{kv}{\omega_p}\right)^2 - 1\right]^{1/2} T_s(k) P_s{}^1\left(\frac{\omega_p}{kv}\right), \tag{43}$$

$$R_{nl} = e^2\omega_p{}^2 \frac{p}{v^3} \sum_{s=|n-l|}^{n+l} \frac{2n+1}{2s(s+1)} \lambda^{(1)}{}_{lsn} A_{nsl} \int_{\omega_p/v}^{\infty} \frac{dk}{k}\left[\left(\frac{kv}{\omega_p}\right)^2 - 1\right]^{1/2} T_s(k) P_s{}^1\left(\frac{\omega_p}{kv}\right). \tag{44}$$

The quantity $\lambda^{(1)}{}_{nsl}$ is defined by (Appendix E):

$$\int_{-1}^{1} P_n(x) P_s{}^1(x) P_l{}^1(x) dx = \lambda^{(1)}{}_{nsl} A_{nsl}, \qquad \lambda^{(1)}{}_{nsl} = \tfrac{1}{2}[s(s+1) + l(l+1) - n(n+1)]. \tag{45}$$

Note the difference in index sequence to $\lambda^{(1)}$ in (43) and (44). Finally we have

$$S_{nl} = -\frac{e^2\omega_p{}^2}{v^3} \sum_{s=|n-l|}^{n+l} \frac{2n+1}{2} A_{nsl} \int_{\omega_p/v}^{\infty} \frac{dk}{k}\frac{1}{2}\left[\left(\frac{kv}{\omega_p}\right)^2 - 1\right] T_s(k)\left[\lambda^{(1)}{}_{snl} P_s\left(\frac{\omega_p}{kv}\right) + \frac{(s-2)!}{(s+2)!}\lambda^{(2)}{}_{snl} P_s{}^2\left(\frac{\omega_p}{kv}\right)\right] \tag{46}$$

TABLE 1

NONZERO VALUES OF A_{nsl} FOR $0 \leq n \leq s \leq l \leq 3$

$n\ s\ l$	A_{nsl}	
0 0 0	2	2.00
0 1 1	2/3	0.667
1 1 2	4/15	0.267
0 2 2	2/5	0.400
2 2 2	4/35	0.114
1 2 3	6/35	0.171
0 3 3	2/7	0.286
2 3 3	8/105	0.0762

with (Appendix E):

$$\int_{-1}^{1} P_s^2(x)P_n^1(x)P_l^1(x)dx = \lambda^{(2)}{}_{snl}A_{nsl}\,,$$

$$\lambda^{(2)}{}_{snl} = \tfrac{1}{2}\{s(s+1)[n(n+1)+l(l+1)] - [n(n+1)-l(l+1)]^2\}\,. \tag{47}$$

V. DISCUSSION

The new results derived in this paper are the explicit expressions (16), (22), together with the fully expanded equations (32) and (35), supplemented with (39), (40), (42)–(44), and (46). These equations apply only to unmagnetized particles interacting with Langmuir waves.

One of the motivations for the expansion of the quasi-linear equations in §§ III and IV was the hope that it might facilitate discussion of quasi-linear relaxation. There is clearly a need for this, because quasi-linear relaxation in three dimensions is inadequately understood and there is even disagreement on the qualitative form of the asymptotic state. The calculations of Ivanov, Soboleva, and Yushmanov (1975) and Appert, Tran, and Vaclavik (1976) suggest that the angular distributions of waves and particles do not become very narrow, indicating that one must expect to gain from an expansion in Legendre polynomials, in particular if the gross features of the angular distributions can be caught with the first few terms.

Equations (1) and (2), under axisymmetric conditions, require advancing in time of two functions, each depending on three other coordinates (p, θ_p, z and k, θ_k, z). Legendre expansion removes the coordinates θ_p and θ_k, at the expense of dealing with two vector equations (32) and (35), instead. The largest gain is likely to be in the area of numerical studies, where one can expect the dimension of these vector equations (i.e., the order at which the expansion is truncated) to be smaller than the number of points to be considered on a θ_p and θ_k mesh, otherwise. Also, one does no longer have to do a (nasty) numerical evaluation of (7), (9), and (10); instead, the integrations in (39), (40), (42)–(46) are one-dimensional and numerically straightforward.

In order to obtain a rough estimate of how many terms are required in the Legendre expansion, let us consider a streaming instability. In one dimension it is known that quasi-linear relaxation becomes an important effect after 10 to 20 growth times (Grognard 1975). The qualitative reason for this was given by Melrose (1974): the waves which are amplified are those generated in spontaneous emission, and the number of growth times required is roughly one (for spontaneous emission to build up the initial distribution) plus the natural logarithm of the ratio of the energy density in the stream to the energy density in the growing waves when amplifications first become effective, i.e., for

$$T_0(k) \approx pv|_{k=\omega_p/v} \qquad \text{or} \qquad T_0(k) \approx \kappa T_e(k_D/k)^2$$

this logarithm is typically between 10 and 20. From a semiquantitative viewpoint, it is clear that this argument should apply equally well to the three-dimensional case. Now, in three dimensions, when the initial distribution of particles is expanded in Legendre polynomials, it is reasonable to suppose that the terms $f_0(p)$ and $f_1(p)$ are the dominant ones for a streaming distribution. (This would not be so for a nearly one-dimensional stream, but streams in practice are unlikely to be highly collimated.) It is relatively simple to deduce how the instability develops initially in this case using (35), (39), and (40). The spontaneously emitted waves are dominated by the term $T_0(k)$ and $T_1(k)$. The growth coefficients $\Gamma_{nl}(k)$ are significant only for $n = l$ and $n = l \pm 1$ and decrease slowly with increasing l. Consequently $T_l(k)$ with $l > 1$ appear in stepwise fashion, with each term appearing about a growth time after the previous term. By the time quasi-linear relaxation becomes important, only terms with $l \lesssim 10$ to 20 are significant.

The foregoing argument leads to the tentative conclusion that it would be reasonable to truncate the expansions at $l \simeq 10$–20 at most. Consider what information would be lost by doing so. For particles confined to the outside of a shell ($p \geq p_0$) in $\boldsymbol{p}$-space (a "gap" distribution) with an angular distribution corresponding to $f_n(p) = 0$ for

$n \geq 2$, it is elementary to find the angular distribution of the waves in the gap as a function of time:

$$T(k, \theta_k) = \frac{\{\sigma_0(k) + \sigma_1(k) \cos \theta_k\}}{\gamma_1(k) \cos \theta_k} [\exp \{\gamma_1(k)t \cos \theta_k\} - 1] . \quad (48)$$

(Neglect of the relativistic correction described by $\gamma_k{}^R$ is unjustified for $\cos \theta_k \approx 0$.) The quantity $\gamma_1(k)$ is the P_1-term from (27). A qualitative feature is that, as time evolves, the waves grow in the forward hemisphere ($\cos \theta_k > 0$) but quickly reach a steady state in the backward hemisphere. It would appear that the development of significant components with higher and higher n as time evolves may be attributed to the increasing asymmetry between forward and backward hemispheres. By truncating the expansion one would expect to obtain a reasonable description of the dominant distribution in the forward hemisphere and a poorer description of the distribution in the backward hemisphere. It seems unlikely that significant errors in the overall description of quasi-linear relaxation would result from this loss of information on the actual distribution in the backward hemisphere.

Finally, inspection of (32) indicates that due to the extra term $v\mathbf{H}\, \partial f/\partial z$, the evolution of an inhomogeneous stream may be quite different from that of a homogeneous stream. In particular, free propagation competes with quasi-linear diffusion in causing a stepwise appearance of higher and higher l-values. No discussion of the realistic case of an inhomogeneous three-dimensional stream seems to exist.

VI. APPLICATION TO TYPE III STREAMS

The most important motivation for the extension of the theory presented here is the application to the propagation of electron streams of astrophysical interest, such as streams generating solar flare hard X-ray emission or type III solar radio bursts. We shall now discuss briefly some implications for the latter. Most discussions of quasi-linear relaxation of type III streams have been based on a one-dimensional treatment, either on the assumption that the growth rate is of order $\omega_p(n_s/n_0)(v/\Delta v)^2$ for a stream with number density n_s and spread Δv about the mean speed v, or explicitly in studying the evolution numerically (Grognard 1975; Magelssen 1976; Takakura and Shibahashi 1976). However, as pointed out by Melrose and Stenhouse (1977) and Melrose (1977), when a realistic pitch-angle distribution is taken into account, the growth rate can be much smaller than estimated in the one-dimensional case. For quite plausible pitch-angle distributions the growth rate could be less than $\omega_p(n_s/n_0)$. In other words, existing discussions of the propagation of type III streams are based on an oversimplified theory which predicts too large a growth rate by perhaps several orders of magnitude. Consequently, it is important to explore the quasi-linear relaxation of streams with realistic pitch-angle distributions. As mentioned above, we believe that the expansion in Legendre polynomials should facilitate numerical treatment of quasi-linear relaxation in the general case. Currently we are using the equations derived here to develop a numerical code to treat the quasi-linear relaxation of streams with realistic pitch-angle distributions with the application to electron streams generating emission of type III solar radio bursts or solar flare hard X-rays in mind.

By the way of illustration, the dilemma pointed out by Sturrock (1964) may be irrelevant in practice. Specifically Sturrock suggested that for a stream with $n_s/n_0 \approx 10^{-4}$ and $(v/\Delta v)^2 \approx 10^2$, the growth rate at about the 100 MHz plasma level would imply that the stream should lose all its energy to Langmuir waves after propagating only about a meter. However, for a realistic pitch-angle distribution giving a growth rate of order $\omega_p(n_s/n_0)$, with a more plausible choice of $n_s/n_0 \approx 10^{-7}$ and with at least 100 growth times before quasi-linear relaxation nears completion, the time required for quasi-linear relaxation to become important is about 1 second. The estimate of 100 times for quasi-linear relaxation to lead to a substantial loss of energy by the stream is based on the one-dimensional treatment of Grognard (1975), and in three dimensions the relaxation may require significantly more growth times. If such be the case, then Sturrock's dilemma does not exist in its original form because the stream should take only a second or so to pass any fixed point. The various nonlinear suppression mechanisms which have been invoked to overcome Sturrock's dilemma (cf. Kaplan and Tsytovich 1967; Papadopoulos, Goldstein, and Smith 1974; and the reviews by Smith 1974 and Kaplan, Pikel'ner, and Tsytovich 1974) would then not be directly relevant. Similarly, there would be insufficient time available for the initial quasi-linear relaxation to occur in the inhomogeneous beam model of Zaitsev, Mityakov, and Rapoport (1972). Briefly, if it proves correct that quasi-linear relaxation is much slower than existing one-dimensional treatments imply, a radical rethinking of our ideas on how type III streams propagate may well be required.

The work reported in § III was done in collaboration with Mr. J. E. Stenhouse. One of us (D. B. M.) would like to thank Dr. R. J.-M. Grognard of CSIRO Division of Radiophysics, Sydney, for helpful comments.

APPENDIX A

COULOMB INTERACTIONS

A single electron in a fully ionized plasma may be regarded as experiencing interactions with its neighbors for impact parameters smaller than λ_D (Coulomb interactions) and as interacting collectively with all other electrons having impact parameters $> \lambda_D$. The latter interaction can be visualized as a (single) electron–(Langmuir) wave interaction. In this Appendix we restrict ourselves to the nonrelativistic case, because so far only nonrelativistic treatment of Coulomb interactions are available. Also, the nonrelativistic formulae do not lend themselves to any obvious relativistic generalization.

It is understood, therefore, that the formulae below are limited to the case $\gamma = 1$, $c = \infty$, $p = mv$. Coulomb and wave interaction have been shown to give rise to two additive terms in the Fokker-Planck coefficients **D** and A (Davidson 1972, § 14.6; Montgomery and Tidman 1964, eq. [2.26] or [7.60]). Addition of the Coulomb part gives

$$\mathbf{D}^{c+w} = \mathbf{D} + \frac{(me\omega_p)^2}{p}\left[(\mathbf{I} - \hat{p}\hat{p}) + \left(\frac{v_e}{v}\right)^2 \hat{p}\hat{p}\right] \log \Lambda\,, \qquad A^{c+w} = A + (e\omega_p/v)^2 \hat{p} \log \Lambda\,, \tag{A1}$$

where **D** and A are given by (7) and (8), respectively. **I** is the unit tensor and log Λ the Coulomb logarithm; $\Lambda = 4\pi n_0 \lambda_D{}^3$, a measure of the number of electrons in one Debye sphere.

The expressions describing Coulomb interaction are relatively simple, because (A1), like equation (1), is restricted to suprathermal electrons ($v \geq v_e$). In that case Coulomb interactions between the suprathermal electrons themselves can be neglected. Consequently, (A1) describes Coulomb interaction of a $v > v_e$ electron with the (fixed) body of thermal particles, and hence the Coulomb operator is linear.

As a result of (A1), (16) and (22) transform into:

$$A_p \to A_p + (e\omega_p/v)^2 \log \Lambda\,, \qquad D_{pp} \to D_{pp} + \frac{(me\omega_p)^2}{p}\left(\frac{v_e}{v}\right)^2 \log \Lambda\,, \qquad D_{\theta\theta} \to D_{\theta\theta} + \frac{(me\omega_p)^2}{p} \log \Lambda\,.$$

A_θ and $D_{p\theta}$ remain unchanged; **R** from (44) does not change either, but

$$P_{nl} \to P_{nl} + \frac{(m^2 e\omega_p v_e)^2}{p}\,\delta_{nl} \log \Lambda\,, \qquad Q_{nl} \to Q_{nl} + (me\omega_p)^2\,\delta_{nl} \log \Lambda\,, \qquad S_{nl} \to S_{nl} - \frac{(me\omega_p)^2}{p}\,\delta_{nl} \log \Lambda n(n+1)\,.$$

APPENDIX B

ELECTRIC FIELD ASSOCIATED WITH REVERSE CURRENT

The reverse current $J_r = n_0 e v_r$ is driven by an electric field E_r and in a quasi-stationary state E_r is generated by small charge imbalances, which in turn are sustained by the beam electrons. The process is self-regulating, such that the total current $J_r + J_{\text{beam}}$ vanishes. The reverse current is carried by thermal electrons, and, assuming $v_e \ll c$ and quasi-stationary conditions, E_r and J_r are simply related through the Spitzer conductivity (Spitzer 1962), whence E_r can be expressed in terms of v_r and the Dreicer field $E_\text{D} = 4\pi e k_\text{D}{}^2$:

$$E_r \approx 2^{-6}(2\pi)^{-1/2} \log \Lambda \frac{v_r}{v_e} E_\text{D} \approx 0.1 \frac{v_r}{v_e} E_\text{D}$$

(log Λ = Coulomb logarithm; Appendix A). The corresponding force term $eE_r(\partial/\partial p_z)f$ in equation (1) is negligible with respect to the dynamic friction term $(\partial/\partial \boldsymbol{p})\cdot \boldsymbol{A} f$ if $eE_r \ll A_z$, or if, cf. (8):

$$\left(\frac{v}{v_e}\right)^2 \ll \frac{16}{\log \Lambda}\left(\frac{2}{\pi}\right)^{1/2} \frac{v_e}{v_r} |\mu_p| \log\left(\frac{v}{v_e}\right) \approx 0.6 \frac{v_e}{v_r} |\mu_p| \log\left(\frac{v}{v_e}\right) \tag{B1}$$

($\mu_p = \cos\theta_p$). Little is known about the value of v_r/v_e in beams of astrophysical interest. For an electron beam causing type III solar radio emission one finds, with due reserve, $v_r/v_e < 10^{-4}$ at 80 MHz (taking $T_e = 10^6$ K, a beam area of 10^{20} cm^2, and a beam strength of 10^{32} electrons s^{-1}. Consequently, (B1) would hold up to $v = c$ for μ_p of order unity.

Whether the reverse current electric field can be neglected for beams causing solar flare hard X-ray emission is at present uncertain, and we refer the reader to Hoyng, Knight, and Spicer (1978) for further details.

Note that only the spontaneous wave emission contribution to $\boldsymbol{A}$ was considered above, and, as a consequence, it is possible to reinforce our conclusion in specific cases: In the nonrelativistic case (at least) Coulomb interactions have a much larger contribution to $\boldsymbol{A}$, cf. (8) and (A1), since log $\Lambda \gg \log(v/v_e)$. As a consequence, the right-hand side of (B1) would be larger by a factor log $\Lambda/\log(v/v_e)$. Likewise, we have refrained from considering terms $\partial\mathbf{D}/\partial\boldsymbol{p}$ in equation (1), whose magnitude depends on the wave distribution $T(\boldsymbol{k})$.

APPENDIX C

LEGENDRE POLYNOMIALS

A few results on Legendre polynomials $P_n(x)$, associated Legendre functions $P_n^m(x)$ and spherical harmonics $Y_n^m(\theta, \phi)$ are collected here. Their definition is (Jackson 1963, chap. 3; Abramowitz and Stegun 1968):

$$P_n(x) = (2^n n!)^{-1} \frac{d^n}{dx^n} (x^2 - 1)^n , \qquad -1 \le x \le 1 ; \tag{C1}$$

$$P_n^m(x) = (-)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) , \qquad P_n^0 = P_n ; \tag{C2}$$

$$P_n^{-m}(x) = (-)^m \frac{(n-m)!}{(n+m)!} P_n^m(x);$$

$$Y_n^m(\theta, \phi) = \left[\frac{2n+1}{4\pi} \frac{(n-m)!}{(n+m)!}\right]^{1/2} P_n^m(\cos\theta) \exp(im\phi) . \tag{C3}$$

The following recursion relations are needed ($P'_n = dP_n/dx$):

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1} , \tag{C4}$$

$$(1 - x^2)P'_n = -nxP_n + nP_{n-1} = \frac{n(n+1)}{2n+1} [P_{n+1} - P_{n-1}] , \tag{C5}$$

$$P'_{n+1} = xP'_n + (n+1)P_n , \tag{C6}$$

$$(2n+1)P_n = P'_{n+1} - P'_{n-1} \tag{C7}$$

(Jackson 1963, p. 59). The P_n^m are *orthogonal*; that is,

$$(P_n^m, P_{n'}^{m'}) = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{nn'} \delta_{mm'} , \tag{C8}$$

where the usual inner product notation is used, defined by

$$(f, g) = \int_{-1}^{1} dx f(x) g(x) . \tag{C9}$$

The Legendre polynomials are not only orthogonal, $(P_n, P_m) = 2\delta_{nm}/(2n+1)$, but also *complete*. That is, any function $\phi(x, \ldots)$ defined on $-1 \le x \le 1$ has a unique expansion in terms of Legendre polynomials

$$\phi(x, \ldots) = \sum_{l=0}^{\infty} a_l(\ldots) P_l(x) . \tag{C10}$$

Other parameters on which ϕ may depend are collectively indicated by dots (...). By taking the inner product of (C10) with P_n one finds a_n:

$$a_n(\ldots) = \frac{2n+1}{2} (P_n, \phi) = \frac{2n+1}{2} \int_{-1}^{1} dx P_n(x) \phi(x, \ldots) . \tag{C11}$$

APPENDIX D

ALTERNATIVE INTEGRATION METHOD

The integration over $\boldsymbol{k}$ can also be done in spherical coordinates k, θ_k, ϕ_k, see Figure 2. Due to the axial symmetry, $\mathbf{D}(\boldsymbol{p})$ cannot depend on ϕ_p, and this allows one to take $\phi_p \equiv 0$ in the following. Instead of (17), one now obtains from (7), (12), and (15):

$$\begin{Bmatrix} D_{pp} \\ D_{p\theta} \\ D_{\theta\theta} \end{Bmatrix} = \frac{e^2}{2\pi v} \sum_{n=0}^{\infty} \int_0^{\infty} k dk T_n(k) \int_{-1}^{1} d\mu_k P_n(\mu_k) \begin{Bmatrix} (\omega_p/kv)^2 \\ (\omega_p/kv)A \\ A^2 \end{Bmatrix} \int_0^{2\pi} d\phi_k \delta(\hat{k}\cdot\hat{v} - \omega_p/kv) . \tag{D1}$$

Factors $\hat{k}\cdot\hat{e}_p$ and $\hat{k}\cdot\hat{e}_\theta$ were taken outside the ϕ_k-integral noting that

$$\hat{k}\cdot\hat{e}_p = \boldsymbol{k}\cdot\boldsymbol{v}/kv = \omega_p/kv$$

$$A \equiv \hat{k}\cdot\hat{e}_\theta|_{k\cdot v=\omega_p} = [(\omega_p/kv)\cos\theta_p - \cos\theta_k]/\sin\theta_p \tag{D2}$$

Melrose and Stenhouse (1977) have derived the identity

$$\int_0^{2\pi} d\phi_k \delta(\hat{k}\cdot\hat{v} - \omega_p/kv) = 2/F(\mu_p, \mu_k, \omega_p/kv) \quad (\mu_k^- < \mu_k < \mu_k^+)$$
$$= 0 \qquad \text{(otherwise)}$$

with $$F(a, b, c) = [1 + 2abc - a^2 - b^2 - c^2]^{1/2}, \qquad \mu_k^\pm = \cos[\theta_p \pm \arccos(\omega_p/kv)]. \tag{D3}$$

Putting $a = \omega_p/kv$ and $b = \cos\theta_p$, it is seen that the integration over μ_k in (D1) involves integrals of the type

$$I_n^m = \frac{1}{\pi}\int_{x_-}^{x_+} dx P_n(x)(ab - x)^m/F(x, a, b), \qquad m = 0, 1, 2;$$

$$x_\pm = ab \pm [(1 - a^2)(1 - b^2)]^{1/2}, \qquad F(x_\pm, a, b) = 0. \tag{D4}$$

The following result was obtained by Melrose and Stenhouse (1977):

$$I_n^0 = P_n(a)P_n(b); \tag{D5}$$

and this leads directly to (22), top line.

I_n^1 can be expressed in terms of I_n^0, writing $(ab - x)P_n = abP_n - xP_n$ and then inserting (C4) to obtain

$$I_n^1 = abP_n(a)P_n(b) - \frac{1}{2n + 1}[(n + 1)P_{n+1}(a)P_{n+1}(b) - nP_{n-1}(a)P_{n-1}(b)].$$

Now use (C4) again on $aP_n(a)$ and $bP_n(b)$, and rearrange terms to find (use [C5] for second equality):

$$I_n^1 = -\frac{n(n + 1)}{(2n + 1)^2}[P_{n+1}(a) - P_{n-1}(a)]\cdot[P_{n+1}(b) - P_{n-1}(b)] = -\frac{(1 - a^2)(1 - b^2)}{n(n + 1)}P'_n(a)P'_n(b), \tag{D6}$$

and this establishes (22), middle line, via (C2).

It is clear that I_n^m can be expressed in I_n^{m-1} following the same method; specifically, for I_n^2 one obtains:

$$I_n^2 = abI_n^1 - \frac{1}{2n + 1}[(n + 1)I_{n+1}^1 + nI_{n-1}^1]. \tag{D7}$$

In other words, I_n^2 contains P'_n, P'_{n+1}, and P'_{n-1} (of argument a and b). I_n^2 can be rewritten in terms of P_n and P''_n, and the strategy is as follows: From the differential equation for the Legendre polynomial P_n one expresses P'_n in terms of P''_n and P_n, symbolically, $P'_n \to P_n, P''_n$. Next, via (C6), $P'_{n+1} \to P_n, P'_n$ and hence $P'_{n+1} \to P_n, P''_n$. Further, $P'_{n-1} \to P_n, P'_{n+1}$ via (C7) and hence also $P'_{n-1} \to P_n, P''_n$. Passing by details, the result is

$$I_n^2 = \tfrac{1}{2}(1 - a^2)(1 - b^2)\left[P_n(a)P_n(b) + \frac{(1 - a^2)(1 - b^2)}{(n - 1)n(n + 1)(n + 2)}P''_n(a)P''_n(b)\right]$$
$$= \tfrac{1}{2}(1 - a^2)(1 - b^2)\left[P_n(a)P_n(b) + \frac{P_n^2(a)P_n^2(b)}{(n - 1)n(n + 1)(n + 2)}\right], \tag{D8}$$

where (C2) was used. This completes the derivation of (22).

APPENDIX E

3 − j SYMBOLS

To establish the identities (41), (45), and (47), it is convenient to use 3 − j symbols, as was done by Hoyng (1977) in a discussion of nonlinear Landau damping. One has the relation

$$\iint d\Omega\, Y_n^{a*} Y_s^b Y_l^c = (-)^a\left[\frac{(2n + 1)(2s + 1)(2l + 1)}{4\pi}\right]^{1/2}\begin{pmatrix} n & s & l \\ -a & b & c\end{pmatrix}\begin{pmatrix} n & s & l \\ 0 & 0 & 0\end{pmatrix} \tag{E1}$$

(Brink and Satchler 1968; Rotenberg 1959). Since integration over ϕ gives zero if $-a + b + c \neq 0$, (E1) can be rewritten in terms of associated Legendre functions, substituting (C3):

$$\int_{-1}^{1} dx P_n{}^a P_s{}^b P_l{}^c = 2(-)^a \left[\frac{(n+a)!(s+b)!(l+c)!}{(n-a)!(s-b)!(l-c)!}\right]^{1/2} \begin{pmatrix} n & s & l \\ -a & b & c \end{pmatrix} \begin{pmatrix} n & s & l \\ 0 & 0 & 0 \end{pmatrix} . \tag{E2}$$

(Contrary to appearance, the entry a does not play an asymmetric role.) With $a = b = c = 0$ one obtains (41). The explicit expression for *nonzero* A_{nsl} is (Brink and Satchler 1968, § 2.7 and Appendix I):

$$A_{nsl} = 2\begin{pmatrix} n & s & l \\ 0 & 0 & 0 \end{pmatrix}^2 = 2\,\frac{(n+s-l)!(n+l-s)!(s+l-n)!}{(n+s+l+1)!}\left[\frac{\Sigma!}{(\Sigma-n)!(\Sigma-s)!(\Sigma-l)!}\right]^2 , \tag{E3}$$

where $2\Sigma = n + s + l$ should be even. The overall sign of the $3 - j$ symbol can remain undefined here. Equation (E2) implies

$$\int_{-1}^{1} dx P_s{}^1 P_n P_l{}^1 = -2\left[\frac{(s+1)!(l+1)!}{(s-1)!(l-1)!}\right]^{1/2} \begin{pmatrix} s & n & l \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} s & n & l \\ 0 & 0 & 0 \end{pmatrix} , \tag{E4}$$

$$\int_{-1}^{1} dx P_s{}^2 P_n{}^1 P_l{}^1 = 2\left[\frac{(s+2)!(n+1)!(l+1)!}{(s-2)!(n-1)!(l-1)!}\right]^{1/2} \begin{pmatrix} s & n & l \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} s & n & l \\ 0 & 0 & 0 \end{pmatrix} . \tag{E5}$$

Because of the symmetry implied by the left-hand sides of both (E4) and (E5), one can arbitrarily permute the columns in any of the $3 - j$ symbols on the right (in general only true if $s + n + l$ is even).

Brink and Satchler (1968, Appendix I) quote the result

$$\begin{pmatrix} l & s & n \\ 1 & -1 & 0 \end{pmatrix} = \frac{n(n+1) - l(l+1) - s(s+1)}{2[l(l+1)s(s+1)]^{1/2}} \begin{pmatrix} l & s & n \\ 0 & 0 & 0 \end{pmatrix} \tag{E6}$$

for $l + s + n$ even, and this allows one to establish (45).

To establish (47) use the recursion formula (Brink and Satchler 1968, Appendix I)

$$[(s+c)(s-c+1)]^{1/2} \begin{pmatrix} n & l & s \\ a & b & c-1 \end{pmatrix} + [(n+a)(n-a+1)]^{1/2} \begin{pmatrix} n & l & s \\ a-1 & b & c \end{pmatrix} + [(l+b)(l-b+1)]^{1/2} \begin{pmatrix} n & l & s \\ a & b-1 & c \end{pmatrix} = 0$$

with $a = b = 1$ and $c = -1$, together with (E6) and the fact that $n + l + s$ even permits column permutation, to obtain

$$\begin{pmatrix} n & l & s \\ 1 & 1 & -2 \end{pmatrix} = \frac{s(s+1)\{n(n+1) + l(l+1)\} - \{n(n+1) - l(l+1)\}^2}{2[(s-1)s(s+1)(s+2)]^{1/2}[n(n+1)l(l+1)]^{1/2}} \begin{pmatrix} n & l & s \\ 0 & 0 & 0 \end{pmatrix} \tag{E7}$$

and insert (E7) in (E5).

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