Contributions

Theoretical

The Kinematics of Cyclotron Emission for Quantized Electrons

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Introduction

Since the work of Wu and Lee (1979) there has been renewed interest in the classical theory of electron cyclotron masers (Lee and Wu 1980, Lee *et al.* 1980, Wu *et al.* 1981, 1982, Hewitt *et al.* 1981, 1982, Melrose *et al.* 1982, Omidi and Gurnett 1982, Melrose and Dulk 1982). A useful idea in these recent developments of the classical theory concerns a geometric interpretation of the classical gyroresonance condition

$$\omega - s\Omega_e/\gamma - k_{\|}v_{\|} = 0 , \quad (1)$$

where Ω_e is the nonrelativistic gyrofrequency, $s = 0, \pm 1, \pm 2, \ldots$ is the harmonic number, $\gamma = (1 - v_\perp^2/c^2 - v_\|^2/c^2)^{-1/2}$ is the Lorentz factor and $\|$ and $\perp$ denote components parallel and perpendicular to the magnetic field. In $v_\perp - v_\|$ space (1) represents an ellipse with centre $v_\| = v_c$, $v_\perp = 0$, eccentricity e_0 and semi-major axis V parallel to the $v_\perp$ axis, with

$$\frac{v_c}{c} = \frac{k_\| c\omega}{k_\|^2c^2 + s^2\Omega_e^2} , \quad e_0^2 = \frac{k_\|^2c^2}{k_\|^2c^2 + s^2\Omega_e^2} ,$$

$$\frac{V^2}{c^2} = 1 - \frac{\omega^2}{k_\|^2c^2 + s^2\Omega_e^2} . \quad (2)$$

In this paper we generalize (1) and (2) to the quantum case. A quantum treatment is required for the theory of gyroemission in super-strong magnetic fields such as those of neutron stars. One of our specific interests is in the possibility of electron cyclotron maser emission of hard X-rays from neutron stars. However, in this paper we restrict our discussion to the kinematics of gyroemission in superstrong magnetic fields.

Energy and Momentum conservation

The energy of an electron in a magnetic field is

$$\varepsilon(n,p_\|) = \{m^2c^4 + p_\|^2c^2 + 2n\Omega_e\hbar mc^2\}^{\frac{1}{2}} \quad (3)$$

with $n = 0,1,2,\ldots$, and where m is the rest energy of the electron. The eigenvalues $n \geq 1$ are doubly degenerate with spin $s = \pm 1$ and orbital quantum number $\ell = n - \frac{1}{2}(s + 1)$; from the kinematic viewpoint it is irrelevant whether s changes or not during a transition, i.e. spin-flip and non-spin-flip transitions may be treated together.

Conservation of energy and momentum in gyroemission requires

$$\varepsilon(n,p_\|) = \varepsilon(n - s, p_\| - \hbar k_\|) + \hbar\omega . \quad (4)$$

In the classical limit ($\hbar \to 0$, $n \to \infty$, $\hbar n \to p_\perp^2/2m\Omega_e$), (4) reduces to (1). Squaring (4) twice leads to

$$(p_\| - p_c)^2c^2 = \frac{\omega^2}{4\hbar^2(\omega^2 - k_\|^2c^2)^2} \times$$

$$\times \left[\{\varepsilon_n^2 - \varepsilon_{n-s}^2 + \hbar^2(\omega^2 - k_\|^2c^2)\}^2 - \right.$$

$$\left. - 4\varepsilon_n^2\hbar^2(\omega^2 - k_\|^2c^2)\right] \quad (5)$$

with

$$\varepsilon_n^2 := m^2c^4 + 2n\Omega_e\hbar mc^2 \quad (6)$$

and with

$$p_c := \frac{k_\|\{\varepsilon_n^2 - \varepsilon_{n-s}^2 + \hbar^2(\omega^2 - k_\|^2c^2)\}}{2\hbar(\omega^2 - k_\|^2c^2)} , \quad (7)$$

where ":=" denotes a definition.

One infers that solution are real only for either

$$\hbar^2(\omega^2 - k_\|^2c^2) \leq (\varepsilon_n - \varepsilon_{n-s})^2 \quad (8)$$

or

$$\hbar^2(\omega^2 - k_\|^2c^2) \geq (\varepsilon_n + \varepsilon_{n-s})^2 . \quad (9)$$

The condition (8) applies to gyroemission. The regime (9) is incompatible with (4) and has appeared as a spurious solution after squaring. In fact (9) is the physical region for an equation obtained from (4) by changing the sign of the term $\epsilon(n - s, p_\| - \hbar k_\|)$, and this corresponds to pair creation. We are not interested in pair creation here and do not discuss the regime (9) below.

The inequality (8) is automatically satisfied for $\omega^2 < k_\|^2c^2$. For $\omega^2 > k_\|^2c^2$ emission is allowed only at $s > 0$, as in the classical case, and then $\epsilon_n > \epsilon_{n=s}$ in (8) implies

$$\hbar(\omega^2 - k_\|^2c^2)^{\frac{1}{2}} \leq \varepsilon_n - \varepsilon_{n-s} . \quad (8')$$

Velocity Space: The *n*-Ellipses

In order to introduce the idea of a resonant ellipse we need to define a velocity space. Unlike the classical case, our velocity space is now of no physical significance in itself. It is merely a convenient space for geometric constructions.

We introduce $v_\perp$ and $v_\parallel$ by writing

$$p_\parallel =: \gamma m v_\parallel , \qquad 2n\Omega_e \hbar m =: \gamma^2 m^2 v_\perp^2 , \quad (10a,b)$$

so that in the classical limit $v_\parallel$ and $v_\perp$ become the physical velocity components. The condition $\epsilon(n,p_\parallel) = \gamma mc^2$ then defines a set of physical curves, one for each value of n:

$$\frac{v_\parallel^2}{c^2} + \frac{v_\perp^2}{c^2}\,\frac{\varepsilon_n^2}{\varepsilon_n^2 - m^2c^4} = 1 . \quad (11)$$

These curves are ellipses which depend only on the value of n and of $\hbar\Omega_e$. We refer to the curve (11) as the n-ellipse. It represents the locus in velocity space of the physical states for a given n and $-\infty < p_\parallel < \infty$. Each n-ellipse has a semi-major axis equal to c along the $v_\perp = 0$ axis, and a semi-minor axis equal to $c\{(2nB/B_c)/(1 + 2nB/B_c)\}^{1/2}$ along the $v_\parallel = 0$ axis, where the critical field $B_c = 4.4 \times 10^9$ T $(= 4.4 \times 10^{13}$ G) may be defined by writing

$$\frac{\hbar\Omega_e}{mc^2} =: \frac{B}{B_c} . \quad (12)$$

The classical limit corresponds to $B_c \to \infty$ when the n-ellipses become densely packed and fill velocity space. In a superstrong magnetic field, e.g. for $B/B_C \gtrsim 0.1$ expected on neutron stars, the n-ellipses are well separated for small $n(= 0,1,2$ say) and the discreteness of the states is important.

Initial and Final Ellipses

The condition (4) may be used to derive two resonant ellipses, one for the initial state and one for the final state. Let the variables for the final state be denoted by primes. We have $p'_\parallel = p_\parallel - \hbar k_\parallel$ and $n' = n - s$. The variables $v'_\parallel$ and $v'_\perp$ are defined by analogy with (10a,b), and they satisfy (11) with each of $v_\parallel$, $v_\perp$ and n replaced by the corresponding primed quantity. Then squaring (4) one finds that it may be rewritten either in the form

$$\omega - \frac{1}{\gamma}\left\{s\Omega_e + \frac{\hbar(\omega^2 - k_\parallel^2c^2)}{2mc^2}\right\} - k_\parallel v_\parallel = 0 \quad (13a)$$

or in the form

$$\omega - \frac{1}{\gamma'}\left\{s\Omega_e - \frac{\hbar(\omega^2 - k_\parallel^2c^2)}{2mc^2}\right\} - k_\parallel v'_\parallel = 0. \quad (13b)$$

It follows by comparison with (2) and (4), that the initial (+) and final (−) ellipses have

$$\frac{v_{c\pm}}{c} = \frac{k_\parallel c\omega}{k_\parallel^2c^2 + (s\Omega_e)_\pm^2}, \qquad e_\pm^2 = \frac{k_\parallel^2c^2}{k_\parallel^2c^2 + (s\Omega_e)_\pm^2},$$

$$\frac{V_\pm^2}{c^2} = 1 - \frac{\omega^2}{k_\parallel^2c^2 + (s\Omega_e)_\pm^2} , \quad (14)$$

with

$$(s\Omega_e)_\pm := s\Omega_e \pm \frac{\hbar(\omega^2 - k_\parallel^2c^2)}{2mc^2} . \quad (15)$$

The allowed initial states are defined by the intersection of the initial ellipse with the n-ellipse, and similarly the allowed final states are defined by the intersection of the final ellipse with the (n-s)-ellipse.

Particular Cases

(a) $\omega^2 > k_\parallel^2c^2$

The case of most interest in practice is $\omega^2 > k_\parallel^2c^2$. In this case both ellipses lie entirely within the circle $v = c$, and for $\omega^2 >> k_\parallel^2c^2$ they lie well inside this circle. The final ellipse lies entirely within the initial ellipse.

No transitions are allowed if the initial ellipse does not intersect the n-ellipse. The threshold for emission is defined by the condition that the two ellipses touch. After some algebra one finds that this condition reduces to (8′). Similarly the condition that the final ellipse touches the (n-s)-ellipse also reduces to (8′). When the inequality in (8′) is satisfied, there are two allowed initial states and two allowed final states, as illustrated in Figure 1 (when the equality in (8′) is satisfied the

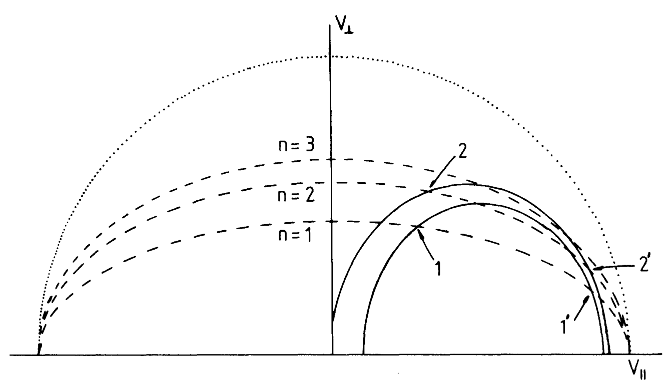

Figure 1. Velocity space is bordered by the axis $v_\perp = 0$, which is also the 0-ellipse, and a semi-circle $v = c$ shown dotted. The n-ellipses for $B/B_c = 1/8$ and $n = 1, 2, 3$ are shown as dashed curves. The initial and final ellipses are plotted as the outer and inner, respectively, solid curves. The parameters chosen for the resonant ellipses are $s = 1$, $\hbar\omega/mc^2 = 1.03\, B/B_c$ and $\hbar k_\parallel/mc = 0.7\, B/B_c$, corresponding to $\omega^2 k_\parallel^2 c^2 = 2.17$. The allowed transitions from $n = 2$ to $n = 1$ are from the points labelled 2 to 1 and from 2′ to 1′.

two initial and the two final states merge into a single initial and a single final state.) There are two allowed transitions in general. These are between the allowed initial and final states on the left in Figure 1, and between the allowed initial and final states on the right in Figure 1. Transitions are allowed only for $0 < s \leq n$.

(b) $\omega^2 = k_\parallel^2 c^2$

For $\omega^2 = k_\parallel^2 c^2$ the initial and final ellipses coincide and touch the circle $v = c$ at $v_\perp = 0$. This single ellipse necessarily cuts all n-ellipses. Transitions between any initial and a final state defined by such intersections are allowed. However again only values of s in the range $0 < s \leq n$ are permitted.

(c) $\omega^2 > k_\parallel^2 c^2$

In the case of $\omega^2 > k_\parallel^2 c^2$ the initial ellipse lies inside the final ellipse for $s > 0$ and outside the final ellipse for $s < 0$. For $s = 0$, the two ellipses are identical. Both ellipses touch the circle $v = c$ at the same point $v_\parallel/c = \omega/k_\parallel c$, $v_\perp/c = (1 - \omega^2/k_\parallel^2 c^2)^{1/2}$. This point separates each ellipse into two sections which we refer to as the inner section (nearer the centre $v_\parallel = 0$, $v_\perp = 0$) and an outer section. Only one section of each ellipse defines an allowed initial or final state. For $(s\Omega_e)_+$ or $(s\Omega_e)_-$ positive it is the inner section which defines an allowed state, and for $(s\Omega_e)_+$ or $(s\Omega_d)_-$ negative it is the outer section which defines an allowed state. These cases

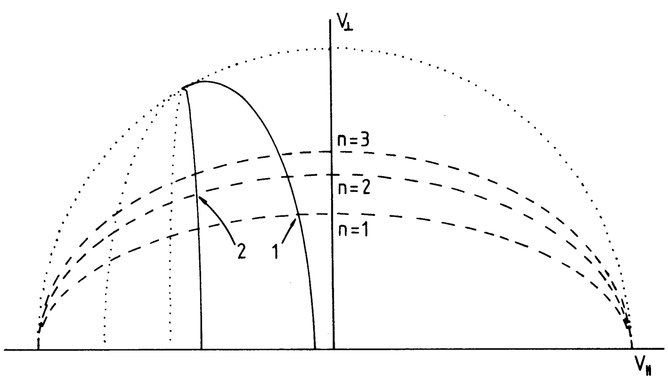

Figure 2. As for Figure 1 with $\hbar\omega/mc^2 = 2B/B_c$, $\hbar k_\parallel/mc = -4\ B/B_c$ and hence $\omega^2/k_\parallel^2 c^2 = 0.25$, $(s\Omega_e)_+ > 0$ and $(s\Omega_e)_- > 0$. Only one transition from $n = 2$ to $n = 1$ is allowed in this case due to the dotted sections of the resonant ellipses being non-physical.

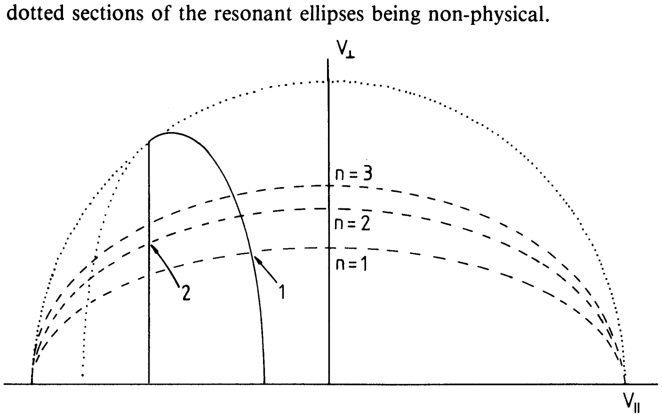

Figure 3. As for Figure 2 with $\hbar\omega/mc^2 = 3\ B/B_c$, $\hbar k_\parallel/mc = -5\ B/B_c$ and hence $\omega^2/k_\parallel^2 c^2 = 0.36$, $(s\Omega_e)_+ = 0$ and $(s\Omega_e)_- > 0$. In this case the initial ellipse has degenerated to a vertical line.

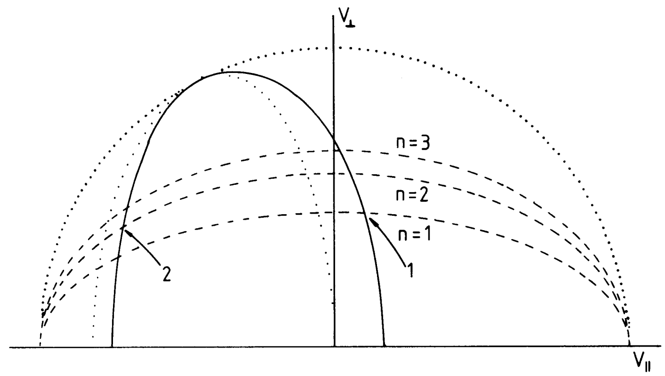

Figure 4. As for Figure 2 with $\hbar\omega/mc^2 = 5\ B/B_c$, $\hbar k_\parallel/mc = -11\ B/B_c$ and hence $\omega^2/k_\parallel^2 c^2 = 0.21$, $(s\Omega_e)_+ < 0$ and $(s\Omega_e)_- > 0$. In this case the initial ellipse has its physical section to the left of its non-physical section.

are separated by the special case $(s\Omega_e)_+$ or $(s\Omega_e)_-$ equal to zero when the relevant ellipse becomes a vertical line. Negative values of s are allowed in this case, and $(s\Omega_e)_+$ and $(s\Omega_e)_-$ can vanish only for $s > 0$ and $s < 0$ respectively. Some examples are illustrated in Figures 2-4.

The case $k_\parallel^2 c^2 > \omega^2$ is not allowed for waves *in vacuo*, and thus is only of academic interest for gyroemission as such. It is relevant to the case of Compton scattering in a superstrong magnetic field. In this case the kinematics are the same as for gyroemission with ω and $k_\parallel$ reinterpreted as the differences between the frequency and parallel wavenumber, respectively, of the final and initial photons. One can then have any value of the ratio $\omega^2/k_\parallel^2 c^2$.

Maser Action in the Quantum Case

In the classical limit maser action for $\omega^2 >> k_\parallel^2 c^2$ is possible for a loss-cone distribution, as illustrated in Figure 5, of Hewitt *et al.* (1982). In the quantum case for transitions between small n the requirement for maser action reduces to a condition on the occupation number $N_n(p_\parallel)$ of electrons in the two states n and n-s involved in the transition. This is also illustrated in Figure 5 for

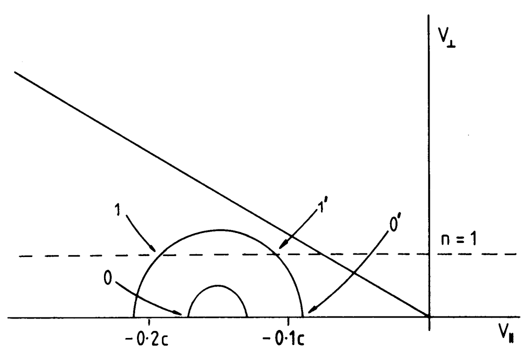

Figure 5. Resonant ellipses in a region near the origin of velocity space are plotted schematically. A line corresponding to a classical loss cone at $v_\perp/v < 1/2$, $v_\parallel < 0$ is shown; such a loss cone occurs in the classical application to the terrestrial kilometric radiation. The outer ellipse may be interpreted as a classical resonant ellipse which would correspond to maser action. In the quantum case transition from $n = 1$ to $n = 0$ are allowed only for the points $1 \rightarrow 0$ and $1' \rightarrow 0'$, and maser action requires that condition (16) be satisfied.

transitions from $n = 1$ to $n = 0$. There are two allowable initial states, labelled 1 and 1′, and two allowed final states labelled 0 and 0′. The values of $p_\parallel$ ($= p_1, p_1', p_0$ and p_0') may be found from the values of $v_\parallel$ at the intersections of the initial and final ellipses with the 1-ellipse and the 0-ellipse (the $v_\parallel$-axis) respectively by using (10a). The condition for maser action is

$$N_1(p_1) + N_1(p_1') > N_0(p_0) + N_0(p_0'). \qquad (16)$$

We propose to discuss the possibility of cyclotron maser emission from neutron stars in a future publication.

Hewitt, R. G., Melrose, D. B., and Rönnmark, K. G., *Proc. Astron. Soc. Aust.*, **4**, 221 (1981).

Hewitt, R. G., Melrose, D. B., and Rönnmark, K. G., 'The Loss-Cone Driven Electron-Cyclotron Maser' Aus. J. Phys. (in press 1982).

Lee, L. C., Kan, J. R., and Wu, C. S., *Planet. Space Sci.*, **28**, 703 (1980).

Lee, L. C., and Wu, C. S., *Phys. Fluids*, **23**, 1348 (1980).

Melrose, D. B., and Dulk, G. A., *Astrophys. J.*, **259**, 844 (1982).

Melrose, D. B., Rönnmark, K. G., and Hewitt, R. G., *J. Geophys. Res.* **87**, 5140 (1982).

Omidi, N., and Gurnett, D. A., *J. Geophys. Res.* **87**, 2377 (1982).

Wu, C. S., and Lee, L. C., *Astrophys. J.*, **230**, 621 (1979).

Wu, C. S., Tsai, S. T., Xu, M. J., and Shen, J. W., *Astrophys. J.*, **248**, 384 (1981).

Wu, C. S., Wong, H. K., Gorney, D. J., and Lee, L. C., J. *Geophys Res* **87**, 4476 (1982).

Coherent Gyromagnetic Emission

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Introduction

Observations of extremely high brightness temperatures in astrophysical objects imply that coherent emission processes must be occurring. An emission process may be coherent due to an inverted energy distribution leading to maser action, or due to bunching of particles in a region less than or of order of the wavelength of the radiation. Calculations of emission by bunches have generally used either a fluid model, e.g. Buschauer and Benford (1976), or a single-particle approach which requires all particles to have the same velocity, e.g. Saggion (1975) uses elements of both. In this paper we investigate gyromagnetic emission by bunches using a single-particle approach which includes the effects of differing particle velocities on the radiation.

Emission by a bunch may be regarded as consisting of two components. One is the sum of the incoherent powers that each particle would emit in the absence of the others. The other is the radiation due to interference between the particles, taken in pairs. Each pair emits a spike of radiation as the particles come together. Pairs which do not approach closely do not produce significant radiation.

We explore the properties of the interference radiation. We start by explaining the idea of interference more formally and then identify the kinematic requirements for it to occur. We neglect radiation reaction, and hence exclude the possibility of absorption or maser action. We show the dependence of the interference energy on the particle velocities, and then present the current associated with each particle. These describe the incoherent radiation of each particle. The remaining term is a cross-product between the two currents which describes the effect of interference between the two particles on the radiation.

For a single particle gyromagnetic emission is continuous. The energy radiated is then identified as the power radiated times the duration of emission and this power is the quantity of interest. If the two particles have identical velocities then their interference radiation is also continuous, and is of the same form as the incoherent power, times a "coherence factor" which depends on the orbital phase difference and geometrical separation of the particles. For N particles the maximum value which this coherence factor can attain is $N(N-1)$. If the velocities of the particles differ then the interference radiation is significant only when they are close. Consequently their interaction time is effectively finite, and their interference provides a finite energy contribution even over an infinite time. This energy contribution may be positive or negative. If positive, the coherent power during the interaction is greater than the incoherent power; if negative the coherent power is less than the incoherent power.

Frequency and Angle of Emission

The current density of a single particle in a magnetic field can be resonant with only those waves whose wavevector **k** and frequency ω satisfy the condition (Melrose 1980, p. 101) numerical calculations of emission by bunches. Finally we discuss the relation of this work to curvature emission by bunches.

Interference Radiation

Two charges q_1 and q_2 moving along orbits $\mathbf{r} = \mathbf{r}_1(t)$ and $\mathbf{r} = \mathbf{r}_2(t)$ with instantaneous velocities $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ give a current density

$$\mathrm{j}(\mathrm{r},t) = q_1\mathrm{v}_1(t)\ \delta(\mathrm{r}-\mathrm{r}_1(t)) + q_2\mathrm{v}_2(t)\ \delta(\mathrm{r}-\mathrm{r}_2(t)) \qquad (1)$$

The energy radiated by this current over an infinite time is found by integrating $-\mathbf{j}(\mathbf{r},t).\ \mathbf{E}(\mathbf{r},t)$ over all time and space, where $\mathbf{E}(\mathbf{r},t)$ is the electric field generated by $\mathbf{j}(\mathbf{r},t)$. Using the power theorem for Fourier transforms the energy radiated is

$$U = -\int \frac{d^3k\ d\omega}{(2\pi)^4}\ \mathrm{j}(\mathrm{k},\omega).\mathrm{E}^*(\mathrm{k},\omega) \qquad (2)$$