INVERSION OF SYNCHROTRON SPECTRA

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Abstract. The problem of synchrotron radiation spectra is treated from the viewpoint of deconvolving the spectrum of ultrarelativistic source electrons from the observed photon spectrum. It is shown that for homogeneous sources the problem amounts to inversion of a Meijer transform with a modified Bessel function kernel. A precise analytic inversion is only possible in the complex plane but Meijer transform tables are available for a wide range of functions. More convenient inversion formulae prove possible by use of a Laplace transform approximation or by analysing the spectra in terms of their integral moments.

The filtering property of the transform is also established showing that the contribution to the synchrotron spectrum of high frequency components in the electron spectrum declines exponentially with their frequency. Thus, as with other Laplace-like transforms, only a few terms in an electron spectrum expansion can be deconvolved for any plausible noise level in the synchrotron spectrum.

1. Introduction and Formulation of the Problem

Synchrotron radiation arises in a wide variety of physical situations particularly in astrophysics where it provides an explanation of the emissions from many cosmic radio sources but ranging in some cases over the entire spectrum up to gamma rays. Synchrotron spectra of such sources obtained over a limited spectral range can usually be interpreted by assumption of some simple trial form for the spectrum of ultra-relativistic electrons in the source – most commonly a power law spectrum. Synchrotron spectra observed over wide intervals (e.g., the Crab nebula spectrum) deviate from single power laws but are usually rather loosely discussed in terms of piecewise fits to several power laws in a number of separate intervals. Study of the form of the source electron spectrum is important in providing insight into the little understood processes of particle acceleration which occur in a wide range of cosmic objects.

This normal approach is not entirely satisfactory since the problem in hand is basically one of inverting an integral transform to deconvolve electron spectra from photon spectra. This means firstly that the entire spectral range of the convolution should be considered at once, rather than piecemeal, to avoid truncation errors which can be large (cf. Brown, 1978). Secondly, and more importantly, the adoption of empirical trial electron spectra to fit the (noisy) observed photon spectra can be misleading since it neglects the problem of non-uniqueness associated with the filtering

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property of the convolution, i.e., the instability of the deconvolution (cf. Craig and Brown, 1976). Proper assessment of the knowledge we can gain of electron spectra from their synchrotron emission can thus only be obtained from a study of the integral deconvolution involved and of its associated noise properties. We have already investigated these properties for continuum spectra obtained from collisional bremsstrahlung in both thermal (Craig and Brown, 1976) and non-thermal (Brown, 1971; Craig, 1979) plasmas, from the inverse Compton process (Craig and Brown, 1980), and from a distribution of black bodies (Craig and Brown, 1980). In this paper we consider the inversion and noise properties of synchrotron spectra, which are the other main form of continuum radiation encountered in nature.

The process of synchrotron emission has been discussed in many texts (e.g., Zheleznyakov, 1964; Ginzburg and Syrovatskii, 1969; Krüger, 1979; or Melrose, 1980). For the purposes of the present paper, we restrict ourselves to situations in which the source region comprises a magnetic field **B** and an isotropic electron velocity distribution both of which are homogeneous. (In inhomogeneous cases the following equations will apply to some weighted mean electron spectrum.) Any effects of photon propagation out of the source are neglected. We denote by θ the angle between **B** and the line of sight and describe the relativistic electron energy (E) spectrum by the number $N(\gamma)$ of electrons per unit Lorentz factor $\gamma = E/m_e c^2$. Then the synchrotron power radiated from the source per steradian per unit frequency (ν) range can be written as

$$I(\nu) = \frac{1}{\sqrt{3}} \frac{e^2}{c} \nu_0 \int_0^\infty N(\gamma) f(\nu/\gamma^2 \nu_0)\, \mathrm{d}\gamma\,, \tag{1}$$

where

$$\nu_0 = \frac{3}{4\pi} \frac{eB \sin\theta}{m_e c}\,; \tag{2}$$

with e, m_e the electronic charge (e.s.u.) and mass and c the speed of light while

$$f(x) = x \int_x^\infty K_{5/3}(x')\, \mathrm{d}x'\,, \tag{3}$$

with $K_{5/3}$ denoting a modified Bessel function of the second kind (Macdonald function). We are thus concerned with the inversion properties of transform (1) to yield $N(\gamma)$ from $I(\nu)$ with f as kernel.

2. Formal Inversion

We write (1) and (3) explicitly as

$$J(\nu) = \frac{\sqrt{3}\, c}{e^2} \frac{I(\nu)}{\nu} = \int_0^\infty \gamma^2 M(\gamma) \int_{\nu/\gamma^2 \nu_0}^\infty K_{5/3}(t)\, \mathrm{d}t\, \mathrm{d}\gamma\,, \tag{4}$$

where

$$M(\gamma) = \gamma^{-4} N(\gamma) ; \tag{5}$$

and differentiate with respect to ν to obtain

$$H(\nu) = \frac{-\sqrt{3}\,c}{\nu_0 e^2} \frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{I(\nu)}{\nu}\right) = \int_0^\infty M(\gamma) K_{5/3}(\nu/\gamma^2 \nu_0)\,\mathrm{d}\gamma , \tag{6}$$

which may be further reduced to standard form by setting

$$F(s) = H(\nu_0 s) ; \qquad G(t) = \tfrac{1}{2} t^{-3/2} M t^{-1/2} , \tag{7}$$

with

$$s = \nu/\nu_0 ; \qquad t = \gamma^{-2} , \tag{8}$$

and ν_0 being a parameter. Then

$$F(s) = \int_0^\infty G(t) K_{5/3}(st)\,\mathrm{d}t , \tag{9}$$

which is a standard modified Bessel transform, (of order 5/3) sometimes known as the Meijer transform (cf. Meijer, 1940), which has been tabulated (e.g., Oberhettinger, 1972; Erdélyi *et al.*, 1954) for an extensive range of 'source' functions $G(t)$. Since the kernel is a function of st only we can obtain an inversion formula by the standard procedure of taking the Mellin transform $\mathscr{M}\{F(s); z\} = \tilde{F}(z)$ (cf. Sneddon, 1972) of (9) with respect to z from which it follows that

$$\tilde{F}(z) = \tilde{G}(1 - z)\tilde{K}_{5/3}(z) ; \tag{10}$$

and, consequently, the formal solution of (9) is, using Sneddon (1972, p. 295)

$$G(t) = \mathscr{M}^{-1}\left(\frac{\tilde{F}(1-\zeta)}{\tilde{K}_{5/3}(1-\zeta)} ; t\right) = \mathscr{M}^{-1}\left(\frac{2^{\zeta+1}\tilde{F}(1-\zeta)}{\Gamma(\frac{4}{3} - \frac{1}{2}\zeta)\Gamma(-\frac{1}{3} + \frac{1}{2}\zeta)} ; t\right), \tag{11}$$

where $\zeta = 1 - z$ and Γ is the gamma function. Unfortunately because of the asymptotic behaviour of $K_{5/3}$ it is not feasible to write (1) to yield $G(t)$ as an explicit real integral of F over s with any ordinary function as inverse kernel. However, the solution can be written as a contour integral by extension of $F(s)$ into the complex plane viz. (cf. Sneddon, 1972; or Oberhettinger, 1972)

$$G(t) = \frac{t}{\pi i} \int_{c-i\infty}^{c+i\infty} s F(s) I_{5/3}(st)\,\mathrm{d}s , \tag{12}$$

where $I_{5/3}$ is the modified Bessel function of the first kind. This formal inversion formula enables derivation of electron spectra from a wide range of a suitable analytic forms $F(s)$

which may be fit to the photon spectrum observed (e.g., Oberhettinger, 1972, Chapter II, tabulates 124 such forms). The well-known case that a power-law spectrum $I \sim \nu^{-\alpha}$ leads to a power-law electron spectrum follows directly. The formal inversion (12) is also a useful basis for analysis for the stability and uniqueness of the derivation of electron spectra from noisy synchrotron spectra. We return to this topic in Section 4. For practical purposes it is convenient, however, to have simpler approximate inversion formulae, some of which we derive in Section 3. In the meantime we note the role of the parameter ν_0 occurring in the above. It is clear on writing inversion (11) in terms of the original variables (ν, γ), or indeed from the original transform (6) that the solution for $N(\gamma)$ depends on ν_0 implicitly. That is for any observed $I(\nu)$ we must prescribe a ν_0 value before we can attempt to deconcolve $N(\gamma)$. We may, therefore, in general find a range of possible functional forms $N(\gamma)$ from the same $I(\nu)$ for different values adopted for ν_0 (i.e., of $B \sin\theta$), subject to the constraint that only ν_0 producing positive definite $N(\gamma)$ are permitted. This may be contrasted with the approximate result obtained in Section 3.1 where ν_0 occurs only as a scaling factor.

3. Approximate Methods of Inversion

3.1. Laplace Transform Approximations

If, with s, t as defined in (8) we write

$$\mathscr{I}(s) = \frac{\sqrt{3}\,c}{e^2}\, I(\nu_0 s) \tag{13}$$

and

$$Q(t) = \tfrac{1}{2} t^{-3/2} N(t^{-1/2}), \tag{14}$$

then the transform (1) becomes

$$\mathscr{I}(s) = \int_0^\infty Q(t) f(st)\, \mathrm{d}t\,. \tag{15}$$

Now the function (3) can be quite well represented by

$$f(x) \simeq f_0(x) = a x^b\, e^{-x}, \tag{16}$$

where $a \simeq 1.8$, $b \simeq 0.3$. In this approximation (15) becomes

$$[a^{-1} s^{-b} \mathscr{I}(s)] = \int_0^\infty [t^b Q_0(t)]\, e^{-st}\, \mathrm{d}t \tag{17}$$

which we recognise as a Laplace transform $\mathscr{L}$ and so can at once write an inversion to yield a first approximation Q_0 to Q, or in terms of the original variables

$$N_0(\gamma) = \left(\frac{2\sqrt{3}c}{ae^2} \nu_0^{b-1}\right)\gamma^{2b-3} \mathscr{L}^{-1}\left\{\nu^{-b} I(\nu); \frac{1}{\gamma^2}\right\}, \tag{18}$$

where the inverse transform is widely tabulated for many functional forms. We also note that in this approximation ν_0 only appears in (18) as a scaling factor. Thus even if B, θ are unknown we can determine the shape of $N_0(\gamma)$. In addition the fact that (1) approximates to a Laplace transform means that we must expect its inversion to be unstable to data errors, a property well established for the Laplace transform (see, e.g., Bohm, 1960; Kunacz *et al.*, 1973; Craig and Brown, 1976; Craig, 1977; for astrophysical examples) – cf. Section 4.

A better approximation than (17) is obtained by writing from (13) $\Delta f = f - f_0$ *and noting that* $\Delta f / f_0 \ll 1$ over all relevant x. Then defining the correction $\Delta Q = Q(t) - Q_0(t)$ where Q is the exact solution of (15) and Q_0 of (17) we obtain

$$\int_0^\infty \Delta Q(t) f_0(st)\,\mathrm{d}t \simeq -\int_0^\infty Q_0 \Delta f(st)\,\mathrm{d}t, \tag{19}$$

where we have neglected the term in $\Delta f \Delta Q$ and the right-hand side can be computed once Q_0 is determined from inversion of (17). The correction $\Delta Q(t)$ to Q_0 can then be evaluated by substitution of $f_0(st)$ and inversion of the resulting Laplace transform.

3.2. Solution in Terms of Integral Moments

This is a standard procedure applicable to kernels depending only on st (or $s - t$) – e.g., Chandrasekhar and Münch (1950), Brown (1978). In the present case it is simplest to deal directly with (1). We multiply by ν^n and integrate from zero to infinity in ν. Then defining the nth moment of $I(\nu)$ as

$$\langle I \rangle_n = \frac{\sqrt{3}c}{e^2 \nu_0} \int_0^\infty I(\nu)\nu^n\,\mathrm{d}\nu, \tag{20}$$

and reversing the order of integration on the right side we obtain using (3), after straightforward reduction

$$\langle I \rangle_n = \alpha_n(\nu_0) \langle N \rangle_{2(n+1)}, \tag{21}$$

where

$$\langle N \rangle_{2(n+1)} = \int_0^\infty N(\gamma)\gamma^{2(n+1)}\,\mathrm{d}\gamma \tag{22}$$

defines the moments of $N(\gamma)$, and the coefficients α_n in (19) are defined by

$$\alpha_n = \frac{v_0^{n+1}}{n+2} \int_0^{\infty} K_{5/3}(x) x^{n+2}\, dx\,. \tag{23}$$

The usefulness and limitations of the moment method are discussed in Brown (1978), with reference to truncation and discretisation errors as well as data errors. It should be noted that moments may not be a good form of representation of spectra which are power-law like. In such cases it may be more appropriate to use moment analysis to analyse deviations from the best fit power law. We also observe that in the present problem the moment coefficient α_n depends on v_0^{n+1} so that the relative contribution of different moments of $N(\gamma)$ depends on the choice of v_0.

4. Smoothing and Stability Properties

We have already noted that we expect the inversion of (1) to yield $N(\gamma)$ to be unstable against perturbations in $I(v)$ due to the similarity to a Laplace transform. That is, the width of the kernel function $f(st)$ filters any high frequencies present in $N(\gamma)$ and renders them indiscernible above the noise in $I(v)$. The frequencies in $N(\gamma)$ above which no information can be deduced from $I(v)$ at any given noise level can be found by examination of the asymptotic behaviour of the exact inversion (11) as follows (cf. Craig and Brown, 1980, for similar discussion of inverse Compton spectra).

The analytic inversion (12) of (11) can be written as

$$G(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\tilde{F}(1-s)}{\tilde{K}_{5/3}(1-s)}\, t^{-s}\, ds\,; \tag{24}$$

and the same equation applies to a perturbation δG arising from a perturbation δF in the data. That is, replacing s by $1-s$,

$$\delta G(t) = \frac{1}{2\pi i} \int_{1-c-i\infty}^{1-c+i\infty} \frac{\delta \tilde{F}(s)}{\tilde{K}(s)}\, t^{s-1}\, ds\,. \tag{25}$$

If we set $s = \sigma + i\tau$ where σ, τ are real so that τ indicates the frequency (wave number) of oscillations in F, then using the fact that

$$\tilde{K}(s) = 2^{s-2}\, \Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\mu)\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\mu) \tag{26}$$

for $\mathrm{Re}\, s > \mu > 0$ we have the asymptotic behaviour for high frequencies ($\tau \to \infty$)

$$|\tilde{K}(\sigma + i\tau)| \sim 2\pi\, 2^{\sigma-2}\, |\tau|^{\sigma-1} \exp(-\tfrac{1}{2}\pi\, |\tau|)\,, \tag{27}$$

and $|\tilde{K}(s)| \to 0$ exponentially as $\tau = \mathrm{Im}(s) \to \infty$. Consequently, δG will become unbounded for high frequency noise ($\tau \to \infty$) in F unless the data noise (perturbation) function $\delta\tilde{F} \to 0$ faster than $|\tilde{K}|$ as $\tau \to \infty$, which will never be the case in practice. For example even if we consider the very steeply decreasing noise function

$$\delta F(x) \sim e^{-x^{\alpha}}, \tag{28}$$

where x, $\alpha > 0$ we have

$$\delta\tilde{F}(s) \sim \frac{1}{\alpha}\,\Gamma(s/\alpha) \tag{29}$$

(for $\mathrm{Re}\, s > 0$), so that

$$\delta\tilde{F}((\alpha + i\tau)/\alpha) \sim \frac{(2\pi)^{1/2}}{\alpha}\,|\tau|^{-1/2}\exp(-\tfrac{1}{2}\pi\tau/\alpha) \tag{30}$$

and

$$\left|\frac{\delta\tilde{F}}{\tilde{K}}\right| \sim \frac{2^{2-\alpha}}{(2\pi)^{1/2}\,\alpha}\,|\tau|^{-\sigma+(\sigma/\alpha)+1/2}\exp(\tfrac{1}{2}\pi\tau(1-1/\alpha))\,; \tag{31}$$

so that as $\alpha \to \infty$ the variation with τ tends to

$$\left|\frac{\delta\tilde{F}}{\tilde{K}}\right| \sim \tau^{1/2-\sigma}\exp(\tfrac{1}{2}\pi\tau)\,. \tag{32}$$

This equation shows that with increasing frequency (τ) of perturbation in F the corresponding oscillations in the deconvoluted G become exponentially large even for such a rapid exponential decrease in data noise as defined by (28) with arbitrarily large α. Consequently, we must reach precisely the same conclusion concerning instability of the synchrotron transform as for inverse Compton spectra (Craig and Brown, 1980) and for thermal bremsstrahlung spectra (Craig and Brown, 1976). That is the number of terms in any Fourier expansion of an electron spectrum derived from deconvolution of its synchrotron radiation spectrum increases only logarithmically with increasing precision in the spectral data, and so for any plausible data accuracy is limited to the first few terms.

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