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Reactive and resistive nonlinear instabilities

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It is argued that familiar nonlinear instabilities, such as three-wave decay instabilities and scattering off quasi-modes, have reactive and resistive versions. In appropriate limits these versions correspond to fixed-phase parametric instabilities and to random phase growth in weak turbulence theory, respectively. Using a simplified model it is shown that the reactive version passes over into the resistive version when the band-width of the pump is comparable with the growth rate.

1. Introduction

The theory of parametric instabilities has developed in two compleinentary ways. One approach applies to fixed-phase or monochromatic waves, and the other to random-phase or bioad-band waves. Following the terminology of Briggs (1964) for beam-driven instabilities, these are referred to as reactive and resistive nonlinear instabilities, respectively. The interrelation between them is discussed in this paper. Two (related) specific nonlinear instabilities are used in developing the ideas; these are scattering of Langmuir waves off thermal ions, and three-wave processes involving two Langmuir waves and an ion sound wave. The reactive and resistive versions of the former are called 'parametric decay into a quasi-mode' and 'nonlinear Landau damping' (NLD) respectively, and the two versions of the latter are described more loosely as the parametric decay instability and the three-wave decay process. To avoid any possible confusion, the terminology 'reactive' and 'resistive' is used exclusively in the following discussion.

In conventional treatments of parametric instabilities (e.g. Nishikawa & Liu 1976; Liu & Kaw 1976; Cap 1978), one wave, called the pump, is assumed to be of large amplitude and to be monochromatic or fixed phase. Anonlinear dispersion equation is derived for low-frequency fluctuations including terms proportional to the square of the amplitude of the pump. Reactive nonlinear instabilities correspond to situations where the dispersion equation is assumed to be real and has complex solutions for frequency ω with real wave vector k. The instability involves the pump decaying into a daughter wave and this low-frequency fluctuation. In practice the low-frequency fluctuation may be a quasi-mode, a strongly damped density fluctuation or it may involve waves in an otherwise weakly damped mode (e.g. the ion sound mode) driven unstable by the non-linear interaction. When the low-frequency disturbance is strongly damped, the instability is resistive, and the imaginary part of the dispersion equation then plays an essential role. More generally, resistive nonlinear instabilities may be regarded as consequences of nonlinear wave–particle and wave–wave interactions treated in the random-phase approximation (RPA). Kinetic equations are derived in terms of a spectral density (often the wave action) in k space (e.g. Sagdeev & Galeev 1969; Tsytovich 1970; Davidson 1972; Melrose 1980). This spectral density includes a spread in k and hence implicitly requires non-zero band-widths. Besides the band-widths being non-zero, another difference between this so-called weak turbulence theory and the theory of parametric instabilities is that no distinction is made between pump and daughter waves.

Three time-scales may be relevant for the waves involved in nonlinear instabilities. These are the growth time t_g, which is the inverse of the growth rate, the damping or loss time t_d, and the phase mixing time t_M, which is the inverse of the effective band-width $\Delta\omega$. Reactive instabilities involve fixed-phase growth, and require that t_M be longer than t_g. In practice one usually assumes $t_M = \infty$. Effects of finite t_M/t_g, that is of non-zero band-widths, have been considered by, for example, Valeo & Oberman (1973), Thomson & Karush (1974), Thomson (1975) and Laval, Pellat & Pesme (1976); the non-zero band-width reduces the growth rate for $t_M/t_g \lesssim 1$ (where t_g refers to the growth time for $t_M = \infty$) and increases the threshold above which growth is possible. For resistive instabilities, the use of the PRA implies that t_M is much less than other times of interest, i.e. one effectively has $t_M = 0$.

In this paper it is argued that for relevant nonlinear instabilities there are reactive and resistive versions which pass over continuously into each other for $t_g \simeq t_M$, i.e. when the growth rate is roughly equal to the band-width of the growing waves. For $t_g < t_M$, growth occurs faster than phase mixing, the growing wave has a fixed phase and the reactive version is relevant. For $t_M < t_g$, phase mixing occurs faster than wave growth, and the growth must be independent of the phase, as in the RPA, so that the resistive version is appropriate. An implication is that the reduced growth rates found for parametric instabilities due to non-zero band-widths (cf. the literature cited above) should correspond to the growth rate for the resistive version. Here this is shown for the two most familiar nonlinear instabilities, the three-wave decay $P \rightarrow D + s$ of a 'pump' Langmuir wave P into another 'daughter' Langmuir wave D and an ion sound wave s, and the decay $P \rightarrow D$ involving a resistive or reactive low-frequency quasi-mode (the resistive version of which corresponds to nonlinear Landau damping). It seems that not all nonlinear instabilities can be paired in this way, owing to the apparent absence of resistive counterparts for modulational instabilities. Conversely, one could argue that the reduced growth rate for $t_M < t_g$ for such instabilities, e.g. for the oscillating two-stream instability, actually corresponds to a hitherto unrecognized resistive counterpart.

2. Nonlinear wave equation

We are concerned with three longitudinal fields, a pump field $\phi_P(k)$, a daughter field $\phi_D(k)$ and a low-frequency field resulting from the beat between these two high-frequency fields. Here $\phi_P(k)$ or $\phi_D(k)$ denotes the electrostatic potential at

$k=(\omega,\mathbf{k})$. The nonlinear wave equation for the daughter field is (SI units)

$$|\mathbf{k}|^2 K(k)\,\phi_D(k) = \frac{1}{\epsilon_0}\int d\lambda^{(3)}\,\alpha^{(3)}_{\text{eff}}(k,k_1,k_2,k_3)\,\phi_P(k_1)\,\phi_D(k_2)\,\phi_P(k_3), \tag{1}$$

where only the effect of the pump in driving the daughter field is retained. On the left-hand side of (1),

$$K(k) = 1+\chi_e(k)+\chi_i(k) \tag{2}$$

denotes the longitudinal dielectric function in terms of the electron and ion susceptibilities. On the right-hand side of (1),

$$d\lambda^{(n)} = \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_n}{(2\pi)^4}(2\pi)^4\delta^4(k-k_1-\,\ldots\,-k_n) \tag{3}$$

denotes the convolution integral and $\alpha^{(3)}_{\text{eff}}(k,k_1,k_2,k_3)$ is an effective cubic response function which is assumed symmetric under interchange of k_1 and k_3. In terms of the actual nonlinear response functions, defined by the weak turbulence expansion of the charge density for any species, i.e. by

$$\rho(k) = -\epsilon_0|\mathbf{k}|^2\chi(k)\,\phi(k)+\sum_{n=2}^{\infty}\int d\lambda^{(n)}\,\alpha^{(n)}(k,k_1,\ldots,k_n)\,\phi(k_1)\ldots\phi(k_n), \tag{4}$$

one has

$$\alpha^{(3)}_{\text{eff}}(k,k_1,k_2,k_3) = 3\alpha^{(3)}(k,k_1,k_2,k_3) + 2\alpha^{(2)}(k,k_1,k-k_1)\,D(k-k_1)\,2\alpha^{(2)}(k-k_1,k_2,k_3). \tag{5}$$

The factors 3 and 2 in (5) appear only when the nonlinear response functions in (4) are assumed to be symmetrized over all arguments; these factors are to be omitted when unsymmetrized approximate forms are used.

The relevant approximations to the nonlinear response functions which apply when $k-k_1$ is a low-frequency beat are

$$\alpha^{(2)}(k,k_1,k-k_1) = [\alpha^{(2)}(k-k_1,-k_1,k)]^* \simeq \frac{\epsilon_0 e}{m_e}\frac{\mathbf{k}\cdot\mathbf{k}_1}{\omega\omega_1}|\mathbf{k}-\mathbf{k}_1|^2\chi_e(k-k_1), \tag{6}$$

and

$$\alpha^{(3)}(k,k_1,k_2,k_3) = -\frac{\epsilon_0 e^2}{m_e^2}\frac{\mathbf{k}\cdot\mathbf{k}_1}{\omega\omega_1}\frac{\mathbf{k}_2\cdot\mathbf{k}_3}{\omega_2\omega_3}|\mathbf{k}-\mathbf{k}_1|^2\chi_e(k-k_1), \tag{7}$$

where only the electronic contributions are retained. Then (5) implies

$$\alpha^{(3)}_{\text{eff}}(k,k_1,k_2,k_3) = -\frac{\epsilon_0 e^2}{m_e^2}\frac{\mathbf{k}\cdot\mathbf{k}_1}{\omega\omega_1}\frac{\mathbf{k}_2\cdot\mathbf{k}_3}{\omega_2\omega_3}|\mathbf{k}-\mathbf{k}_1|^2\frac{\chi_e(k-k_1)\{1+\chi_i(k-k_1)\}}{1+\chi_e(k-k_1)+\chi_i(k-k_1)}. \tag{8}$$

3. Nonlinear dispersion equation

Following an approach used by Falk & Tsytovich (1975), (1) may be reduced to a nonlinear dispersion equation by performing a statistical average (denoted by angular brackets $\langle\,\rangle$) over the pump field. The statistical average

$$\phi_P(k_1)\,\phi_P(k_3)$$

is non-zero only for $k_1 = -k_3$. Consequently we may write

$$\langle \phi_P(k_1)\,\phi_P(k_3)\rangle = (2\pi)^4\delta^4(k_1+k_3)\,|\phi_P|^2(k_1). \tag{9}$$

Then *(1)* implies the nonlinear dispersion equation, for the daughter waves,

$$-|\mathbf{k}|^2 K(k) + \frac{1}{\epsilon_0}\int \frac{d^4k_1}{(2\pi)^4}\,|\phi_P|^2(k_1)\,\alpha^{(3)}_{\text{eff}}(k, k_1, k, -k_1) = 0. \tag{10}$$

Two different models for $|\phi_P|^2(k)$ allow us to treat resistive and reactive instabilities using (10), as we now show.

3.1. *Resistive nonlinear instabilities*

In resistive instabilities the waves may be regarded as having random phases. A semi-classical formalism can then be justified, with the waves regarded as a collection of quanta with energy $\hbar\omega$, momentum $\hbar\mathbf{k}$ and occupation number $N(\mathbf{k})$, so that the energy density in the elemental range $d^3\mathbf{k}/(2\pi)^3$ is

$$\hbar\omega(\mathbf{k})\,N(\mathbf{k})\,d^3\mathbf{k}/(2\pi)^3.$$

Let $R(\mathbf{k})$ be the ratio of electric to total energy, and let the labels P, D, etc. denote the pump, daughter, etc. as above. Then

$$|\phi_P|^2(k) = \frac{\hbar\omega_P(\mathbf{k})\,N_P(\mathbf{k})R_P(\mathbf{k})}{\epsilon_0|\mathbf{k}|^2}\,2\pi\delta(\omega-\omega_P(\mathbf{k})), \tag{11}$$

where negative frequencies are included implicitly through $\omega_P(-k) = -\omega_P(\mathbf{k})$.

The daughter waves satisfy the dispersion equation $K(k) = 0$ with solution $\omega = \omega_D(\mathbf{k})$ in the absence of the pump. Let the linear damping of the waves be described by the absorption coefficient $\gamma_D(\mathbf{k})$, such that the wave energy decays as $\exp[-\gamma_D(\mathbf{k})\,t]$. For k nearly satisfying the dispersion relation for the D waves in the absence of the pump, *(1)* reduces to

$$\begin{aligned}\omega-\omega_D(\mathbf{k})+\tfrac{1}{2}i\gamma_D(\mathbf{k}) = {} & -\frac{\hbar e^2}{\epsilon_0 m_e^2}\frac{R_D(\mathbf{k})}{\omega_D(\mathbf{k})}\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\frac{R_P(\mathbf{k}_1)\,N_P(\mathbf{k}_1)}{\omega_P(\mathbf{k}_1)}\\ & \times|\boldsymbol{\kappa}\,.\,\boldsymbol{\kappa}_1|^2\,|\mathbf{k}-\mathbf{k}_1|^2\,\frac{\chi_e(k-k_1)\{1+\chi_i(k-k_1)\}}{1+\chi_e(k-k_1)+\chi_i(k-k_1)},\end{aligned} \tag{12}$$

where $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}_1$ are unit vectors along k and $\mathbf{k}_1$ respectively, and where

$$\omega-\omega_1 = \omega_D(\mathbf{k})-\omega_P(k_1)$$

is implicit in $k = k_1$.

To treat nonlinear absorption processes, we replace $\omega-\omega_D(\mathbf{k})$ on the left-hand side of *(12)* by $i\gamma_D^{NL}(\mathbf{k})/2$, where $\gamma_D^{NL}(\mathbf{k})$ is the nonlinear absorption coefficient, and take the appropriate imaginary part of the final combination of terms on the right-hand side. A resistive nonlinear instability corresponds to a negative value of $\gamma_D^{NL}(\mathbf{k})$. The two examples discussed in §4 are induced scattering off ions, due to the imaginary part of χ_i in the numerator of (12), and the three-wave decay, due to the imaginary part of the denominator in *(12)*.

3.2. *Reactive nonlinear instabilities*

For strictly monochromatic waves we may write

$$\phi_P(x) = a_P e^{-ik_P x} + a_P^* e^{ik_P x}, \tag{13}$$

where k is the wave 4-vector for the pump. Then we find

$$|\phi_P|^2(k) = \frac{W_P R_P}{\epsilon_0 |\mathbf{k}_P|^2}[(2\pi)^4\delta^4(k-k_P) - (2\pi)^4\delta^4(k+k_P)], \tag{14}$$

where W_P is the energy density in the pump, and where R_P denotes $R_P(\mathbf{k}_P)$. For most purposes only the contribution from the term with $k = k_P$ in *(14)* need be retained, and then *(10)* with *(8)* reduces to

$$K(k) + \frac{e^2}{\epsilon_0 m_e^2}\frac{W_P R_P}{\omega^2\omega_P^2}|\boldsymbol{\kappa}.\boldsymbol{\kappa}_P|^2|\mathbf{k}-\mathbf{k}_P|^2\frac{\chi_e(k-k_P)\{1+\chi_i(k-k_P)\}}{K(k-k_P)} = 0. \tag{15}$$

Reactive nonlinear instabilities correspond to solutions of *(14)* with complex ω for real k.

The reactive instabilities of interest here are the parametric decay instability and decay into a reactive quasi-mode. These are derived by setting

$$\chi_e(k-k_1) \simeq 1/(\mathbf{k}-\mathbf{k}_1)^2\lambda_{De}^2 \quad \text{and} \quad \chi_i(k-k_1) \simeq -\omega_{pi}^2/(\omega-\omega_1)^2,$$

with $|\omega-\omega_1|^2 \ll \omega_{pi}^2$ and $|\omega-\omega_1|^2 \gg \omega_{pi}^2$ distinguishing the two cases, respectively.

4. Effect of non-zero bandwidths

It is plausible that induced scattering off thermal ions and decay into a reactive quasi-mode are the resistive and reactive versions, respectively, of a single non-linear instability, and that the three-wave interaction and the parametric decay instability are the resistive and reactive versions, respectively, for a different but related nonlinear instability. In principle this could be confirmed by retaining both the real and imaginary parts of the nonlinear term in *(10)* and solving for the dispersion relation including both effects simultaneously. However, this approach is complicated by the fact that the reactive versions are derived only in the limit of zero band-width. In this section, a simple model for non-zero band-widths is introduced and is used to establish the stated relation between the resistive and reactive nonlinear instabilities.

The model involves first generalizing *(13)* to

$$\phi_P(x) = a_P(x)\, e^{ik_P x} + a_P^*(x)\, e^{ik_P x}, \tag{16}$$

where $a_P(x)$ describes the envelope of the pump field. Then in place of *(14)* we have

$$|\phi_P|^2(k) = \frac{W_P R_P}{\epsilon_0 |\mathbf{k}_P|^2}[\zeta(k-k_P) + \zeta(k+k_P)], \tag{17}$$

with

$$\zeta(k) = \int d^4\xi e^{ik\xi}\langle a_P(x)\, a_p^*(x-\xi)\rangle. \tag{18}$$

Next it is assumed that the pump waves may be regarded as having a spread in frequencies but not in direction:

$$\zeta(k) = \zeta_P(\omega)\,(2\pi)^3\delta^3(\mathbf{k}). \tag{19}$$

Finally a specific form for $\zeta_P(\omega)$ is assumed. Two different forms are convenient for analytic purposes. These are

$$\zeta_P(\omega) = \begin{cases} 2\pi^{\frac{1}{2}} \exp\left[-\dfrac{(\omega-\omega_P)^2}{(\Delta\omega)^2}\right], & (20a) \\ \dfrac{2\Delta\omega}{(\omega-\omega_P)^2+(\Delta\omega)^2}. & (20b) \end{cases}$$

These correspond to the temporal correlation function varying with time as $\exp[-\frac{1}{4}(\Delta\omega)^2t^2]$ and $\exp[-\Delta\omega|t|]$, respectively. In either case, *Aw* may be interpreted as the band-width of the waves. The cases (20a) and (20b) are referred to as Gaussian and Lorentzian profiles, respectively.

4.1. Three-wave interaction

The resistive and reactive versions of the three-wave decay may be treated together by considering an approximate dispersion equation derived from *(10)* by inserting *(17)* and *(19)* and making appropriate approximations to the susceptibilities in (8). For a thermal distribution of particles (plasma frequency $\omega_{p\alpha}$, thermal speed V_α, Debye length $\lambda_{D\alpha} = V_\alpha/\omega_{p\alpha}$) we have

$$\chi_\alpha(k) = \frac{1}{|\mathbf{k}|^2\lambda_{D\alpha}^2}\{1-\bar{\phi}(z_\alpha)+\pi^{\frac{1}{2}}z_\alpha \exp[-z_\alpha^2]\} \tag{21}$$

with $z_\alpha = \omega/2^{\frac{1}{2}}|\mathbf{k}|V_\alpha$ and

$$\bar{\phi}(z) = 2ze^{-z^2}\int_0^z dt e^{t^2} \simeq \begin{cases} 2z^2 & z^2 \ll 1, \\ 1+\dfrac{1}{2z^2} & z^2 \gg 1. \end{cases} \tag{22}$$

The appropriate approximations for three-wave processes are

$$\chi_e(k-k_1) \simeq 1/|\mathbf{k}-\mathbf{k}_1|^2\lambda_{De}^2 \quad \text{and} \quad \chi_i(k-k_1) \simeq -\omega_{pi}^2/(\omega-\omega_1)^2$$

in the numerator and, in the denominator,

$$K(k-k_1) \simeq \frac{1+|\mathbf{k}-\mathbf{k}_1|^2\lambda_{De}^2}{(\omega-\omega_1)^2|\mathbf{k}-\mathbf{k}_1|^2\lambda_{De}^2}[\omega-\omega_1-\omega_s(\mathbf{k}-\mathbf{k}_1)+\tfrac{1}{2}i\gamma_s(\mathbf{k}-\mathbf{k}_1)]$$
$$\times[\omega-\omega_1+\omega_s(\mathbf{k}-\mathbf{k}_1)+\tfrac{1}{2}i\gamma_s(\mathbf{k}-\mathbf{k}_1)], \tag{23}$$

where ω_s and γ_s denote the frequency and absorption coefficient for ion sound waves. The poles implied by *(23)* occur near $w-\omega_1 = \pm\omega_s(\mathbf{k}-\mathbf{k}_1)$. Assuming only one of the poles is important, and omitting arguments $k \mp \mathbf{k}_1$ for simplicity, *(10)* then gives

$$\omega-\omega_D+\tfrac{1}{2}i\gamma_D \mp \frac{\omega_{pi}^2}{2\omega_s}\frac{e^2}{\epsilon_0 m_e^2}\frac{R_DR_PW_P}{\omega_D\omega_P^2}|\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}_P|^2|\mathbf{k}-\mathbf{k}_P|^2\int\frac{d\omega_1}{2\pi}\frac{\zeta_P(\omega_1)}{w-w_1\mp\omega_s+\frac{1}{2}i\gamma_s} = 0, \tag{24}$$

where we assume $\omega_1 \simeq \omega_P$ in taking a factor $1/\omega_P^2$ outside the integral. For the two cases ($20a$, b) the integral in (24) gives

$$\int \frac{d\omega_1}{2\pi} \frac{\zeta_P(\omega_1)}{\omega-\omega_1 \mp \omega_s + \frac{1}{2}i\gamma_s} = \begin{cases} \pm \dfrac{\bar{\phi}(z_\pm)}{\Delta\omega z_\pm}, & (25a) \\[2ex] \pm \dfrac{1}{\omega-\omega_P \mp \omega_s + i(\Delta\omega + \frac{1}{2}\gamma_s)}, & (25b) \end{cases}$$

with

$$z_\pm = \frac{\omega-\omega_P \mp \omega_s + \frac{1}{2}i\gamma_s}{\Delta\omega}. \tag{26}$$

The resistive version of the three-wave decay may be treated by taking the imaginary part of the integral ($25a$, b), inserting this in (24) and equating the resulting contributions to the imaginary part of w. The results for the two cases ($25a$, b) are

$$\gamma_D^{NL}(\mathbf{k}) = \sum_\pm \pm \frac{\omega_{pi}^2}{\omega_s} \frac{e^2}{\epsilon_0 m_e^2} \frac{R_D R_P W_P}{\omega_D \omega_P^2} |\boldsymbol{\kappa} . \boldsymbol{\kappa}_P|^2 |\mathbf{k}-\mathbf{k}_P|^2 \times \begin{cases} \dfrac{\Delta\omega}{(\omega-\omega_P \mp \omega_s)^2 + (\frac{1}{2}\gamma_s)^2}, & (27a) \\[2ex] \dfrac{\Delta\omega + \frac{1}{2}\gamma_s}{(\omega-\omega_P \mp \omega_s)^2 + (\Delta\omega + \frac{1}{2}\gamma_s)^2}, & (27b) \end{cases}$$

where $|z_\pm|^2 \gg 1$ is assumed in ($27a$).

For comparison, the result from conventional weak turbulence theory, which follows from (12), is

$$\gamma_D^{NL}(\mathbf{k}) = \sum_\pm \pm \frac{\pi e^2 \hbar}{\epsilon_0 m_e^2} \frac{R_D(\mathbf{k})}{\omega_D(\mathbf{k})\,|\mathbf{k}|^2 \lambda_{De}^2} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{R_P(\mathbf{k}_1)\, N_P(\mathbf{k}_1)}{\omega_P(\mathbf{k}_1)} \times |\boldsymbol{\kappa} . \boldsymbol{\kappa}_1|^2 \omega_s(\mathbf{k}-\mathbf{k}_1)\, \delta(\omega_D(\mathbf{k}) - \omega_P(\mathbf{k}_1) \mp \omega_s(\mathbf{k}-\mathbf{k}_1)), \tag{28}$$

where the damping of the ion sound waves is neglected ($\gamma_s = 0$). The main difference between (28) and (24) is in the descriptions of the pump spectrum. These are related by

$$\hbar \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{R_P(\mathbf{k}_1)\, N_P(\mathbf{k}_1)}{\omega_P(\mathbf{k}_1)} = \frac{R_P W_P}{\omega_P^2} \int \frac{d\omega_1}{2\pi} \zeta_P(\omega_1). \tag{29}$$

Otherwise, one is to take the limit $\gamma_s \to 0$ in (24), corresponding to neglecting damping in (28), and hence to replacing the resonant denominator in (24) by $-i\pi\delta(\omega-\omega_1 \mp \omega_s)$ in the integrand, and using $\omega_s^2(\mathbf{k}-\mathbf{k}_1) \simeq |\mathbf{k}-\mathbf{k}_1|^2 \omega_{pi}^2 \lambda_{De}^2$.

Two implications for the resistive version of the three-wave decay follow from (24) with ($25a$, b). One involves the neglect of Im ω in the nonlinear (final) term in (24): this is justified only for Im w $< \Delta\omega + \frac{1}{2}\gamma_s$, and implies an upper limit to the growth rate for the resistive version. The other is that the line profile of the growing waves is similar to that of the pump, and, in particular, Aw may be interpreted as the band-width of the growing waves. Thus the resistive version applies only if the growth rate is less than about the band-width of the growing waves.

Note that it is only the decay $P \to D + s$ which has a negative absorption coefficient and so leads to a resistive instability. The other process (upper sign) corresponds to a coalescence P + s → D and does not cause the daughter waves to grow exponentially.

The reactive version of the three-wave decay may be described by (24) with only the real part retained in ($25a$, b). For the Gaussian profile ($25a$), in the limit $|z_{\pm}|^2 \gg 1$ the resulting dispersion equation is identical to that for Aw = 0. This is

$$(\omega-\omega_D+\tfrac{1}{2}i\gamma_D)\,(\omega-\omega_P\mp\omega_s+\tfrac{1}{2}i\gamma_s)\mp\frac{\omega_{pi}^2}{2\omega_s}\frac{\hbar e^2}{\epsilon_0 m_e^2}\frac{R_D R_P W_P}{\omega_D\omega_P^2}\,|\boldsymbol{\kappa}\,.\,\boldsymbol{\kappa}_P|^2\,|\mathbf{k}-\mathbf{k}_P|^2=0, \tag{30}$$

with the well-known solution (for the lower sign)

$$\omega=\tfrac{1}{2}[-i(\tfrac{1}{2}\gamma_s+\tfrac{1}{2}\gamma_D)+i\{(\tfrac{1}{2}\gamma_D-\tfrac{1}{2}\gamma_s)^2+4c\}^{\frac{1}{2}}] \tag{31}$$

with

$$c=\frac{\omega_{pi}^2}{2}\frac{e^2}{\epsilon_0 m_e^2}\frac{R_D R_P W_P}{\omega_s\omega_D\omega_P^2}\,|\boldsymbol{\kappa}\,.\,\boldsymbol{\kappa}_P|^2\,|\mathbf{k}-\mathbf{k}_p|^2. \tag{32}$$

Growth occurs only above the threshold

$$c>\tfrac{1}{2}\gamma_D\cdot\tfrac{1}{2}\gamma_s. \tag{33}$$

The condition $|z_+|^2 \gg 1$ requires that the growth rate (contained in the imaginary part of ω in (26)) be much greater than the band-width Aw. A similar conclusion follows for the Lorentzian profile ($25b$); in this case $\frac{1}{2}i\gamma_s$ in (30), (31) and (33) is replaced by $i(\Delta\omega+\frac{1}{2}\gamma_s)$.

The result (e.g. Thomson & Karush 1974) that a non-zero band-width decreases the growth rate (from the value for Aw = 0) follows trivially from (31), as modified for the Lorentzian profile. Then for Aw $\gg 4c$ the growth rate corresponds to $|\mathrm{Im}\,\omega| \simeq c/\Delta\omega$. The new point made here is that this corresponds to the growth rate for the resistive version, e.g. to (27b) for $\Delta\omega \gg \frac{1}{2}\gamma_s$, $|\omega-\omega_P\mp\omega_s|$. Thus the reduced growth rate for the parametric three-wave decay found by Thomson & Karush (1974) for Aw = 0 corresponds to the resistive or weak-turbulence version of the decay instability.

Qualitatively, the inclusion of Aw $\neq 0$ increases the threshold for the reactive version for Aw $> \frac{1}{2}\gamma_s$, and causes the reactive version to pass over into the resistive version for Aw $> c^{\frac{1}{2}}$.

4.2. *NLD* and decay into a quasi-mode

NLD, due to induced scattering off thermal ions, and the parametric decay into a quasi-mode may be derived from the dispersion equation (10) with (8) by retaining only the imaginary part of $\chi_i(k-k_1)$ and the real part of

$$\chi_i(k-k_1) \simeq -\omega_{pi}^2/(\omega-\omega_1)^2,$$

respectively. A minor complication is that, in the former case, $\mathrm{Re}\,\chi_i(k-k_1)$ should not be neglected; its inclusion leads to a factor $(1+T_e/T_i)^2$ in the denominator. Let us ignore this factor for the purpose of argument and assume a dispersion equation

$$\omega-\omega_D+\tfrac{1}{2}i\gamma_D+\frac{e^2}{\epsilon_0 m_e^2}\frac{R_D R_P W_P}{\omega_D\omega_P^2}\,|\boldsymbol{\kappa}\,.\,\boldsymbol{\kappa}_P|^2\,|\mathbf{k}-\mathbf{k}_P|^2\int\frac{d\omega_1}{2\pi}\,\zeta_P(\omega_1)\,\chi_i(k-k_1)=0. \tag{34}$$

(It is not difficult to include an interpolation factor to reproduce the factor $(1+T_e.T_i)^2$ for NLD, but this merely complicates the algebra without affecting the argument.) For the two forms ($20a$, b) the integral in (34), with the form (21) for the susceptibility, reduces to

$$\int \frac{d\omega_1}{2\pi}\,\zeta_P(\omega_1)\,\chi_i(k-k_1) = \begin{cases} \dfrac{\omega_{pi}^2}{\overline{\omega}^2}\left\{1-\phi\left(\dfrac{\omega-\omega_P}{\overline{\omega}}\right)\right\}, & (35a)\\[2ex] \dfrac{\omega_{pi}^2}{\omega_V^2}\left\{1-\phi\left(\dfrac{\omega-\omega_P+i\Delta\omega}{\omega_V}\right)\right\}, & (35b)\end{cases}$$

with

$$\phi(z) = \overline{\phi}(z) - i\pi^{\frac{1}{2}} z e^{-z^2}, \tag{36}$$

and with

$$\overline{\omega}^2 = \omega_V^2 + (\Delta\omega)^2, \quad \omega_V^2 = 2|\mathbf{k}-\mathbf{k}_P|^2\,V_i^2. \tag{37}$$

The analysis of this case is similar to that of the three-wave interaction. NLD is treated by inserting the imaginary part of ($35a$) or ($35b$) in (34) and solving it for $\operatorname{Im}\omega$, neglecting the real terms. For the Gaussian case ($35a$) the argument $(w-\omega_P)/\overline{\omega}$ is real and one may use (36) to evaluate the imaginary part. Comparing the cases Aw $= 0$ and Aw $\neq 0$, one finds that the band-width of the growing waves is determined by the Doppler spread ω_V for Aw $\lesssim \omega_V$, and is determined by the band-width of the pump waves for Aw $< \omega_V$. The same conclusion follows for the Lorentzian case (35b).

For the decay into a reactive quasi-mode, the growth rate for Aw $= 0$ may be found from (15) with

$$\chi_e(k-k_P) \simeq 1/|\mathbf{k}-\mathbf{k}_P|^2\lambda_{De}^2, \quad |\chi_i(k-k_P)| \simeq \omega_{pi}^2/(\omega-\omega_P)^2 \gg 1.$$

One finds

$$\operatorname{Im}\omega = \frac{3^{\frac{1}{2}}}{2}(2\omega_s c)^{\frac{1}{3}}, \tag{38}$$

with c defined by (32). This result is reproduced by (34) with ($35a$, b) provided the imaginary part of ϕ is ignored (cf. (36)) and the argument of ϕ is much greater than unity. This is the case for $(w-\omega_P)^2 \gg \omega_V^2(\Delta\omega)^2$, when the right-hand side of ($35a$) or ($35b$) may be approximated by $-\omega_{pi}^2/(\omega-\omega_P)^2$. In particular, this implies that the result (38) applies only for $|\operatorname{Im}\omega| >$ Aw, ω_V, i.e. only when the growth rate exceeds the band-width of the growing waves. For

$$\operatorname{Im}\omega < \max[\text{Aw}, \omega_V],$$

$|\omega-\omega_P|$ is less than $\max[\text{Aw}, \omega_V]$ and the reactive version (i.e. decay into a reactive quasi-mode) is replaced by the resistive version (NLD).

5. The oscillating two-stream instability

The effect of a non-zero band-width on the oscillating two-stream instability (OTSI) has been discussed by Thomson & Karush (1974) and by Smith, Goldstein & Papadopoulos (1979). These two discussions differ in that in the former it was found that the growth rate involves the statistical average over a product of four pump fields, whereas in the latter only two pump fields are averaged. From a practical viewpoint this difference is not very important, involving only a factor of two. The discussion of OTSI here is restricted to identifying the source of this inconsistency.

OTSI involves a low-frequency fluctuation with a frequency whose real part is negligible. Then both the positive and negative frequency parts of $\phi_D(k)$ need to be retained in (1), which becomes

$$\left[-|\mathbf{k}|^2 K(k)+\frac{1}{\epsilon_0}\int\frac{d^4k_1}{(2\pi)^4}\alpha^{(3)}_{\rm eff}(k,k_1,k,-k_1)\,\phi_P(k_1)\,\phi_P(-k_1)\right]\phi_D(k)$$
$$+\left[\frac{1}{\epsilon_0}\int\frac{d^4k_1}{(2\pi)^4}\alpha^{(3)}_{\rm eff}(k,k_1,k_1,-k)\,\phi_P(k_1)\,\phi_P(k_1)\right]\phi_D(-k)=0. \quad (39)$$

One then derives a pair of coupled equations for $\phi_D(k)$ and $\phi_D(-k)$. These equations are of the form

$$[K(k)+K^{NL}(k)]\,\phi_D(k)+S(k)\,\phi_D(-k)=0. \quad (40a)$$

$$[K(-k)+K^{NL}(-k)]\,\phi_D(-k)+S(-k)\,\phi_D(k)=0. \quad (40b)$$

The determinant of the coefficients leads to the dispersion equation

$$\{K(k)+K^{NL}(k)\}\{K(-k)+K^{NL}(-k)\}-S(k)\,S(-k)=0. \quad (41)$$

According to (8) with the approximations made here, $\alpha^{(3)}_{\rm eff}(k,k_1,k_1,-k)$ in (39) is equal to $\alpha^{(3)}_{\rm eff}(k,k_1,k,-k_1)$. It then follows that $K^{NL}(k)K^{NL}(-k)=S(k)S(-k)$ for a monochromatic pump, and (41) reduces to the familiar form for the dispersion equation for OTSI, i.e. to

$$1+\frac{K^{NL}(k)}{K(k)}+\frac{K^{NL}(-k)}{K(-k)}=0 \quad (42)$$

in the notation used here.

The two different dispersion equations for a statistically averaged pump are effectively derived as follows. Thomson & Karush's form corresponds to incorporating $K^{NL}(k)$ into $K(k)$ in (41), then statistically averaging, and Smith, Goldstein & Papadopoulos' form corresponds to statistically averaging (42). The latter follows from (41) provided

$$1 \quad P \quad -k_1)\,\phi_P(k_1')\;P \quad -k_1')\rangle=\langle\phi_P(k_1)\,\phi_P(k_1)\,\phi_P(-k_1')\,\phi_P(-, \quad (43)$$

which is the case whenever the fourth-order correlation function factorizes into second-order correlation functions.

It may be concluded that the effect of a non-zero band-width on OTSI is adequately described by the results obtained by Smith, Goldstein & Papadopoulos (1979). As for the other reactive nonlinear instabilities, the growth rate is reduced for $t_M < t_1$. However, for OTSI there is no recognized random phase counterpart, owing, presumably, to the role played by the ponderomotive force in this and other 'modulational' instabilities (e.g. Nishikawa 1968; Goldman 1984).

6. Discussion and conclusions

It is shown in this paper that four familiar instabilities involving Langmuir turbulence may be paired into reactive and resistive versions of two instabilities. It is apparent that this result applies in a wider context whenever reactive and resistive counterparts exist. The reactive counterpart is a conventional parametric instability for which the band-width of the pump is negligible. The band-width of the growing waves is then formally less than the growth rate (although growth itself causes a frequency spread), and this band-width increases proportional to that of the pump. The reactive version passes over into the resistive version when the band-width of the growing waves exceeds the growth rate. The random phase approximation can then be justified, and the weak turbulence theory of wave–particle and wave–wave processes is appropriate. There are, however, some reactive nonlinear instabilities which have no accepted resistive counterpart. These include modulation instabilities, notably the oscillating two-stream instability which involves the ponderomotive force. As the spread in frequency is increased above the growth rate (for zero spread) such instabilities are suppressed, and it could be argued that the reduced growth rate corresponds to a resistive counterpart of the modulational instability.

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