Astron. Astrophys. 156, 268–276 (1986)

ASTRONOMY AND ASTROPHYSICS

Two-photon emission in X-ray pulsars

I. Basic formulas

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Received May 10, accepted July 3, 1985

Summary. It is suggested that the following processes may determine the spectra of X-ray pulsars such as Her X-1. (i) Non-radiative collisional excitation of electrons to their first Landau level, followed by radiative decay, is the only important source of photons. (ii) These cyclotron photons are transferred through the source by being absorbed and re-emitted ("resonant" scattering) implying a mean occupation level of the electrons in their first Landau level consistent with the population of cyclotron photons. (iii) Non-cyclotron photons below the cyclotron frequency can be produced by the re-emission process occurring through two-photon emission, rather than single photon emission, with the sum of the two frequencies equal to the cyclotron frequency. (iv) Non-cyclotron photons above the cyclotron frequency can be produced by a related process.

Specific formulas describing the two-photon emission process are derived and expressed in the form of an averaged emission probability. Compound probabilities are defined for absorption plus re-emission and for non-radiative collisional excitation plus cyclotron emission. Interpolation formulas relate these compound probabilities to the probabilities for resonant scattering and resonant bremsstrahlung respectively.

Key words: X-rays: binaries – stars: neutron – accretion – plasmas – radiation transfer

1. Introduction

The discovery of a cyclotron feature in the spectrum of the X-ray pulsar Her X-1 (Trümper et al., 1978) appeared, initially, to confirm a prediction by Basko and Sunyaev (1975) for the presence of such a line in the spectrum of an accreting, magnetized neutron star. The prediction involved two arguments relating, respectively, to the "non-cyclotron" and "cyclotron" photons. The non-cyclotron photons were assumed to be generated as in an unmagnetized X-ray source (e.g. Felten and Rees, 1972). The ultimate source of these photons is bremsstrahlung, but because the absorption per unit length κ_B for this process is smaller than the corresponding quantity κ_s for (Thomson) scattering, the resulting spectrum is below the black-body level by a factor $(\kappa_B/\kappa_s)^{1/2}$. On the other hand the cyclotron absorption per unit length κ_c is greater than κ_s, and hence near the cyclotron frequency the spectrum should rise to the black-body level, producing a cyclotron line in emission. This second argument has been rejected by subsequent authors on the grounds that cyclotron absorption is followed rapidly by re-emission of the photon, and hence should be regarded as resonant scattering. For electrons in their ground state, the (Thomson) scattering cross-section formally has a divergence at the cyclotron frequency $\omega = \Omega_e$, and this resonance is treated by replacing the resonant denominator $\omega - \Omega_e$ by $\omega - \Omega_e + i\Gamma/2$ where Γ is the decay rate of the first excited state due to cyclotron emission (e.g. Ventura, 1979). There is also a resonance in the cross-section for bremsstrahlung and this has been treated in an analogous way (e.g. Kirk and Mészáros, 1980). Most discussions of the formation of the spectra of X-ray sources with cyclotron features have concentrated on various aspects of the transfer of radiation through the strongly scattering source region (e.g. Bonazzola et al., 1979; Nagel, 1980; Kaminker et al., 1983). Although the assumed ultimate source of photons is usually implicit, the implication is that all photons are generated by bremsstrahlung.

Our primary purpose in this paper and in an accompanying paper (Kirk and Melrose, 1985; hereinafter, Paper II) is to explore the possible role of two-photon cyclotron emission in the formation of X-ray spectra. To this end it is necessary to derive compound expressions for resonant scattering, resonant bremsstrahlung and related processes, and also to derive the transport equation which governs the transfer of radiation according to these processes. We propose and discuss the following alternative viewpoint on the origin of cyclotron and non-cyclotron photons: the only important source of photons is non-radiative collisional excitation of electrons to their first Landau level, with cyclotron photons being produced in the subsequent radiative decay; non-cyclotron photons are produced by a small fraction of the resonant scattering events involving two-photon emission. A preliminary discussion of these ideas has been given by Kirk, et al. (1984), who argued that two-photon emission is a more important source of non-cyclotron photons than is bremsstrahlung.

The distinction between "cyclotron" and "non-cyclotron" photons is made here simply in order to facilitate the discussion. In the transport equation itself such a distinction is unnecessary. The role of two-photon emission envisaged here is one of effectively converting a cyclotron photon, which is absorbed by an electron, into two non-cyclotron photons when the subsequent

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re-emission is via the two-photon process. This can produce a spectrum of non-cyclotron photons below the cyclotron line, i.e. at $\omega < \Omega_e$. Non-cyclotron photons at $\omega > \Omega_e$ can be produced by a related process. First an electron absorbs a cyclotron photon so that it is in its first excited state. The excited electron then scatters a non-cyclotron photon at $\omega < \Omega_e$ with simultaneous transition to the ground state so that the scattered photon has frequency $\omega' = \omega + \Omega_e$. In this way it is possible in principle for the entire spectrum of non-cyclotron photons to arise through secondary processes involving the cyclotron photons. Feynman diagrams for all relevant processes are illustrated in Fig. 1.

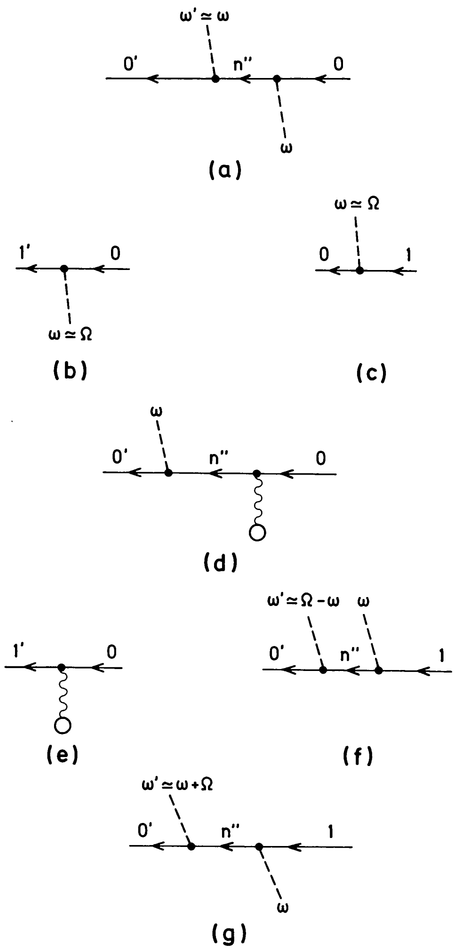

Fig. 1. Feynman diagrams. The initial state is on the lower right, the final state is on the upper left; a solid line with an arrow denotes an electron with the numbers labelling the value of n; the dashed lines indicate photons labelled with their frequency; the squiggly line and the circle denote an interaction with the field of another particle. In Compton scattering (a) there are significant contributions from $n'' = 0$, 1 and 2 in general. The probability exhibits a resonance at $\omega \cong \Omega$ due to the intermediate state $n'' = 1$. In the centre of the line the intermediate state is real in the sense that the "scattering" separates into two uncorrelated processes, namely cyclotron absorption (b) and cyclotron emission (c). Bremsstrahlung (d) is analogous to Compton scattering (a) with the initial photon replaced by an interaction with the field of another particle. There is again a resonance at $\omega \cong \Omega$ due to $n'' = 1$. In its centre this resonance corresponds to a real intermediate state: the collisional interaction causes a non-radiative transition to the first excited state (e), and there is an uncorrelated decay, usually through process (c). Non-resonant photons may be produced by a small fraction of the decays occuring through the two-photon process (f) or the compound scattering-decay process (g) rather than through process (c)

We adopt a viewpoint which is qualitatively different from that adopted by earlier authors, e.g. Herold (1979), Mészáros and Ventura (1979), Kirk and Mészáros (1980), Nagel (1981), who used expressions which incorporate both the resonant and non-resonant terms. The physical distinction which we make is between processes which involve real and virtual intermediate electron states. There are several notable implications of the intermediate states being real for resonant scattering and resonant bremsstrahlung. First, for an intermediate state to be real, the electron must spend a non-negligible time in it, implying a non-negligible mean occupation number for the excited state. As the dominant excitation and de-excitation processes are cyclotron absorption and emission, respectively, the mean occupation number for the excited state is proportional to the mean occupation number of the cyclotron photons (Melrose 1981). Second, for a real intermediate state, the excitation and de-excitation processes are uncorrelated. This allows us to separate the discussion of the excitation processes, which include non-radiative collisional excitation, and the de-excitation processes, which include two-photon emission. Third, although it is not important in the present application, it is essential to regard the intermediate state as real to include the effects of induced or stimulated emission correctly (Melrose, 1981). Fourth, the kinematic conditions are different when the intermediate is real and when it is virtual. For a real intermediate state in resonant scattering, energy must be conserved separately in the excitation and de-excitation processes, whereas for a virtual intermediate state energy need be conserved only between the initial and the final states. Some of the implications of these kinematic restrictions have been discussed by Wassermann and Salpeter (1980). In describing the resonant processes in terms of compound probabilities we include the induced processes and the two separate energy conservation conditions (expressed through the product of two δ-functions) explicitly, and we show that the usual procedure for treating resonant denominators provides a convenient interpolation formula between the regimes of true scattering and absorption emission.

In Sect. 2 general formulas relevant to these various processes are written down and in Sect. 3 an approximate expression describing two-photon emission is derived. In Sect. 4 we define and derive the compound probabilities, and show that the usual procedure of treating resonant denominators provides a convenient interpolation formula between the regime of true scattering and that of the emission-absorption. In Paper II we demonstrate that such compound probabilities arise naturally in the transport equation, and we discuss the relative importance of bremsstrahlung and the two-photon process in the accretion columns of X-ray pulsars.

2. General expressions for probabilities

We introduce the notation used here and then write down probabilities (rates per unit time and per unit volume of k-space) for the processes of relevance here. Most of the following results have been written down previously by Melrose and Parle (1983a,b) who gave further details of their derivations.

2.1. *Summary of notation*

We use natural units $\hbar = c = 1$, except where stated otherwise, and SI units in describing the electromagnetic field; gaussian units are obtained by setting $\varepsilon_0 (= 1/\mu_0 c^2$ before setting $c = 1)$ equal to $1/4\pi$.

2.1.1. Electrons

Electron (or positron) states are denoted by the sign ϵ of the energy ($\epsilon = 1$ for electrons, $\epsilon = -1$ for positrons) and a set q of quantum numbers. This set includes the principal quantum number $n = 0, 1, 2, \ldots$ and the orbital $l = 0, 1, 2, \ldots$ and spin $\sigma = \pm 1$ quantum numbers, which are related by

$$n = l + \tfrac{1}{2}(1 + \sigma) . \tag{1}$$

There is an arbitrariness in the choice of σ, corresponding to an arbitrariness in the choice of the spin operator which we choose to be the z-component (i.e. the component along **B**) of the magnetic moment operator. Sokolov and Ternov (1968) referred to this choice as "transverse polarization". This is different from the spin operator implicit in the use of the wave functions of Johnson and Lippmann (1949) for example. Another quantum number included in the set q is the z-component of momentum p_z. We choose the sign of p_z by writing the plane wave functions as $\exp[-i\epsilon(\varepsilon t - p_z z)]$. Specifically, p_z (and not $-p_z$) is the physical (parallel) momentum for a positron. The energy eigenvalues are ε_q with

$$\varepsilon_q = (m^2 + p_z^2 + 2neB)^{1/2} . \tag{2}$$

There is a further quantum number which depends on the choice of coordinate axes and of the gauge for the background magnetic field. If we choose Cartesian coordinates with the vector potential **A** for the magnetostatic field depending only on x, called the "Landau gauge", then p_y is an appropriate choice of this additional quantum number. With the plane wave function now including a factor of the form $\exp[ip_y y]$, p_y specifies the center of gyration as being at $x = -\epsilon p_y/eB$. Another convenient choice (Sokolov and Ternov 1968) is in cylindrical polar coordinates with the gauge such that **A** is independent of the azimuthal angle, called the "cylindrical gauge". The relevant quantum number is s (with $n - s = 0, 1, 2, \ldots$) which is interpreted as specifying the distance of the center of gyration from the axis in units of the gyroradius. In practice we average over the positions of the gyrocenters and the resulting expressions then do not depend on the specific choice of this additional quantum number. Melrose and Parle (1983b) formulated rules for writing down probabilities averaged in this way.

2.1.2. The vertex function

A quantity which plays a central role in our theoretical development is the vertex function $[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^\mu$. This is obtained from a matrix element of the Dirac matrix γ^μ between an initial electron (ϵ, q) and a final electron (ϵ', q'), with a wave with 3-vector $\mathbf{k} = (k_\perp \cos\psi, k_\perp \sin\psi, k_\parallel)$ emitted at the vertex. The 4-tensor notation has μ running over 0 to 3 with signature $+---$. There are sixteen components of the 4×4 γ-matrices for each value of μ, the sixteen components of the Γ-functions correspond to the choices $\epsilon = \pm 1$, $\epsilon' = \pm 1$, $\sigma = \pm 1$, $\sigma' = \pm 1$. An explicit expression for the vertex function for "transverse polarization" is

$$\begin{aligned}
[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^\mu = C^*_{q'} C_q [& \delta_{\sigma'\sigma}\{\alpha^{\epsilon'\epsilon}_{q'q}(J^l_{l'-l} + \rho'_{n'}\rho_n J^{l+\sigma}_{l'-l}), \\
& \epsilon\beta^{\epsilon'\epsilon}_{q'q}(-\rho_n e^{i\sigma\psi} J^{l+\sigma}_{l'-l-\sigma} - \rho'_{n'} e^{-i\sigma\psi} J^l_{l'-l+\sigma}), \\
& i\epsilon\sigma\beta^{\epsilon'\epsilon}_{q'q}(\rho_n e^{i\sigma\psi} J^{l+\sigma}_{l'-l-\sigma} - \rho'_{n'} e^{-i\sigma\psi} J^l_{l'-l+\sigma}), \\
& \eta^{\epsilon'\epsilon}_{q'q}(J^l_{l'-l} + \rho'_{n'}\rho_n J^{l+\sigma}_{l'-l})\} \\
& - \epsilon\sigma\delta_{\sigma'-\sigma}\{a^{\epsilon'\epsilon}_{q'q}(-\rho_n e^{i\sigma\psi} J^{l+\sigma}_{l'-l-\sigma} + \rho'_{n'} e^{i\sigma\psi} J^l_{l'-l-\sigma}), \\
& \epsilon b^{\epsilon'\epsilon}_{q'q}(J^l_{l'-l} + \rho'_{n'}\rho_n e^{2i\sigma\psi} J^{l+\sigma}_{l'-l-2\sigma}), \\
& i\epsilon\sigma b^{\epsilon'\epsilon}_{q'q}(J^l_{l'-l} + \rho'_{n'}\rho_n e^{2i\sigma\psi} J^{l+\sigma}_{l'-l-2\sigma}), \\
& d^{\epsilon'\epsilon}_{q'q}(-\rho_n e^{i\sigma\psi} J^{l+\sigma}_{l'-l-\sigma} + \rho'_{n'} e^{i\sigma\psi} J^l_{l'-l-\sigma})\}]
\end{aligned} \tag{3}$$

with

$$C_q := \left[\frac{(\varepsilon_q + \varepsilon^0_q)(\varepsilon^0_q + m)}{4\varepsilon_q \varepsilon^0_q}\right]^{1/2} (ie^{i\psi})^l , \tag{4}$$

and with

$$\begin{aligned}
\alpha^{\epsilon'\epsilon}_{q'q} &= \delta_{\epsilon'\epsilon}(1 + \rho'_z\rho_z) + \sigma\delta_{\epsilon'-\epsilon}(\rho'_z + \rho_z) \\
\beta^{\epsilon'\epsilon}_{q'q} &= \delta_{\epsilon'\epsilon}(1 - \rho'_z\rho_z) + \sigma\delta_{\epsilon'-\epsilon}(\rho'_z - \rho_z) \\
\eta^{\epsilon'\epsilon}_{q'q} &= \delta_{\epsilon'\epsilon}(\rho'_z + \rho_z) + \sigma\delta_{\epsilon'-\epsilon}(1 + \rho'_z\rho_z) \\
a^{\epsilon'\epsilon}_{q'q} &= \delta_{\epsilon'\epsilon}(\rho'_z + \rho_z) - \sigma\delta_{\epsilon'-\epsilon}(1 + \rho'_z\rho_z) \\
b^{\epsilon'\epsilon}_{q'q} &= \delta_{\epsilon'\epsilon}(\rho'_z - \rho_z) - \sigma\delta_{\epsilon'-\epsilon}(1 - \rho'_z\rho_z) \\
d^{\epsilon'\epsilon}_{q'q} &= \delta_{\epsilon'\epsilon}(1 + \rho'_z\rho_z) - \sigma\delta_{\epsilon'-\epsilon}(\rho'_z + \rho_z) .
\end{aligned} \tag{5}$$

where we use the notations

$$\begin{aligned}
&\rho_z := \frac{p_z}{\varepsilon_q + \varepsilon^0_q}, \qquad \rho_n := \frac{p_n}{\varepsilon^0_q + m} \\
&\rho'_z := \frac{p'_z}{\varepsilon_{q'} + \varepsilon^0_{q'}}, \qquad \rho'_n := \frac{p_{n'}}{\varepsilon^0_{q'} + m} \\
&p_n \equiv (2neB)^{1/2}, \qquad \varepsilon^0_q \equiv (m^2 + p_n^2)^{1/2} .
\end{aligned} \tag{6}$$

The functions $J^n_\nu(x)$ have argument $x = k^2_\perp/2eB$ and are defined by

$$\begin{aligned}
J^n_\nu(x) &= \{n!/(n+\nu)!\}^{1/2} \exp(-\tfrac{1}{2}x) x^{1/2\nu} L^\nu_n(x) \\
&= (-)^\nu J^{n+\nu}_{-\nu}(x) ,
\end{aligned} \tag{7}$$

where $L^\nu_n(x)$ is a generalized Laguerre polynomial.

2.1.3. Waves

The response of the medium is described in terms of a response 4-tensor $\alpha^{\mu\nu}(k)$ or in terms of the dielectric 3-tensor $K_{ij}(\omega, \mathbf{k})$, which is such that $K_{ij}(\omega, \mathbf{k}) - \delta_{ij}$ is equal to minus $c/\omega^2\varepsilon_0$ (before setting $c = 1$) times the $\mu = i$, $\nu = j$ component of $\alpha^{\mu\nu}(k)$. The inhomogeneous wave equation

$$\Lambda^{\mu\nu}(k) A_\nu(k) = -\mu_0 J^\mu_{\text{ext}}(k) , \tag{8}$$

with

$$\Lambda^{\mu\nu}(k) = k^2 g^{\mu\nu} - k^\mu k^\nu + \mu_0 \alpha^{\mu\nu}(k) , \tag{9}$$

has solution

$$A^\mu(k) = D^\mu_\nu(k) J^\nu_{\text{ext}}(k) , \tag{10}$$

where $D^{\mu\nu}(k)$ is the "photon propagator" and where $J^\mu_{\text{ext}}(k)$ is an arbitrary extraneous current. Here $D^{\mu\nu}(k)$ is the photon propagator in the medium, be it a plasma, the birefringent vacuum or a combination of the two.

The natural wave modes correspond to poles in $D^{\mu\nu}(k)$. Wave modes are labelled M, M', etc., with (i) dispersion relations $\omega =$

$\omega_M(\mathbf{k})$ $(=-\omega_M(-\mathbf{k}))$ etc., with (ii) polarization 4-vectors $e^\mu_M(\mathbf{k}) = e^{*\mu}_M(-\mathbf{k})$ normalized by

$$e^\mu_M(\mathbf{k})e^*_{M\mu}(\mathbf{k}) = -1\,, \tag{11}$$

where * denotes complex conjugation, and with (iii) ratio $R_M(\mathbf{k})$ of electric to total energy in the waves. For most purposes here we assume that the waves are approximately transverse with $\omega_M \cong |\mathbf{k}|$, $R_M(\mathbf{k}) \cong \frac{1}{2}$.

2.2. *Specific probabilities*

We now write down general expressions (Melrose and Parle, 1983b) for the relevant probabilities (average over the positions of the gyrocenters where relevant).

2.2.1. Single-photon emission

The probability $w^M_{q'q}(\mathbf{k})$ that an electron in state ϵ, q emit a photon in the mode M with transition to a state ϵ', q' is

$$w^M_{q'q}(\mathbf{k}) = \frac{e^2 R_M(\mathbf{k})}{\varepsilon_0|\omega_M(\mathbf{k})|}\,|e^*_{M\mu}(\mathbf{k})[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^\mu|^2 2\pi\delta(\epsilon\varepsilon_q - \epsilon'\varepsilon_{q'} - \omega_M(\mathbf{k}))\,. \tag{12}$$

This probability includes all the relevant crossed processes: the signs $\epsilon = \pm 1$ and $\epsilon' = \pm 1$ correspond to electron and positron in the initial and final state, respectively, and absorption of the photon corresponds to the expression (12) with $\mathbf{k} \to -\mathbf{k}$,

$$[\Gamma^{\epsilon'\epsilon}_{q'q}(-\mathbf{k})]^\mu = [\Gamma^{\epsilon\epsilon'}_{qq'}(\mathbf{k})]^{*\mu} \tag{13}$$

One-photon pair annihilation is included through $\epsilon = 1, \epsilon' = -1$, and the probability for the inverse process of one-photon pair creation follows from (12) with $\epsilon = -1$, $\epsilon' = 1$ and $\mathbf{k} \to -\mathbf{k}$.

2.2.2. Two-photon emission

The probability $w^{M'M}_{q'q}(\mathbf{k}'\,\mathbf{k})$ that two photons are emitted in the transition $\epsilon, q \to \epsilon', q'$ is

$$w^{M'M}_{q'q}(\mathbf{k}', \mathbf{k}) = \frac{e^4 R_{M'}(\mathbf{k})R_M(\mathbf{k})}{\varepsilon_0^2|\omega_{M'}(\mathbf{k}')\omega_M(\mathbf{k})|}\,|e^*_{M\mu}(\mathbf{k})e^*_{M'\nu}(\mathbf{k}')M^{\mu\nu}(\mathbf{k}, \mathbf{k}')|^2$$
$$\times\, 2\pi\delta(\epsilon\varepsilon_q - \epsilon'\varepsilon_{q'} - \omega_M(\mathbf{k}) - \omega_{M'}(\mathbf{k}'))\,; \tag{14a}$$

$$\begin{aligned} M^{\mu\nu}(\mathbf{k}, \mathbf{k}') = {} & \sum_{q''} \frac{[\Gamma^{\epsilon'\epsilon''}_{q'q''}(\mathbf{k})]^\mu [\Gamma^{\epsilon''\epsilon}_{q''q}(\mathbf{k}')]^\nu}{\epsilon\varepsilon_q - \omega_{M'}(\mathbf{k}') - \epsilon''\varepsilon_{q''}} \exp\left(-\frac{i}{2eB}(\mathbf{k}' \times \mathbf{k})_z\right) \\ & + \sum_{q''} \frac{[\Gamma^{\epsilon'\epsilon''}_{q'q''}(\mathbf{k}')]^\nu [\Gamma^{\epsilon''\epsilon}_{q''q}(\mathbf{k})]^\mu}{\epsilon\varepsilon_q - \omega_M(\mathbf{k}) - \epsilon''\varepsilon_{q''}} \exp\left(\frac{i}{2eB}(\mathbf{k}' \times \mathbf{k})_z\right) \\ & + (2/e)[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k} + \mathbf{k}')]^\theta D_{\theta\eta}(k_M + k'_{M'}) \\ & \times \alpha^{\eta\mu\nu}(k_M + k'_{M'}, -k_M, -k'_{M'}) \end{aligned} \tag{14b}$$

where k_M denotes $(\omega_M(\mathbf{k}), \mathbf{k})$ and similarly for $k'_{M'}$. The final term involving the photon propagator $D^{\mu\nu}(k)$ and the quadratic response tensor $\alpha^{\mu\nu\rho}(k, k', k'')$ for the medium is neglected here; its contribution for Compton scattering has been discussed by Stoneham (1980a,b).

The crossed processes included in (14) are two-photon emission by an electron or a positron, Compton scattering by an electron or a positron and two-photon pair annihilation and creation.

In paper II we also mention double Compton scattering. The probability for this may be obtained from the probability for three-photon emission through a crossing symmetry. The probability for three-photon emission may be written down in a form analogous to (12) and (14). We shall not require an explicit expression for this probability.

2.2.3. Non-radiative collisional excitation by an external field (an ion)

The probability for a transition ϵ, q to ϵ', q' due to a given external field with 4-potential $A^\mu(k)$ is

$$P_{q'q} = e^2 \int \frac{d^4k}{(2\pi)^4}\, 2\pi\delta(\epsilon\varepsilon_q - \epsilon'\varepsilon_{q'} - \omega)|[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^\mu A^*_\mu(k)|^2 \tag{15}$$

In practice we are interested in a transition due to the field of an ion. Simple arguments suggest that for many purposes the ion (charge Q, mass M) may be regarded as classical and unmagnetized. Let its velocity and Lorentz factor be $\mathbf{V}$ and Γ respectively, with 4-velocity $U = (\Gamma, \Gamma\mathbf{V})$. Then we have

$$A^\mu(k) = Q e^{ikx_0} D^{\mu\nu}(k) U_\nu 2\pi\delta(kU)\,, \tag{16}$$

where $x_0 = (t_0, \mathbf{x}_0)$ denotes the initial conditions. The recoil of the ion may be included by making the replacement

$$\delta(kU) \to \frac{M}{E}\,\delta(E' - E - \omega) \tag{17}$$

with $E = E(\mathbf{P}) = (M^2 + |\mathbf{P}|^2)^{1/2}$, $E' = E(\mathbf{P}')$, $\mathbf{P} = \Gamma M\mathbf{V}$ and $\mathbf{P}' = \mathbf{P} - \mathbf{k}$. Here we shall consider only the simplest case where the ion is at rest and its recoil is neglected. Then in the Coulomb gauge we have $A^\mu(k) = (\Phi(k), 0)$ with

$$\Phi(k) = \frac{Q 2\pi\delta(\omega)e^{ikx_0}}{4\pi\varepsilon_0|\mathbf{k}|^2 K^L(\omega, \mathbf{k})} \tag{18}$$

where $K^L(\omega, \mathbf{k})$ is the longitudinal part of the dielectric tensor. The probability per unit time $w^c_{q'q}$ of a non-radiative collisional transition then becomes

$$w^c_{q'q} = \frac{e^2Q^2}{(4\pi\varepsilon_0)^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{2\pi\delta(\epsilon\varepsilon_q - \epsilon'\varepsilon_{q'})}{\{|\mathbf{k}|^2K^L(0, \mathbf{k})\}^2}\,|[\Gamma^{\epsilon'\epsilon}_{q'q}(\mathbf{k})]^0|^2\,. \tag{19}$$

2.2.4. Electron-ion bremsstrahlung

The probability $p^M_{q'q}(\mathbf{k})$ that a photon is emitted in the transition $q \to q'$ in the external field is

$$P^M_{q'q}(\mathbf{k}) = \frac{e^4}{\varepsilon_0}\frac{R_M(\mathbf{k})}{|\omega_M(\mathbf{k})|} \int \frac{d^4k'}{(2\pi)^4}\,|e^*_{M\mu}(\mathbf{k})M^{\mu\nu}(\mathbf{k}, \mathbf{k}')A^*_\nu(k')|^2$$
$$\times\, 2\pi\delta(\epsilon\varepsilon_q - \epsilon'\varepsilon_{q'} - \omega_M(\mathbf{k}) - \omega') \tag{20}$$

with $M^{\mu\nu}(\mathbf{k}, \mathbf{k}')$ given by (14). With the assumptions made above for $A^\mu(k)$, the probability per unit time that a photon be emitted becomes

$$w^c_{q'q}(\mathbf{k}) = \frac{e^4Q^2}{\varepsilon_0(4\pi\varepsilon_0)^2}\frac{R_M(\mathbf{k})}{|\omega_M(\mathbf{k})|} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,|e^*_{M\mu}(\mathbf{k})M^{\mu 0}(\mathbf{k}, \mathbf{k}')|^2$$
$$\times\, \frac{\delta(\epsilon\varepsilon_q - \epsilon'\varepsilon_{q'} - \omega_M(\mathbf{k}))}{\{|\mathbf{k}|^2K^L(0, \mathbf{k})\}^2}\,. \tag{21}$$

2.2.5. Electron-electron collisions

The method used above may be extended to treat electron-electron collisions. An expression for the cross-section for this process was derived by Langer (1981). Apart from differences in notation, Langer's result differs from that obtained using the present method in two notable ways. First, Langer used Johnson and Lippmann's (1949) wavefunctions which correspond to eigenfunctions of a physically obscure spin operator. As a consequence only spin-averaged forms of his results are of physical significance. The choice of spin eigenfunctions made here corresponds to a spin operator which commutes with the Hamiltonian (Sokolov and Ternov, 1968), as one requires of any physically meaningful spin operator. In the standard representation this operator is

$$\mu_z = m\sigma_z + \rho_y[\boldsymbol{\sigma} \times (\mathbf{p} + e\mathbf{A})]_z . \tag{22}$$

Second, Langer ignored the effects of the medium through his choice of photon propagator, which corresponds to $D^{\mu\nu}(k) = -g^{\mu\nu}/\varepsilon_0 k^2$ in the present notation. Further investigation of electron scattering is in progress and will be reported elsewhere.

3. Approximate probability for two-photon emission

In this section we derive an approximate expression for the probability of two-photon emission, and of the related process of Compton scattering with a transition $n = 1$ to $n' = 0$ or $n = 0$ to $n' = 1$. First, we introduce some of the approximations by deriving the known approximate expression for one-photon emission. We then derive the approximate probability for two-photon emission and use a crossing symmetry to derive the probability for Compton scattering $0 \to 1'$ from it.

3.1. One-photon emission

In treating one-photon emission by an electron ($\epsilon = \epsilon' = 1$) we start from the probability (12) and make the following approximations:

(i) Dispersion in the medium is neglected, implying $\omega_M(\mathbf{k}) = |\mathbf{k}|$ and $R_M(\mathbf{k}) = \frac{1}{2}$. We later sum over the two states of transverse polarization.

(ii) The magnetic field is "weak", i.e. $eB \ll m^2$. This corresponds to $B \ll B_c = 4.4\,10^9\,\mathrm{T} = 4.4\,10^{13}\,\mathrm{G}$.

(iii) The electron is non-relativistic, implying $n \ll m^2/2eB = B_c/2B$ and $p_3^2 \ll m^2$.

(iv) The photon is of "low" frequency, i.e. $\omega \ll m$, corresponding to a photon energy $\ll \frac{1}{2}\,\mathrm{MeV}$.

With these approximations we may set $\varepsilon_q, \varepsilon_q^0, \varepsilon_{q'}$ and $\varepsilon_{q'}^0$ equal to m, except where a difference between them appears explicitly, as in the δ-function in (12), and also we may ignore ρ_z, ρ_z', ρ_n and ρ_n', in comparison with unity, cf. (6). Also the argument of the functions $J_\nu^n(x)$ is small ($x \ll 1$) so that they may be approximated according to

$$J_\nu^n(x) = \begin{cases} \left\{\dfrac{(n+\nu)!}{n!}\right\}^{1/2} \dfrac{x^{\nu/2}}{\nu!} & \nu > 0 \\[2ex] (-)^\nu \left\{\dfrac{n!}{(n-|\nu|)!}\right\}^{1/2} \dfrac{x|\nu|/2}{|\nu|!} & \nu < 0 \end{cases} \tag{23}$$

We consider only transitions from the first excited state ($n = 1$, $l = 1$, $\sigma = -1$) to the ground state ($n' = 0$, $l' = 0$, $\sigma' = -1$) without a spin flip (i.e. $\sigma = \sigma'$). The 3-vector part of the vertex function for this case may approximated by

$$\boldsymbol{\Gamma}_{0'1}^{++}(\mathbf{k}) \cong -i\left(\frac{B}{2B_c}\right)^{1/2} (1, i, 0) . \tag{24}$$

We choose two transverse polarizations

$$\mathbf{t} = (\cos\theta\cos\psi, \cos\theta\sin\psi, -\sin\theta) \tag{25a}$$

$$\mathbf{a} = (-\sin\psi, \cos\psi, 0) \tag{25b}$$

and write

$$M_t \equiv \mathbf{t}\cdot\boldsymbol{\Gamma}_{0'1}^{++}(\mathbf{k}) = -i\left(\frac{B}{2B_c}\right)^{1/2} e^{i\psi}\cos\theta \tag{26a}$$

$$M_a \equiv \mathbf{a}\cdot\boldsymbol{\Gamma}_{0'1}^{++}(\mathbf{k}) = \left(\frac{B}{2B_c}\right)^{1/2} e^{i\psi} . \tag{26b}$$

The probability for emission in each polarization then reduces to

$$w_{0'1}^{t,a}(\mathbf{k}) = \frac{e^2}{2\varepsilon_0\omega} |M_{t,a}|^2 2\pi\delta\left\{\omega - \Omega - k_z v_z + \frac{k_z^2}{2m}\right\} \tag{27}$$

with $k_z = \omega\cos\theta$ and $v_z = p_z/m$ here. In (27) the first quantum correction to the cyclotron frequency is included

$$\Omega = \frac{eB}{m}\left(1 - \frac{B}{2B_c}\right) \tag{28}$$

and only terms up to the first quantum recoil are retained in the expansion of the argument of the δ-function.

The transition rate Γ from the first excited state to the ground state is found by summing over the two states of polarization and integrating over $\mathbf{k}$-space:

$$\Gamma = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \{w_{0'1}^t(\mathbf{k}) + w_{0'1}^a(\mathbf{k})\} . \tag{29}$$

For an electron at rest ($p_z = 0$) (26) and (27) in (29), with the quantum recoil neglected give the well known result

$$\Gamma = \tfrac{4}{3}\alpha\Omega\frac{B}{B_c} , \tag{30}$$

where $\alpha(= e^2/4\pi\varepsilon_0\hbar c)$ is the fine structure constant.

3.2. Two-photon emission

We now treat two-photon emission from the first excited state ($n = 1, l = 1, \sigma = -1$) to the ground state in a manner analogous to the treatment of one-photon emission in part (a). Except in the argument of the δ-function we set $p_z = 0$ and ignore terms of higher order in B/B_c. The δ-function in (14) may be rewritten, for $\epsilon = \epsilon' = 1$,

$$\delta(\varepsilon_q - \varepsilon_{q'} - \omega - \omega') = \delta\left\{\omega + \omega' - \Omega - (k_z + k_z')v_z + \frac{(k_z + k_z')^2}{2m}\right\} \tag{31}$$

with $k_z = \omega\cos\theta$ and $k_z' = \omega'\cos\theta'$, cf. (27). From a kinematic viewpoint two-photon emission is like one-photon emission with the original ω, k_z replaced by $\omega + \omega'$, $k_z + k_z'$.

3.2.1. Sum over intermediate states

The major complication in treating two-photon emission is in performing the sum over the intermediate state q'' in (14). In (14) the sum over p_z'' and over the additional quantum number (e.g. p_y'' or s'') has already been performed, and the remaining sum is over n'', σ'' and ϵ''. In the transition from $n = 1$ to $n' = 0$ there are contributions from $n'' = 0$, 1 and 2 to lowest order in our expansion parameters; both electron ($\epsilon'' = 1$) and positron ($\epsilon'' = -1$) intermediate states contribute, and both spin flip ($\sigma'' = 1$)

and non-spin-flip ($\sigma'' = -1$) transitions also contribute. Some of the details of the evaluation of the sum are given in the Appendix.

A convenient way of presenting the result after this sum has been performed is in terms of a probability expressed as a polarization tensor. Let α and β run over **t** and **a** and α' and β' run over **t**$'$ and **a**$'$, cf. (25a,b). The approximate probability may then be written as the fourth-rank polarization tensor

$$w_{0'1}^{\alpha\alpha'\beta\beta'}(\mathbf{k}',\mathbf{k}) = \frac{e^4}{4\varepsilon_0^2|\omega\omega'|}(M^{\alpha\alpha'})(M^{\beta\beta'})^* 2\pi\delta\left(\omega+\omega'-\Omega - (k_z+k_z')v_z + \frac{(k_z+k_z')^2}{2m}\right). \quad (32)$$

The lengthy calculation summarized in the Appendix gives

$$M^{tt'} = \frac{i}{2m\sqrt{2eB}}\Big[-\{(\Omega-\omega)\sin\theta'\cos\theta'\cos\theta + \frac{2\omega}{\omega'}\sin\theta' \times(\omega'\sin^2\theta+\Omega\cos^2\theta)\}e^{i\psi} - \{(\Omega-\omega')\sin\theta\cos\theta\cos\theta' + \frac{2\omega'}{\omega}\sin\theta(\omega\sin^2\theta+\Omega\cos^2\theta)\}e^{i\psi'} + \frac{\omega^2}{\Omega+\omega'}\sin\theta\cos\theta \times\cos\theta' e^{2i\psi-i\psi'} + \frac{\omega'^2}{\Omega+\omega}\sin\theta'\cos\theta'\cos\theta e^{i\psi+2i\psi'}\Big] \quad (33a)$$

$$M^{aa'} = \frac{i}{2m\sqrt{2eB}}\Big[(\Omega+\omega)\sin\theta' e^{i\psi} + (\Omega+\omega')\sin\theta e^{i\psi'} + \frac{\omega^2\sin\theta}{\Omega+\omega'}e^{2i\psi-i\psi'} + \frac{\omega'^2\sin\theta'}{\Omega+\omega}e^{-i\psi+2i\psi'}\Big] \quad (33b)$$

$$M^{ta'} = \frac{i}{2m\sqrt{2eB}}\Big[-i(\Omega+\omega)\sin\theta'\cos\theta e^{i\psi} - i\left\{(\Omega-\omega')\sin\theta\cos\theta + \frac{2\Omega\omega'}{\omega}\cos\theta'\sin\theta\right\}e^{i\psi'} - i\frac{\omega^2\sin\theta\cos\theta}{\Omega+\omega'}e^{2i\psi-i\psi'} + i\frac{\omega'^2\sin\theta'\cos\theta}{\Omega+\omega}e^{-i\psi+2i\psi'}\Big] \quad (33c)$$

$$M^{at'} = \frac{i}{2m\sqrt{2eB}}\Big[-i(\Omega+\omega')\sin\theta\cos\theta' e^{i\psi'} - i\left\{(\Omega-\omega)\sin\theta'\cos\theta' + \frac{2\Omega\omega}{\omega'}\cos\theta\sin\theta'\right\}e^{i\psi} - i\frac{\omega'^2\sin\theta'\cos\theta'}{\Omega+\omega}e^{-i\psi+2i\psi'} + i\frac{\omega^2\sin\theta\cos\theta'}{\Omega+\omega'}e^{2i\psi-i\psi'}\Big]. \quad (33d)$$

3.2.2. Sum over polarization and average over angles

If one is not interested in the polarization of the photons then it is appropriate to sum over the states of polarization. This sum corresponds to

$$\sum_{\text{poln}}(M^{\alpha\alpha'})(M^{\beta\beta'})^* = |M^{tt'}|^2 + |M^{aa'}|^2 + |M^{ta'}|^2 + |M^{at'}|^2. \quad (34)$$

In performing this sum considerable simplification occurs if one also averages over the azimuthal angles ψ and ψ' and over the polar angles θ and θ'. In the following we quote results in which the average over ψ and ψ' and an average over forward and backward directions have been performed; that is we neglect terms which vary as odd powers of $\cos\theta$ and $\cos\theta'$. We write

$$y = \frac{\omega}{\Omega}, \qquad y' = \frac{\omega'}{\Omega} = 1-y, \quad (35)$$

and neglect the difference $\omega+\omega'-\Omega$ except in the δ-function in (32). With the average denoted by angular brackets, we have

$$\langle|M^{tt'}|^2\rangle = \frac{\Omega}{8m^3}\Big\langle y'^2\sin^2\theta'\cos^2\theta'\cos^2\theta + \frac{4y^2}{y'^2}(y'^2\sin^4\theta+\cos^4\theta + 2y'\sin^2\theta\cos^2\theta)\sin^2\theta' + y^2\sin^2\theta\cos^2\theta\cos^2\theta' + \frac{4y'^2}{y^2}(y^2\sin^4\theta'+\cos^4\theta'+2y\sin^2\theta'\cos^2\theta')\sin^2\theta + \left(\frac{y^2}{1+y'}\right)^2\sin^2\theta\cos^2\theta\cos^2\theta' + \left(\frac{y'^2}{1+y}\right)^2\sin^2\theta'\cos^2\theta'\cos^2\theta\Big\rangle \quad (36a)$$

$$\langle|M^{aa'}|^2\rangle = \frac{\Omega}{8m^3}\Big\langle(1+y)^2\sin^2\theta' + (1+y')^2\sin^2\theta + \left(\frac{y^2}{1+y'}\right)^2 \times\sin^2\theta + \left(\frac{y'^2}{1+y}\right)^2\sin^2\theta'\Big\rangle \quad (36b)$$

$$\langle|M^{ta'}|^2\rangle = \frac{\Omega}{8m^3}\Big\langle(1+y)^2\sin^2\theta'\cos^2\theta + y^2\sin^2\theta\cos^2\theta + \frac{4y'^2}{y^2} \times\sin^2\theta\cos^2\theta' + \left(\frac{y^2}{1+y'}\right)^2\sin^2\theta\cos^2\theta + \left(\frac{y'^2}{1+y}\right)^2\sin^2\theta'\cos^2\theta\Big\rangle \quad (36c)$$

$$\langle|M^{at'}|^2\rangle = \frac{\Omega}{8m^3}\Big\langle y'^2\sin^2\theta'\cos^2\theta' + \frac{4y^2}{y'^2}\cos^2\theta\sin^2\theta' + (1+y')^2 \times\sin^2\theta\cos^2\theta' + \left(\frac{y'^2}{1+y'}\right)^2\sin^2\theta'\cos^2\theta' + \left(\frac{y^2}{1+y'}\right)^2 \times\sin^2\theta\cos^2\theta'\Big\rangle. \quad (36d)$$

3.2.3. The averaged probability for two-photon emission

The probability for two-photon emission averaged over all angles is obtained by averaging (36a to d) over θ and θ' and inserting the resulting expressions in the sum (34). Note that although we do this we also retain the dependence on $\cos\theta$ and $\cos\theta'$ in the δ-function. The reason is that the Doppler and recoil terms are important in determining the line shape, etc., but they are unimportant in determining the rate photons are emitted. Thus we write down the averaged probability (in normal units)

$$\langle w_{0'1}(\mathbf{k}',\mathbf{k})\rangle = \frac{(2\pi)^3}{45}\alpha^2\left(\frac{B}{B_c}\right)^3\frac{c^6}{\Omega^4}\frac{F(y)}{y^2(1-y)^2} \times\delta\left(\omega+\omega'-\Omega-(k_z+k_z')v_z + \frac{\hbar(k_z+k_z')^2}{2m}\right) \quad (37)$$

with

$$F(y) = |y(1-y)|\left[7\{y^2+(1-y)^2\} + 3\left\{\frac{y^4}{(2-y)^2}+\frac{(1-y)^4}{(1+y)^2}\right\} + 4\left(\frac{y^2}{(1-y)^2}+\frac{(1-y)^2}{y^2}\right) + 2\left(\frac{y^2}{1-y}+\frac{(1-y)^2}{y}\right) + 10\right]. \quad (38)$$

A plot of the function $F(y)$ is shown in Figure 2.

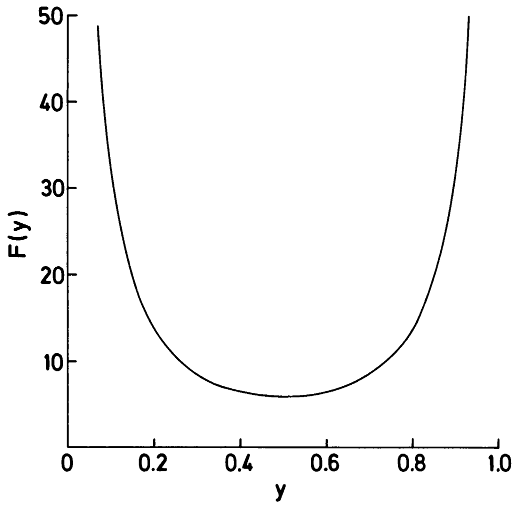

Fig. 2. The function $F(y)$ defined by (38) is plotted for $0 < y < 1$. Kirk et al. (1984) found that in this range the function is well approximated by $F(y) \cong 4[1 + y^3/(1-y) + (1-y)^3/y]$

3.2.4. Compton scattering $0 \to 1$

The probability for Compton scattering with a transition from the ground state to the first excited state is related to two-photon emission by a crossing symmetry. Suppose ω is the initial photon and ω' is the final photon; we require $\omega > \Omega$ for the process to be kinematically allowed. The probability for it is obtained from that for two-photon emission, cf. (14), by $\mathbf{k} \to -\mathbf{k}$. In (37) this is equivalent to replacing ω and k_z by $-\omega$ and $-k_z$ in the argument of the δ-function. Thus the probability for Compton scattering (cs) $0 \to 1$ is

$$\langle w^{cs}_{1'0}(\mathbf{k}', \mathbf{k})\rangle = \frac{(2\pi)^3}{45}\alpha^2\left(\frac{B}{B_c}\right)^3 \frac{c^6}{\Omega^4}\frac{F(-y)}{y^2(1+y)^2} \times \delta\left\{\omega' - \omega - \Omega - (k'_z - k_z)v_z + \frac{\hbar(k_z - k_z)^2}{2m}\right\} \tag{39}$$

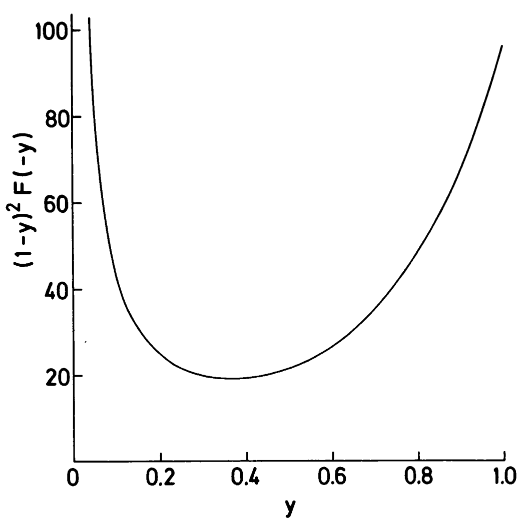

Fig. 3. The function $(1-y)^2F(-y)$ is plotted for $0 < y < 1$. Note that the function $F(-y)$ diverges as $96/(1-y)^2$ for $y \cong 1$. The function also diverges at $y = 2$ due to the resonance at the second harmonic $\omega = 2\Omega$

where $\mathbf{k} \to -\mathbf{k}$ also involves changing the sign of y in (38). The function $F(-y)(1-y)^2$ is plotted in Fig. 3.

4. Resonances and compound probabilities

Melrose (1981) defined compound or effective (Compton) scattering probability $0 \to 0'$ due to absorption $0 \to 1''$ followed by emission $1'' \to 0'$. Near the resonance at the cyclotron frequency it was argued that the probability for Compton scattering overlaps with this compound probability; a Lorentzian profile with decay rate Γ identified as the inverse radiative lifetime (including the effects of induced emission) of the excited state provides an interpolation between the two. Here we establish this result in a more general way and show that an analogous result applies to the resonance at the cyclotron frequency in bremsstrahlung, with the compound probability then being for non-radiative collisional excitation $0 \to 1''$ and radiative decay $1'' \to 0'$.

4.1. Absorption and re-emission as resonant scattering

Consider scattering of a photon $\omega_M(\mathbf{k})$ to $\omega_{M'}(\mathbf{k}')$ by an electron with transition from ϵ, q to ϵ', q'. The probability for this is given by (14) with $\mathbf{k} \to -\mathbf{k}$, $\omega_M(-\mathbf{k}) = -\omega_M(\mathbf{k})$. The probability contains denominators in $M^{\mu\nu}(\mathbf{k}, \mathbf{k}')$ which have resonances at $\epsilon\varepsilon_q - \omega_{M'}(\mathbf{k}') - \epsilon''\varepsilon_{q''} = 0$ and at $\epsilon\varepsilon_q + \omega_M(\mathbf{k}) - \epsilon''\varepsilon_{q''} = 0$ with the δ-function requiring $\epsilon\varepsilon_q + \omega_M(\mathbf{k}) = \epsilon'\varepsilon_{q'} + \omega_{M'}(\mathbf{k}')$. Only the latter of these resonances is relevant for the following discussion. Comparison with the δ-function in (12) shows that $\epsilon\varepsilon_q + \omega_M(\mathbf{k}) - \epsilon''\varepsilon_{q''} = 0 = \epsilon'\varepsilon_{q'} + \omega_{M'}(\mathbf{k}') - \epsilon''\varepsilon_{q''}$ is the condition for a photon $\omega_M(\mathbf{k})$ to be absorbed in the transition $\epsilon, q \to \epsilon'', q''$ or for a photon $\omega_{M'}(\mathbf{k}')$ to be emitted in the transition $\epsilon'', q'' \to \epsilon', q'$. In this case the state ϵ'', q'' is a real state.

Near such resonances the scattering probability diverges, and the "scattering" should be regarded as absorption followed by emission. The scattering probability is defined as a rate per unit time and per unit volumes of $\mathbf{k}$-space and $\mathbf{k}'$-space. At the resonance the absorption probability gives the rate per unit time and per unit volume of $\mathbf{k}$-space that photons $\omega_M(\mathbf{k})$ are absorbed. Once the electron is in the state ϵ'', q'' it decays, say with a total rate per unit time $\Gamma^{\epsilon''}_{q''}$. The probability per unit volume of $\mathbf{k}'$-space that this decay produces a photon $\omega_{M'}(\mathbf{k}')$ is given by the ratio of the emission probability $w^{M'}_{q''q'}(\mathbf{k}')$, cf. (12), to the decay rate $\Gamma^{\epsilon''}_{q''}$. Hence the effective resonant scattering probability, i.e. the rate per unit time and per unit volumes of $\mathbf{k}$-space and $\mathbf{k}'$-space, is given by

$$\bar{w}^{M'M}_{q'q}(\mathbf{k}', -\mathbf{k}) = \sum_{q''} w^M_{q''q}(-\mathbf{k})w^{M'}_{q'q''}(\mathbf{k}')/\Gamma^{\epsilon''}_{q''}, \tag{40}$$

where we sum over all possible resonances.

In the immediate vicinity of a resonance the actual scattering probability (14) (with $\mathbf{k} \to -\mathbf{k}$) is dominated by the term which exhibits the relevant resonance. Inspection of (14) shows that for near-resonant scattering we have (Res = resonant)

$$\mathrm{Res}\, w^{M'M}_{q'q}(\mathbf{k}', -\mathbf{k}) \cong \sum_{q''}\frac{e^4 R_{M'}(\mathbf{k}')R_M(\mathbf{k})}{\varepsilon_0^2|\omega_{M'}(\mathbf{k}')\omega_M(\mathbf{k})|}|e^*_{M'\nu}(\mathbf{k}')[\Gamma^{\epsilon'\epsilon''}_{q'q''}(\mathbf{k}')]^{\nu}|^2 \times |e_{M\mu}(\mathbf{k})[\Gamma^{\epsilon\epsilon''}_{qq''}(\mathbf{k})]^{*\mu}|^2 2\pi \times \frac{\delta(\epsilon\varepsilon_q - \epsilon'\varepsilon_{q'} + \omega_M(\mathbf{k}) - \omega_{M'}(\mathbf{k}'))}{(\epsilon\varepsilon_q + \omega_M(\mathbf{k}) - \epsilon''\varepsilon_{q''})^2} \tag{41}$$

where we use (13). Using (12) in (40), we may write down an interpolation formula which includes both (40) and (41) by replacing the denominator in (41) by a Lorentzian line profile:

$$\mathrm{Res}\, w_{q'q}^{M'M}(\mathbf{k}', -\mathbf{k}) \cong \sum_{q''} \frac{e^4 R_{M'}(\mathbf{k}') R_M(\mathbf{k})}{\varepsilon_0^2 |\omega_{M'}(\mathbf{k}')\omega_M(\mathbf{k})|} |e^{\prime *}_{M'\nu}(\mathbf{k}')[\Gamma^{\epsilon'\epsilon''}_{q'q''}(\mathbf{k}')]^\nu|^2 \times |e_{M\mu}(\mathbf{k})[\Gamma^{\epsilon\epsilon''}_{qq''}(\mathbf{k})]^{*\mu}|^2 \times \frac{\pi\Gamma^{\epsilon''}_{q''}/2}{(\epsilon\varepsilon_q + \omega_M(\mathbf{k}) - \epsilon''\varepsilon_{q''})^2 + (\Gamma^{\epsilon''}_{q''}/2)^2} \times 2\pi\delta(\epsilon\varepsilon_q - \epsilon'\varepsilon_{q'} + \omega_M(\mathbf{k}) - \omega_{M'}(\mathbf{k}')). \quad (42)$$

This effective probability interpolates between the compound probability for absorption and re-emission in the core of the line, and true scattering in the wings of the line. Note however that far in the wings the terms retained in (41) cease to dominate in the true scattering probability (14), and the full expression (14) then needs to be used.

The expression (41) generalizes a result derived by Melrose (1981). Note that the decay rate $\Gamma^{\epsilon''}_{q''}$ in (42) is the inverse lifetime for the excited state ϵ'', q'' to all processes.

4.2. Resonant bremsstrahlung and non-radiative collisional excitation

The resonance in the cross-section for bremsstrahlung was treated by Kirk and Mészáros (1980) using a non-relativistic form of quantum electrodynamics involving the "seagull" diagram. With the view-point adopted here the resonance in bremsstrahlung corresponds to a non-radiative collisional excitation (from state $n = 0$ to state $n'' = 1$) followed by a one-photon decay (from $n'' = 1$ to $n' = 0$). As in the case of scattering, the distinction between resonant bremsstrahlung and non-radiative excitation followed by radiative decay is that the intermediate state is virtual for bremsstrahlung and real at the resonance. By analogy with (42), we may define a probability which interpolates between nearly resonant bremsstrahlung and the non-radiative collisional excitation plus radiative decay.

Inspection of the probability (21) for bremsstrahlung, together with the expression (14) for $M^{\mu\nu}(\mathbf{k}, \mathbf{k}')$, and of the probability (19) for non-radiative collisional excitation leads to the following interpolation formula:

$$\mathrm{Res}\, w_{q'q}^{c}(\mathbf{k}) = \sum_{q''} w^c_{q''q} \frac{e^2 R_M(\mathbf{k})}{\varepsilon_0} |e^*_{M\mu}(\mathbf{k})[\Gamma^{\epsilon'\epsilon''}_{q'q''}(\mathbf{k})]^\mu|^2 \times \frac{\pi\Gamma^{\epsilon''}_{q''}/2}{(\epsilon'\varepsilon_{q'} + \omega_M(\mathbf{k}) - \epsilon''\varepsilon_{q''})^2 + (\Gamma^{\epsilon''}_{q''}/2)^2}. \quad (43)$$

Once account is taken of the differences in notation, (43) reproduces the corresponding result of Kirk and Mészáros (1980).

4.3. Evaluation of the collision probability

For completeness let us outline the evaluation of $w^c_{q'q}$ in the case of collisional excitation 0 to 1′ for non-relativistic electrons. In earlier derivations, e.g. Canuto and Chiu (1971), the term $K^L(0, \mathbf{k})$ in (19) was not included. Here we assume it to be of the form $K^L(0, \mathbf{k}) = 1 + k_D^2/|\mathbf{k}|^2$, where k_D^{-1} plays the role of a Debye length. Then making the non-relativistic approximation and the other approximations discussed in Sect. 3.1., one finds

$$w^c_{q'q} = \frac{e^3 B}{2\pi} \frac{\varepsilon}{|p'_z|} \left(\frac{Ze}{\varepsilon_0}\right)^2 \frac{1}{(2eB)^2} \int_0^\infty dx' \frac{([\Gamma^{++}_{q'q}(\mathbf{k}')]^0|^2}{(x' + x_0)^2} \quad (44)$$

with $x' = k_\perp'^2/2eB$,

$$x_0 = \frac{k_D^2}{2eB} + \frac{(p_z - p'_z)^2}{2eB} \quad (45)$$

and with $p'_z = (p_z^2 - 2eB)^{1/2}$ for a transition $0 \to 1'$. For this transition (without a spin flip) one finds

$$|[\Gamma^{++}_{1'0}(\mathbf{k}')]^0|^2 \cong |J'_0(x')|^2 = x' e^{-x'} \quad (46)$$

and

$$\int_0^\infty dx' \frac{(J'_0(x'))^2}{(x' + x_0)^2} \cong \frac{\partial}{\partial x_0}(x_0 e^{x_0} E(x_0)) \equiv C_1(x_0), \quad (47)$$

with

$$E(x) = \int_x^\infty dx \frac{e^{-x}}{x}. \quad (48)$$

Then we find, cf. Canuto and Chiu (1971),

$$w^c_{1'0} \cong \frac{e^2 B}{2\pi} \frac{m}{|p'_z|} \left(\frac{Ze}{\varepsilon_0}\right)^2 \frac{C_1(x_0)}{(2eB)^2}. \quad (49)$$

For a one-dimensional non-relativistic Maxwellian distribution of electrons interacting with various ions (charge $Z_i e$ in (49), number density n_i), the rate of excitations $0 \to 1'$ is

$$R_1^c = \sum_i n_i \int \frac{dp_z}{(2\pi m T)^{1/2}} e^{-p_z^2/2mT} w^c_{1'0} \quad (50)$$

where the sum is over all ionic species and where T is the electron temperature in energy units.

5. Conclusions

The main results of this paper are the approximate expressions derived in Sect. 3 for two-photon cyclotron emission, and the compound probabilities derived in Sect. 4. These are the basic results required for a quantitative treatment of the processes discussed in the Introduction in formulating a theory for the formation of the spectra in X-ray pulsars with cyclotron features. This application is discussed in Paper II.

Acknowledgements. The visit of JGK to Sydney was funded by a grant under the Australian Research Grants Scheme. We thank Michelle Allen for helpful comments on the manuscript.

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Appendix

Reduction of $M^{\mu\nu}(\mathbf{k},\mathbf{k}')$

We wish to approximate $M^{\mu\nu}(\mathbf{k},\mathbf{k}')$ in (14) in the weak field ($B \ll B_c$), low frequency ($\omega, \omega' \ll m$), non-relativistic ($p_z^2 \ll m^2$) limit for transitions $n = 1$ to $n' = 0$. The final term involving the nonlinear response tensor in (14) is neglected. We write the two remaining terms in $M^{\mu\nu}(\mathbf{k},\mathbf{k}')$ as $M_1^{\mu\nu}(\mathbf{k},\mathbf{k}') + M_2^{\mu\nu}(\mathbf{k},\mathbf{k}')$ and note the identity $M_2^{\mu\nu}(\mathbf{k},\mathbf{k}') = M_1^{\nu\mu}(\mathbf{k}',\mathbf{k})$ so that only $M_1^{\mu\nu}(\mathbf{k},\mathbf{k}')$ need be evaluated explicitly. For $\epsilon = \epsilon' = 1$ we then have

$$\begin{aligned}M_1^{\mu\nu}(\mathbf{k},\mathbf{k}') &= \sum_{\epsilon'',\sigma'',n''} \frac{(\varepsilon_q - \omega' + \epsilon''\varepsilon_{q''})e^{-i(\mathbf{k}'\times\mathbf{k})_z/2eB}}{(\varepsilon_q - \omega')^2 - m^2 - 2n''eB - k_z'^2} \\ &\quad \times [\Gamma_{q'q''}^{+\epsilon''}(\mathbf{k})]^\mu [\Gamma_{q''q}^{\epsilon''+}(\mathbf{k}')]^\nu \\ &= \sum_{n''} \left[\frac{(\varepsilon_{q'} + \varepsilon_{q'}^0)(\varepsilon_{q'}^0 + m)(\varepsilon_q + \varepsilon_q^0)(\varepsilon_q^0 + m)}{4\varepsilon_{q'}\varepsilon_{q'}^0 4\varepsilon_q\varepsilon_q^0}\right]^{1/2} (-ie^{i\psi})^{n'} \\ &\quad \times \frac{(ie^{i\psi'})^n e^{in''}(\psi - \psi')e^{-i(\mathbf{k}'\times\mathbf{k})_z/2eB}}{(\varepsilon_q - \omega')^2 - m^2 - 2n''eB - k_z'^2} F_{q'qn''}^{\mu\nu}(\mathbf{k},\mathbf{k}') \qquad \text{(A1)}\end{aligned}$$

with

$$F_{q'qn''}^{\mu\nu}(\mathbf{k},\mathbf{k}') = \sum_{\epsilon''\sigma''} [g_{q'q''}^{+\epsilon''}(\mathbf{k})]^\mu [g_{q''q}^{\epsilon''}(\mathbf{k}')]^\nu (\varepsilon_q - \omega' + \epsilon''\varepsilon_{q''}) \qquad \text{(A2)}$$

where we define the g's by, cf. (3) and (4),

$$[\Gamma_{q'q''}^{+\epsilon''}(\mathbf{k})]^\mu = \left[\frac{(\varepsilon_{q'} + \varepsilon_{q'}^0)(\varepsilon_{q'}^0 + m)}{\psi\varepsilon_{q'}\varepsilon_{q'}^0}\right]^{1/2} (-ie^{-i\psi})^{n'}(ie^{i\psi})^{n''}[g_{q'q''}^{+\epsilon''}(\mathbf{k})]^\mu \qquad \text{(A3a)}$$

$$[\Gamma_{q''q}^{\epsilon''+}(\mathbf{k}')]^\nu = \left[\frac{(\varepsilon_q + \varepsilon_q^0)(\varepsilon_q^0 + m)}{\psi\varepsilon_q\varepsilon_q^0}\right]^{1/2} (-ie^{-i\psi'})^{n''}(ie^{i\psi})^n[g_{q''q}^{\epsilon''+}(\mathbf{k}')]^\nu. \qquad \text{(A3b)}$$

We need only the space components of $F^{\mu\nu}$, i.e. $\mu, \nu \neq 0$. A straight-forward but lengthy calculation gives

$$\begin{aligned}F_{q'qn''}^{11}(\mathbf{k},\mathbf{k}') &= -(-)^{n''}e^{i(\psi'-\psi)}[J_{n''-1}^0(x)J_{n''-2}^1(x') \\ &\quad \times \{\varepsilon_1 - \omega' + k_z'\rho_z' - m\} \\ &\quad + \rho_1 e^{-2i\psi'}J_{n''-1}^0(x)J_{n''}^0(x')p_{n''}] \qquad \text{(A4a)}\end{aligned}$$

$$\begin{aligned}F_{q'qn''}^{22}(\mathbf{k},\mathbf{k}') &= -(-)^{n''}e^{i(\psi'-\psi)}[J_{n''-1}^0(x)J_{n''-2}^1(x')\{\varepsilon_1 - \omega' \\ &\quad + k_z'\rho_z' - m\} - \rho_1 e^{-2i\psi'}J_{n''-1}^0(x)J_{n''}^0(x')p_{n''}] \qquad \text{(A4b)}\end{aligned}$$

$$\begin{aligned}F_{q'qn''}^{33}(\mathbf{k},\mathbf{k}') &= (-)^{n''}J_{n''}^0(x)J_{n''-1}^1(x')\{\varepsilon_1 - \omega' - k_z'\rho_z' - m\} \\ &\quad - \rho_1 J_{n''}^0(x)J_{n''-1}^0(x')p_{n''}] \qquad \text{(A4c)}\end{aligned}$$

$$\begin{aligned}F_{q'qn''}^{12}(\mathbf{k},\mathbf{k}') &= i(-)^{n'}e^{i(\psi'-\psi)}[J_{n''-1}^0(x)J_{n''-2}^1(x')\{\varepsilon_1 - \omega' \\ &\quad + k_z'\rho_z' - m\} - \rho_1 e^{2i\psi'}J_{n''-1}^0(x)J_{n''}^0(x')p_{n''}] \qquad \text{(A4d)}\end{aligned}$$

$$\begin{aligned}F_{q'qn''}^{21}(\mathbf{k},\mathbf{k}') &= -i(-)^{n''}e^{i(\psi'-\psi)}[J_{n''-1}^0(x)J_{n''-2}^1(x')\{\varepsilon_1 - \omega' \\ &\quad + k_z'\rho_z' - m\} + \rho_1 e^{-2i\psi'}J_{n''-1}^0(x)J_{n''}^0(x')p_{n''}] \qquad \text{(A4e)}\end{aligned}$$

$$\begin{aligned}F_{q'qn''}^{13}(\mathbf{k},\mathbf{k}') &= (-)^{n''}[e^{-i\psi}J_{n''-1}^0(x)\rho_1 J_{n''-1}^0(x')\{-(\varepsilon_1 - \omega')\rho_z' \\ &\quad - k_z' + m\rho_z'\} - e^{-i\psi}J_{n''-1}^0(x)J_{n''-1}^1(x')p_{n''}\rho_z'] \qquad \text{(A4f)}\end{aligned}$$

$$\begin{aligned}F_{q'qn''}^{31}(\mathbf{k},\mathbf{k}') &= (-)^{n''}[-e^{-i\psi'}\rho_1 J_{n''}^0(x)J_{n''}^0(x)\{(\varepsilon_1 - \omega')\rho_z' - k_z' \\ &\quad + m\rho_z'\} - e^{i\psi'}J_{n''}^0(x)J_{n''-2}^1(x')p_{n''}\rho_z'] \qquad \text{(A4g)}\end{aligned}$$

$$\begin{aligned}F_{q'qn''}^{23}(\mathbf{k},\mathbf{k}') &= -i(-)^{n''}[\rho_1 e^{-i\psi}J_{n''-1}^0(x)J_{n''-1}^0(x')\{-(\varepsilon_1 - \omega') \\ &\quad \times \rho_z' - k_z' + m\rho_z'\} + e^{-i\psi}J_{n''-1}^0(x)J_{n''-1}^1(x')p_{n''}\rho_z'] \qquad \text{(A4h)}\end{aligned}$$

$$\begin{aligned}F_{q'qn''}^{32}(\mathbf{k},\mathbf{k}') &= i(-)^{n''}[\rho_1 e^{-i\psi'}J_{n''}^0(x)J_{n''}^0(x')\{(\varepsilon_1 - \omega')\rho_z' - k_z' \\ &\quad + m\rho_z'\} + e^{i\psi'}J_{n''}^0(x)J_{n''-2}^1(x')p_{n''}\rho_z']. \qquad \text{(A4i)}\end{aligned}$$

So far we have made no approximations but we have set $p_z = 0$, giving $p_z' = -k_z - k_z'$, and have relabelled the state q' by 1 where convenient. Also $p_{n''}$ denotes $\sqrt{2n''eB}$.

The following simplifying approximations are now made. The initial square root factor in (A1) is approximated by unity, as is the phase factor $\exp[-i(\mathbf{k}' \times \mathbf{k})_z/2eB]$. The denominator in (A1) is approximated by $-2m[(n'' - 1)\Omega + \omega']$, the J-functions are approximated as in (23) and only the leading terms in expansions in x and x' are retained. Finally ε_1 in (A4) is approximated by $m + \Omega$ and corrections of order ω/m or Ω/m are neglected otherwise.

The resulting expression for the space components $\mu = i$, $\nu = j$ of $M^{\mu\nu}(\mathbf{k},\mathbf{k}')$ is

$$\begin{aligned}M^{ij}(\mathbf{k},\mathbf{k}') &= \frac{i}{2m\sqrt{2eB}}\Big[\delta_3^i\delta_3^j\{-2\omega\sin\theta e^{i\psi} - 2\omega'\sin\theta' e^{i\psi'}\} \\ &\quad + (\delta_1^i + i\delta_2^i)(\delta_1^j + i\delta_2^j)\{-\Omega\sin\theta e^{-i\psi} - \Omega\sin\theta' e^{-i\psi'}\} \\ &\quad + (\delta_1^i + i\delta_2^i)(\delta_1^j - i\delta_2^j)\{(\Omega - \omega')\sin\theta' e^{i\psi'} \\ &\quad + \frac{\omega(\Omega - \omega')}{\Omega + \omega'}\sin\theta e^{i\psi}\} + (\delta_1^i - i\delta_2^i)(\delta_1^j + i\delta_2^j) \\ &\quad \times \{(\Omega - \omega)\sin\theta e^{i\psi} + \frac{\omega'(\Omega - \omega)}{\Omega + \omega}\sin\theta' e^{i\psi'}\} \\ &\quad + \delta_3^i(\delta_1^j + i\delta_2^j)\left\{\frac{\Omega}{\Omega - \omega'}(\omega\cos\theta + 2\omega'\cos\theta') - \Omega\cos\theta\right\} \\ &\quad + (\delta_1^i - i\delta_2^i)\delta_3^j\left\{\frac{\Omega}{\Omega - \omega}(\omega'\cos\theta' \right. \\ &\quad \left. + 2\omega\cos\theta') - \Omega\cos\theta'\right\}\Big]. \qquad \text{(A5)}\end{aligned}$$

The results (33) follow by projecting onto the vectors (25).