

The Zakharov equations: a derivation using kinetic theory

By D. B. MELROSE

School of Physics, University of Sydney, Sydney NSW 2006, Australia

(Received 14 July 1986)

The Zakharov equations are derived using the weak turbulence expansion with approximate forms of the nonlinear response tensors from kinetic theory. The method is used to generalize the equations to the magnetized case. The range of validity of the Zakharov model is discussed briefly.

1. Introduction

The Zakharov equations (Zakharov 1972) are used extensively in discussions of the evolution of Langmuir turbulence when strong turbulence effects are considered; see, for example, the review by Goldman (1984). The original derivation of the equations was based on a simplified model involving fluid concepts. The model leads to two equations: one of these describes the evolution of the envelope of the Langmuir waves with the nonlinearity included through a term involving a density fluctuation, and the other describes the evolution of the density fluctuation due to the ponderomotive force exerted by the Langmuir waves. This model can be modified to include various other effects (cf. §4 below), but the range-of validity of the model itself is not readily apparent. It is desirable to relate the Zakharov theory to a more general theory for nonlinear plasma processes, so that, for instance, the validity of the model may be discussed in the wider context. In particular, it is desirable to use kinetic theory to rederive the Zakharov equations.

The derivation of the Zakharov equations in this paper is effectively in two parts. One part relies on results obtained elsewhere for relevant approximations to nonlinear response tensors derived using kinetic theory. The other part, presented in §§2 and 3 below, involves an extension of a method developed in two earlier papers (Melrose 1986*a*, 1987*a*, hereafter called I & II respectively) on parametric instabilities. The method is generalized in §4 to include the effects of a magnetic field; this generalization also allows for electromagnetic effects. Some remarks on the validity of the Zakharov model are made in §5.

2. Evolution of the Langmuir waves

The first of the Zakharov equations describes the evolution of the envelope of the Langmuir turbulence. Let $\mathbf{E}(t, \mathbf{x}) = -\text{grad } \phi(t, \mathbf{x})$ be the electric field in the Langmuir waves, and let the envelope $\boldsymbol{\varepsilon}(t, \mathbf{x})$ be defined by

$$\mathbf{E}(x) = \frac{1}{2}\{\boldsymbol{\varepsilon}(x) \exp(-i\omega_p t) + \text{c.c.}\}, \quad (1)$$

where ‘c.c.’ means complex conjugate and, where convenient, x denotes t, \mathbf{x} collectively.

A heuristic derivation of the equation for $\boldsymbol{\varepsilon}$ is as follows. The dispersion relation $\omega \simeq \omega_p + 3|\mathbf{k}|^2 V_e^2 / 2\omega_p + \frac{1}{2}i\gamma_L$ for Langmuir waves implies a wave equation

$$\left(\omega - \omega_p - \frac{3|\mathbf{k}|^2 V_e^2}{2\omega_p} + \frac{i\gamma_L}{2} \right) \mathbf{E}(k) = 0,$$

where γ_L is the absorption coefficient and where k denotes ω, k collectively. On including a density fluctuation δn_e , we write $\omega_p \rightarrow \omega_p(1 + \delta n_e / 2n_e)$, and include the term proportional to δn_e as a driving term. Then with the fast variation removed as in (1), the wave equation for $\boldsymbol{\varepsilon}$ may be inverted by making the replacement $\omega - \omega_p \rightarrow i\partial/\partial t$ and $|\mathbf{k}|^2 \rightarrow -\nabla^2$ to find

$$\left\{ \frac{i\partial}{\partial t} + \frac{3V_e^2}{2\omega_p} \nabla^2 + \frac{i\gamma_L}{2} \right\} \boldsymbol{\varepsilon}(x) = \omega_p \frac{\delta n_e(x)}{n_e} \boldsymbol{\varepsilon}(x). \quad (2)$$

The objective here is to rederive (2) in a more rigorous manner.

The Fourier transform $\phi(k) = ik \cdot \mathbf{E}(k) / |\mathbf{k}|^2$ of the electrostatic potential in the waves may be written in the form

$$\phi(k) = \phi_+(k) + \phi_-(k) \quad (3)$$

with
$$\phi_{\pm}(k) = \frac{i}{2|\mathbf{k}|^2} \mathbf{k} \cdot \boldsymbol{\varepsilon}^{\pm}(k) \quad (4)$$

and with
$$\boldsymbol{\varepsilon}^{\pm}(k) = \boldsymbol{\varepsilon}(\omega \mp \omega_p, k). \quad (5)$$

where the Fourier transform of (1) is used. Denoting equations in I and II with the relevant prefixes, (I.1) implies

$$|\mathbf{k}|^2 K(k) \phi_{\pm}(k) = \frac{1}{\epsilon_0} \int d\lambda^{(3)} \alpha_{\text{eff}}^{(3)}(k, k_1, k_2, k_3) \phi_{\pm}(k_1) \phi_{\pm}(k_2) \phi_{\mp}(k_3), \quad (6)$$

where the notation is as in I, with no distinction now drawn between pump and daughter waves. Here the waves are assumed to be longitudinal. Further approximations are made in assuming the form (I.8) for the effective cubic response, viz.

$$\alpha_{\text{eff}}^{(3)}(k, k_1, k_2, k_3) = -\frac{\epsilon_0 e^2 \mathbf{k} \cdot \mathbf{k}_1 \mathbf{k}_2 \cdot \mathbf{k}_3}{m_e^2 \omega_1 \omega_2 \omega_3} |\mathbf{k} - \mathbf{k}_1|^2 \frac{\chi_e(k - k_1) \{1 + \chi_i(k - k_1)\}}{1 + \chi_e(k - k_1) + \chi_i(k - k_1)}. \quad (7)$$

An important step is the identification of the quantity

$$\delta n_e(k) = -\frac{\epsilon_0 \chi_e(k) \{1 + \chi_i(k)\} |\mathbf{k}|^2}{m_e K(k)} \int d\lambda^{(2)} \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{\omega_1 \omega_2} \phi_+(k_1) \phi_-(k_2) \quad (8)$$

as the self-consistent density fluctuation. This identification was made in §3 of II. Here $\delta n_e(k)$ is referred to as 'self-consistent' for the following reason: if one calculates the 'bare' charge density due to the quadratic response to the fields $\phi_+(k_1)$ and $\phi_-(k_2)$, then the 'self-consistent' charge density $-\epsilon \delta n_e(k)$ is related to this in the same way as the self-consistent field is related to the bare field for an individual charge. Henceforth $\delta n_e(k)$ is referred to as the density fluctuation.

Using (8), (6) simplifies to

$$|\mathbf{k}|^2 K(k) \phi_{\pm}(k) = \frac{e^2}{\epsilon_0 m_e} \int d\lambda^{(2)} \frac{\mathbf{k} \cdot \mathbf{k}_1}{\omega_1} \phi_{\pm}(k_1) \delta n_e(k_2). \quad (9)$$

The factor $1/\omega\omega_1$ may be approximated by $1/\omega_p^2$ and taken outside the integral, and then using (4) for longitudinal fields, (9) implies

$$K(k)\epsilon^\pm(k) = \frac{1}{n_e} \int d\lambda^{(2)} \epsilon^\pm(k_1) \delta n_e(k_2). \tag{10}$$

The final approximation applies for $\omega \simeq \omega_p$, with

$$K(k) \simeq \frac{2}{\omega_p} \left\{ \omega - \omega_p - \frac{3}{2} \frac{|\mathbf{k}|^2 V_e^2}{\omega_p} + \frac{i\gamma_L}{2} \right\}. \tag{11}$$

Then on inverting the Fourier transform, (10) reproduces (2), as required, for the upper sign. For the lower sign one assumes $\omega \simeq -\omega_p$ and then (10) reproduces the complex conjugate of (2).

3. Evolution of the density fluctuations

The second of the Zakharov equations describes the evolution of the density fluctuations and follows directly from (8). Let an operator $\hat{G}(x)$ be defined by

$$\hat{G}(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \frac{K(k)}{\chi_e(k) \{1 + \chi_i(k)\}}. \tag{12}$$

Then on inverting the Fourier transforms, (8) implies

$$\hat{G}(x) \delta n_e(x) = \frac{\epsilon_0}{4m_e \omega_p^2} \nabla^2 \epsilon(x) \cdot \epsilon^*(x), \tag{13}$$

where (4) has been used.

If the low-density fluctuation has w, k nearly satisfying the dispersion relation $\omega = |\mathbf{k}|v_s + \frac{1}{2}i\gamma_s$ for weakly damped ion sound waves, then one may make the approximation

$$\frac{K(k)}{\chi_e(k) \{1 + \chi_i(k)\}} \simeq \frac{1}{\omega_{pi}^2} [w^2 + iwy, -|\mathbf{k}|^2 u:] \tag{14}$$

and hence

$$\hat{G}(x) \simeq \frac{1}{\omega_{pi}^2} \left[\frac{\partial^2}{\partial t^2} + \gamma_s \frac{\partial}{\partial t} - v_s^2 \nabla^2 \right]. \tag{15}$$

Noting the time average (cf. (1)),

$$\overline{|\mathbf{E}(x)|^2} = \frac{1}{2} \epsilon(x) \cdot \epsilon^*(x), \tag{16}$$

(12) reduces to

$$\left[\frac{\partial^2}{\partial t^2} + \gamma_s \frac{\partial}{\partial t} - v_s^2 \nabla^2 \right] \delta n_e(x) \simeq \frac{\epsilon_0 \omega_{pi}^2}{4m_e \omega_p^2} \nabla^2 |\epsilon(x)|^2, \tag{17}$$

which is the second of the Zakharov equations. The right-hand term in (17) is effectively the ponderomotive force.

The approximation (14) applies only for $|\mathbf{k}|^2 \lambda_{De}^2 \ll 1$, $w^2 \ll \omega_{pi}^2$ and $T_e \gg T_i$, when weakly damped ion sound waves exist. More generally, the operator $\hat{G}(x)$ should be constructed using (12).

4. Magnetized case

The electromagnetic and magnetized cases may be treated by relaxing the assumption that $\mathbf{E}(x)$ is a longitudinal field, and by replacing (6) by the appropriate tensor equation. The relevant equation is

$$A_{ij}(k) \epsilon_j^\pm(k) = \frac{1}{4} \int d\lambda^{(3)} S_{ijlm}^{\text{eff}}(k, k_1, k_2, k_3) \epsilon_j^\pm(k_1) \epsilon_l^\pm(k_2) \epsilon_m^\mp(k_3) \quad (18)$$

$$\text{with} \quad A_{ij}(k) = \frac{c^2}{\omega^2} (k_i k_j - |\mathbf{k}|^2 \delta_{ij}) + K_{ij}(k). \quad (19)$$

It is assumed that $\mathbf{E}(x)$ is of the form (1) with ω_p replaced by a frequency ω_0 which is arbitrary for the present.

An explicit form for the effective cubic nonlinear conductivity tensor for a magnetized plasma follows from results derived by Melrose (1987b):

$$S_{ijlm}^{\text{eff}}(k, k_1, k_2, k_3) \cong -\frac{i s_0 e^2}{m_e^2 \omega \omega_2} \tau_{ij}(\omega) \tau_{lm}(\omega_2) |\mathbf{k} - \mathbf{k}_1|^2 \frac{\chi_e(k - k_1) \{1 + \chi_i(k - k_1)\}}{K(k - k_1)} \quad (20)$$

with

$$\tau_{ij}(\omega) = \begin{pmatrix} \frac{\omega^2}{\omega^2 - \Omega_e^2} & -\frac{i\omega\Omega_e}{\omega^2 - \Omega_e^2} & 0 \\ \frac{i\omega\Omega_e}{\omega^2 - \Omega_e^2} & \frac{\omega^2}{\omega^2 - \Omega_e^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (21)$$

where the magnetic field is along the third axis and Ω_e is the electron cyclotron frequency. Also, in (20) the approximations $w \simeq \omega_1$ and $\omega_3 \simeq -\omega_2$ are assumed except where the difference $w - \omega_1 = \omega_2 + \omega_3$ appears explicitly. The electromagnetic case in the unmagnetized limit follows by replacing τ_{ij} and τ_{lm} in (20) by δ_{ij} and δ_{lm} , respectively (cf. Melrose 1986b).

It is implicit in the approximations made in deriving (20) that the low-frequency disturbance is assumed longitudinal. It may therefore be described in terms of a density fluctuation. In place of (8), we have

$$\delta n_e(k) = -\frac{\epsilon_0}{4m_e} \frac{\chi_e(k) \{1 + \chi_i(k)\} |\mathbf{k}|^2}{K(k)} \int d\lambda^{(2)} \frac{\tau_{ji}(\omega_1)}{\omega_1^2} \epsilon_i^+(k_1) \epsilon_j^-(k_2) \quad (22)$$

with, as in (20), $\omega_2 = -\omega_1$ except where the difference $w = \omega_1 + \omega_2$ appears explicitly. Then, in place of (10), we find

$$A_{ij}(k) \epsilon_j^\pm(k) = \frac{\tau_{ij}(\omega)}{n_e} \int d\lambda^{(2)} \epsilon_j^\pm(k_1) \delta n_e(k_2). \quad (23)$$

The generalization of the Zakharov equations is obtained by inverting the Fourier transforms in (23) and (22). Assuming $w \simeq \omega_0$ in (23), let us introduce the operator

$$\hat{A}_{ij}(x) = \int \frac{d\omega'}{2\pi} \frac{d^3k}{(2\pi)^3} \exp(-i\omega't + i\mathbf{k} \cdot \mathbf{x}) A_{ij}(\omega_0 + \omega', k). \quad (24)$$

Then (23) and (22) imply

$$\hat{A}_{ij}(x) \epsilon_j(x) = \frac{\tau_{ij}(\omega_0)}{n_e} \epsilon_j(x) \delta n_e(x) \quad (25)$$

and

$$\hat{G}(x) \delta n_e(x) = \frac{\epsilon_0}{4m_e \omega_0^2} \nabla^2 [\tau_{ji}(\omega_0) \epsilon_i(x) \epsilon_j^*(x)], \quad (26)$$

respectively.

Further simplification of (25) involves approximations to the operator $\hat{A}_{ij}(x)$. Consider, for example, an electron plasma with the cold plasma response supplemented by the first term in an expansion in $|\mathbf{k}|^2 V_e^2/\omega^2$, where V_e is the thermal speed:

$$A_{ij}(\omega, \mathbf{k}) = \frac{c^2}{\omega^2} \{k_i k_j - |\mathbf{k}|^2 \delta_{ij}\} + \delta_{ij} - \frac{\omega_p^2}{\omega^2} \tau_{ij}(\omega) - \frac{3\omega_p^2 V_e^2}{\omega^4} k_i k_j. \quad (27)$$

We have

$$\begin{aligned} \hat{A}_{ij}(x) \epsilon_j(x) \simeq & \frac{c^2}{\omega_0^2} \left[-\text{curl curl } \boldsymbol{\epsilon}(x) + \frac{2i\omega_0}{c^2} \frac{\partial \boldsymbol{\epsilon}(x)}{\partial t} \right]_i \\ & + \left[\delta_{ij} - \frac{\omega_p^2}{\omega_0^2} \tau_{ij}(\omega_0) \right] \epsilon_j(x) + \frac{3\omega_p^2 V_e^2}{\omega_0^4} [\text{grad div } \boldsymbol{\epsilon}(x)]_i. \end{aligned} \quad (28)$$

A result quoted by Kuznetsov (1974, equation (14)) follows by inserting (28) in (25), retaining only the terms up to first order in Ω_e/ω_0 and setting $\omega_0 = \omega_p$.

Zakharov (1975) considered collapse near the cut-off frequencies $\omega = \omega_p, \frac{1}{2}[\Omega_e^2 + 4\omega_p^2]^{\frac{1}{2}} \pm \frac{1}{2}\Omega_e$. More generally, if the waves are assumed to be in any specific wave mode M , with dispersion relation $\omega = \omega_M(\mathbf{k})$ and polarization vector $\mathbf{e}_M(\mathbf{k})$, then one may approximate $A_{ij}(k)$ for $\omega \simeq \omega_M(\mathbf{k})$ by

$$A_{ij}(k) \simeq (\omega - \omega_M(\mathbf{k}) + i\gamma_M(\mathbf{k})/2) \frac{R_M(\mathbf{k})}{\omega_M(\mathbf{k})} e_{Mi}(\mathbf{k}) e_{Mj}^*(\mathbf{k}) \quad (29)$$

where $R_M(\mathbf{k})$ is the ratio of the electric to total energy (Melrose 1980, 1986c) and $\gamma_M(\mathbf{k})$ is the absorption coefficient. The inversion of (29) to find $\hat{A}_{ij}(x)$ then simplifies near the cut-off frequencies where one may expand in powers of $|\mathbf{k}|$, with $|\mathbf{k}|c/\omega \ll 1$ by hypothesis. The cases discussed by Zakharov (1975) may be treated by setting ω_0 equal to the relevant cut-off frequency and evaluating $\hat{A}_{ij}(x)$ in this way.

The right-hand side of (26) implies that the ponderomotive force is modified in a magnetized plasma by the appearance of $\tau_{ji}(\omega_0)$. This modification has been noted previously; a particularly elegant derivation of the modified form was given by Cary & Kaufman (1977), cf. also Manheimer (1985).

5. Discussion

As indicated in §4, the Zakharov equations may be modified in various ways to take account of electromagnetic effects, the magnetization of the plasma, etc. This raises the question as to the conditions under which the Zakharov model remains useful, that is, the conditions under which the nonlinear evolution may be described by two coupled equations of the Zakharov form. The derivation presented here allows us to answer this question. From (18), which is quite general, we may construct two Zakharov-like equations provided that the equation may be split as in (22) and (23). This requires that the effective cubic nonlinear response tensor $S_{ijlm}^{\text{eff}}(k, k_1, k_2, k)$ can be approximated by a product

of three functions, these being dependent on ω , ω_2 and $k-k_1$ respectively (cf. (20)). This is the case provided (i) that the high-frequency wave has a high phase speed, so that only the leading term in an expansion in $|\mathbf{k}|V_e/\omega$ need be retained, (ii) that the low-frequency disturbance is longitudinal, so that it may be described in terms of a density fluctuation δn_e , and (iii) that the low-frequency disturbance has a slow phase speed, so that only the leading term in an expansion in $\omega/|\mathbf{k}|V_e$ need be retained. (This third condition follows from the details of the derivation of the form (20); cf. Melrose (1986b, 1987b).)

In summary, the Zakharov model is justified provided that the effective cubic response tensor is of the form (18), and a detailed discussion of the range of validity of the model thus requires a detailed examination of the assumptions involved in the derivation of (18).

This paper arose out of a discussion with M. V. Goldman.

REFERENCES

- CARY, J. R. & KAUFMAN, A. N. 1977 *Phys. Rev. Lett.* **39**, 402.
 GOLDMAN, M. V. 1984 *Rev. Mod. Phys.* **56**, 709.
 KUZNETSOV, E. A. 1974 *Soviet Phys. JETP*, **39**, 1003.
 MANHEIMER, W. M. 1985 *Phys. Fluids*, **28**, 1569.
 MELROSE, D. B. 1980 *Plasma Astrophysics*, vol. 1. Gordon & Breach.
 MELROSE, D. B. 1986a *J. Plasma Phys.* **36**, 269.
 MELROSE, D. B. 1986b *Aust. J. Phys.* (In press.)
 MELROSE, D. B. 1986c *Instabilities in Space and Laboratory Plasmas*. Cambridge University Press.
 MELROSE, D. B. 1987a *J. Plasma Phys.* (To be published.)
 MELROSE, D. B. 1987b *Aust. J. Phys.* **39**, 891.
 ZAKHAROV, V. E. 1972 *Soviet Phys. JETP*, **35**, 908.
 ZAKHAROV, V. E. 1975 *JETP Lett.* **21**, 221.