IMPLICATIONS OF LIOUVILLE'S THEOREM ON THE APPARENT BRIGHTNESS TEMPERATURES OF SOLAR RADIO BURSTS

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Abstract. Liouville's theorem for radiation, of which the generalized étendue is a consequence, implies $\mu^2 \, d^2\Omega \, d^2A$ = constant along the ray path, where μ is the refractive index and $d^2\Omega$ and d^2A are the ranges, respectively, of solid angle and of area that define a ray (actually a bundle of rays). Implications of this concept on the propagation of radio waves from the actual to the apparent source in the solar corona (i.e., the scatter image of the true source) are discussed. The implications for sources of fundamental plasma radiation include: (1) The observed solid angle $\Delta\Omega$ (defining the directivity) and apparent area ΔA of the source are compatible with Liouville's theorem only if the apparent source (the scatter image of the true source) corresponds to the envelope of subsources with a small 'filling factor' f. (2) The brightness temperature T_B of the actual source is greater than that of the apparent source by f^{-1}. (3) For sources of fundamental plasma radiation the factor f is very small ($\lesssim 10^{-2}$). (4) A long-standing discrepancy between the observed low value of T_B at meter/decameter wavelengths for the quiet Sun and the known coronal temperature may be resolved by noting that the implied coronal temperature is given by T_B/f, and that the factor f must be significantly less than unity.

A brief discussion is included of the relation between Liouville's theorem, the generalized étendue, Milne's laws, occupation numbers, extension in phase, and suppression of emission by a medium with refractive index unequal to unity.

1. Introduction

Most solar radio bursts of types I to V have apparent source sizes, heights, and directivities that are attributed to the results of propagation, through the corona, of radio waves originating at the fundamental (F) of the plasma frequency $\omega \approx \omega_p$, its second harmonic (H), or both. The directivity of F radiation has been the subject of considerable attention, both theoretically and observationally, so we commence by summarizing that work and pointing out remaining problems.

Early calculations (Jaeger and Westfold, 1950) on propagation through a spherically-symmetric corona (cf. Appendix A) led to results in serious conflict with observations: the small refractive index $\mu \ll 1$ for the fundamental at the source and Snell's law ($\mu \sin\theta$ = const.) imply that the escaping radiation should be confined to a cone with small half angle ($\theta = \arcsin\mu$) about the radial direction. This half angle is of order a few degrees for expected initial value of μ ($\ll 1$). Thus the large increase in the refractive index along the ray path implies a strong outward collimation of fundamental plasma

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radiation in a spherical corona. Although observed F radiation is moderately directive, it is confined to a cone much broader than a few degrees.

To overcome this and other difficulties, it was proposed that there are small-scale inhomogeneities in the corona that scatter the radiation (Roberts, 1959; Fokker, 1965; Steinberg *et al.*, 1971; Steinberg, 1972; Riddle, 1972a, b, 1974). In these scattering models the observed sources are interpreted as apparent sources that are 'scatter images' of the actual sources. The apparent sources are larger, higher and less directive than the actual sources would be in the absence of scattering. However, to account for the inferred heights and sizes, so much scattering is required that the calculations imply that this directivity imposed by refraction is essentially destroyed, and it is implied that the escaping radiation should be less directive than it is observed to be. Thus, in contrast with the early calculations of Jaeger and Westfold (1950), the scattering models require an additional mechanism to make the radiation more directive. The favored idea for type III bursts is 'ducting' (Duncan, 1979) in which it is postulated that the radiation is guided outwards in underdense ducts, rather like light is guided along an optical fiber. For type I bursts a different model involving overdense 'fibers' has been proposed by Bougeret and Steinberg (1977). In both models the radiation gets into underdense regions and stays there.

It seems plausible that scattering of rays from a source should lead to an apparent source that is both larger and less directive than the actual source would be in the absence of scattering, and this is certainly the case for the 'envelope' of the apparent source and its large scale structure. However, there is another argument that relates to the detailed structure within the envelope, by which we mean brightness variations of small scale, perhaps a sprinkling of spots of high brightness. This detailed structure may be so small and/or complex as to be unresolvable by realistic instruments.

Here we express this argument in terms of the 'generalized étendue' of geometric optics, e.g., Welford and Winston (1978); a derivation of the generalized étendue from Liouville's equation is outlined in Appendix B. The generalized étendue may be expressed as follows. In any optical system, consider a bundle of rays confined to an infinitesimal area d^2A_1 and an infinitesimal range $d^2\Omega_1$ of solid angles at a point P_1 where the refractive index is μ_1. At any other point P_2 along the ray path, where the refractive index is μ_2, this bundle of rays is confined to d^2A_2 and $d^2\Omega_2$ such that one has

$$\mu_1^2 \cos\theta_1 \, d^2A_1 \, d^2\Omega_1 = \mu_2^2 \cos\theta_2 \, d^2A_2 \, d^2\Omega_2 , \tag{1}$$

where θ_1 is the angle between the normal to the element of area d^2A_1 and the direction of the bundle of rays at P_1 and θ_2 is the corresponding angle at P_2.

The path from the actual to the apparent source may be regarded as an optical system, and the result (1) may be applied to it. (From here on we use symbols without primes to denote quantities at the true source, and primes to denote quantities at the apparent source, i.e., the scatter image.) The generalized étendue (1) implies (i) for fixed μ the radiation at the apparent source must become more directive ($d^2\Omega'/d^2\Omega < 1$) in proportion to any increase in size of the apparent source relative to the actual source

($d^2A'/d^2A > 1$), (ii) for fixed A there is a strong collimation ($d^2\Omega'/d^2\Omega \ll 1$) due to the large increase in the refractive index ($\mu'^2/\mu^2 \gg 1$) for propagation through a corona where the density decreases outward, and (iii) an increase in source size should cause a collimation that enhances the one due to the increase in the refractive index. Thus the implications of the scattering models, as outlined above, seem to be in conflict with this basic result of geometric optics. Specifically, the scattering models give large values for ΔA and $\Delta\Omega$ for the scattered image (i.e., for the apparent source), and this suggests that all of μ^2, $\Delta\Omega$, and ΔA increase simultaneously along the ray path, where $\Delta\Omega$ and ΔA denote, respectively, the ranges of solid angle and of area filled by the bundle of rays.

It could be argued that Liouville's theorem or the generalized étendue (1) do not apply in the presence of true scattering. While this is technically the case, coronal 'scattering' is not true scattering; it is the result of small-angle refractions at density inhomogeneities, and geometric optics, Liouville's theorem and (1) are then all valid. Liouville's theorem is not applicable in scattering models (as opposed to propagation through the actual corona) where each ray is assumed to propagate in a statistically different realization of the coronal inhomogeneities. However, the implications of Liouville's theorem on the propagation through the actual corona remain pertinent.

Our purpose in this paper is to discuss the implications of the generalized étendue on the interpretation of solar radio emission. In Section 2 we discuss Milne's law, the generalized étendue and Liouville's theorem, and their relevance to solar radio bursts and to coronal scattering models. The relevant properties of sources are summarized in Section 3. In Section 4 the brightness temperatures of sources are discussed, and the need for a small filling factor for the rays at the apparent source is emphasized. A particularly notable application of the ideas developed here is to the quiet Sun emission. A long-standing discrepancy is that the measured brightness temperature of the quiet Sun is systematically smaller than all other estimates of the coronal temperature. This discrepancy can probably be resolved by taking into account the filling factor $f < 1$ or alternatively the low emissivity of a bremsstrahlung source when $\mu < 1$.

2. Scattering Models, Milne's Laws, and the Generalized Étendue

Rays generated near the plasma level in a spherically symmetric corona are strongly refracted into the radial direction, leading to the highly directed radiation predicted by the earliest ray tracing models (Jaeger and Westfold, 1950). Refraction due to the mean density gradient in the corona was not included in the scattering model of Fokker (1965), who essentially set the refractive index equal to unity outside the inhomogeneities that cause the postulated scattering. As expected in Fokker's model, the scattering reduces any initial directivity. The effect of the mean density gradient and the change in the average value of μ with radial distance was included in the scattering models of Steinberg *et al.* (1971), Steinberg (1972), and Riddle (1972a, b, 1974). Nevertheless these models seem to imply that scattering both increases the source size and decreases the directivity, seemingly violating the constraint required by the generalized étendue (1). The assumptions made in the scattering models and the details of the application of the generalized étendue to solar radio bursts need to be examined to resolve this paradox.

In this section we discuss the applicability of geometric optics to coronal scattering models (Section 2.1), and then summarize the relevant conservation laws, first in terms of one of Milne's laws (Section 2.2), and then in terms of Liouville's theorem and the generalized étendue (Section 2.3).

2.1. Refraction versus scattering

The underlying idea in a scattering model is summarized in the following words of Steinberg *et al.* (1971): "in [the scattering model] a large number of rays are computed which travel through different coronas where the average electron density distribution and the statistics of the inhomogeneities are the same, but where these inhomogeneities are microscopically different [and] an ensemble average of all these rays is made". Thus the statistical assumption is that there is an ensemble of coronas, and that each (calculated) ray propagates through a different realization of the coronal density inhomogeneities. The usefulness of such models is in predicting certain of the statistical properties of the resulting scattering image or apparent source. The distribution of emerging rays enables one to define apparent sizes and directivities.

However, scattering models give no direct information on the detailed distribution of brightness temperature over the apparent source, nor of the true source. A separate algorithm is required to discuss the brightness temperature.

2.2. Milne's laws, occupation numbers, and extension in phase

Milne's Theorem III (Milne, 1930) states that in the absence of emission or absorption processes, the ratio of specific intensity I_ν to the square of the index of refraction μ_ν is constant along a ray path, i.e.,

$$I_\nu/\mu_\nu^2 = \text{const.} \tag{2}$$

Further, Milne showed that if an emitting particle of temperature T is embedded in a medium of refractive index $\mu_{\nu 0}$, then the emitted intensity is not related to the Planck function B_ν by $I_\nu = B_\nu$, but by

$$\frac{I_\nu}{\mu_{\nu 0}^2} = B_\nu = 2kT\,\frac{\nu^2}{c^2}\,, \tag{3}$$

where the second equality is valid for $h\nu \ll kT$. Using (2) and generalizing (3) by putting $\mu_{\nu 0} = \mu_\nu$ and $T = T_B$, then, along the ray path:

$$T_B = \text{const.} \tag{4}$$

In addition, it follows that the emitted intensity I_ν is lower by a factor of $\mu_{\nu 0}^2$ than it would be in vacuum.

By definition, the flux density S_ν is

$$S_\nu = \int \mathrm{d}\Omega\, I_\nu = 2k\,\frac{\nu^2}{c^2}\,\mu_\nu^2 \int \mathrm{d}\Omega\, T_B\,. \tag{5}$$

In order to illustrate a fundamental result, we consider a spherically-symmetric, non-scattering corona, with an observer at a distance r from a source of radius r_0. Then, if the source and observer are in vacuum it is well known that

$$S_\nu(\text{obs}) = 2k \, \frac{\nu^2}{c^2} \, T_B \, \pi \left(\frac{r_0}{r}\right)^2 . \tag{6}$$

However, if the source and observer are embedded in a medium of refractive index $\mu_\nu = \mu_{\nu 0}$, then

$$S_\nu(\text{obs}) = 2k \, \frac{\nu^2}{c^2} \, T_B \mu_{\nu 0}^2 \, \pi \left(\frac{r_0}{r}\right)^2 , \tag{7}$$

which is smaller than the case of vacuum by a factor $\mu_{\nu 0}^2$, i.e., a factor $\ll 1$ if $\mu_{\nu 0} < 1$. The factor $\mu_{\nu 0}^2$ in (7) appears for the same reason as the corresponding factor in (3), and may be interpreted in terms of radiation being suppressed in a medium with $\mu_{\nu 0} < 1$, as is most familiar in the Razin effect.

Lastly, if the source is located where $\mu_\nu = \mu_{\nu 0}$ in a medium where the density decreases as r^{-2} (see Appendix A), and if the observer is in vacuum, then radiation is observed to come only from a small solid angle of the source:

$$S_\nu(\text{obs}) = 2k \, \frac{\nu^2}{c^2} \, T_B \, \pi \left(\frac{\mu_{\nu 0} r_0}{r}\right)^2 , \tag{8}$$

which is again smaller than the case of emission in vacuum by a factor $\mu_{\nu 0}^2 \ll 1$. The interpretation of (8) is different from that of (7), with the reduction by the factor $\mu_{\nu 0}^2$ being attributed to the small apparent size of the source, cf. (A.14). Hence, in the second and third cases (where $\mu_{\nu 0}^2 \ll 1$) the observer receives much less flux than the first case with the source in vacuum.

Generalizing the above result to the case where both refraction and scattering are included, and using the fact that the emitted power of a source embedded in a medium is low, the scatter image must either be uniform and of anomalously low brightness, or sparsely speckled with spots of high (true source) brightness. In no case can the spots be of brightness higher than that of the true source.

In the absence of dissipation there are two independent conservation laws for the radiation, and these are implicit in the foregoing results. One is the constancy of T_B, cf. Equation (4). In a semi-classical notation $T_B(\mathbf{k})$, which is a function of $\mathbf{k}$ in general, is equal to $\hbar\omega(\mathbf{k})$ times the occupation number $N(\mathbf{k})$ of photons. In a time-independent medium, the frequency $\omega(\mathbf{k})$ is constant, and the constancy of $T_B(\mathbf{k})$ and of $N(\mathbf{k})$ are equivalent. The second conservation law is for photons: the number of photons escaping per unit time (over a time-scale in which the source remains constant) is equal to the number of photons emitted per unit time. The number of photons is related to $N(\mathbf{k})$ by the integral over phase space ($d^6\Gamma$ in the notation of Appendix A). The constancy of the number of photons and of $N(\mathbf{k})$ then implies the constancy of the phase-space volume, called the extension in phase. This final result is Liouville's theorem. The results (2) and (7) may be derived directly from the constancy of $N(\mathbf{k})$, $\omega(\mathbf{k})$, and the extension in phase. Furthermore, (7) is related to the fact that, for emission in a medium, the available volume of phase space is considerably less than for vacuum.

We conclude that Liouville's theorem, the generalized étendue of (1), Milne's laws of (2) through (8), and constraints on phase space volume are related. Each form is useful in some circumstances.

2.3. Filling factor for the apparent source

Consider fundamental plasma radiation so that one has $\mu^2 \ll 1$ initially. Refraction narrows the initial cone (half angle ζ) to which this bundle of rays is confined such that when the refractive index approaches unity the half angle is of order $\mu\zeta \ll 1$. This implies that the range of solid angles filled at the apparent source is much smaller than that filled at the actual source by a factor of order μ^{-2}.

It is useful to define a filling factor f' for the radiation at the apparent source. Let an area $\Delta A'$ of the apparent source be sprinkled with bright elements that cover a fraction f' of $\Delta A'$, and let the radiation be confined within a solid angle $\Delta\Omega'$. Let the actual source be described in the same way in unprimed quantities, where a filling factor f (≈ 1) is introduced for symmetry. The integral over (1) then gives

$$\mu^2 \Delta A\, \Delta\Omega f = \mu'^2 \Delta A'\, \Delta\Omega' f' . \tag{9}$$

For emission at the fundamental (F) the filling factor f' at the apparent source is evidently very small. For example, some observations described below suggest that $\Delta A\, \Delta\Omega \approx \Delta A'\, \Delta\Omega'$; thus with $\mu = 0.1$ at the actual source and $\mu' = 1$ at the apparent source, one finds a filling factor $f' \approx 10^{-2}$. The implications of this are discussed in Section 4 below.

3. Sizes, Heights, and Directivities of Observed Sources

In this section we describe the observed properties of several solar radio emissions for which scattering and the generalized étendue probably play significant roles in determining the relationship between the apparent and true source brightness temperatures. We summarize the observed heights, sizes, and directivities of type III bursts (both F and H plasma emission), type I bursts and storms (F emission), and quiet Sun radiation (thermal bremsstrahlung originating near the plasma level) at long metric and decametric wavelengths. A common feature of these emissions is that the heights of the observed sources are all above the plasma levels where the radiation presumably arises, i.e., the observed sources are scatter images of the true sources. It is convenient to express the result in terms of the ratio of the frequency ω of the radiation to the plasma frequency ω_p at the height from which the radiation appears to originate.

3.1. Type III bursts

(1) At meter wavelengths the heights of type III bursts correspond to the location where $\omega \approx 3\omega_p$ (Dulk and Suzuki, 1980), and at kilometer wavelengths the heights correspond to $\omega \approx 2$ to $5\omega_p$ (Steinberg *et al.*, 1984; Steinberg, Hoang, and Dulk, 1985).

(2) F and H radiation at a given frequency from the same disturbance, as it passes successively through the levels $\omega_p \approx \omega$ and $\omega_p \approx \omega/2$, appears to come from similar heights (Stewart, 1972).

(3) At a given frequency, F and H sources at meter wavelengths have similar sizes; the F is somewhat smaller (by some tens of percent) than the H source at kilometer wavelengths (Dulk, Steinberg, and Hoang, 1984).

(4) Source sizes increase with wavelength λ roughly $\sim \lambda^{1.0}$ (Dulk, Melrose, and Suzuki, 1979). Taking into account point (3), this demonstrates that F sources are broadened to a greater extent than are H sources.

(5) Source sizes increase from the center to the limb, by about 30% at meter wavelenghts (Dulk and Suzuki, 1980), and by about 100% at kilometer wavelengths (Steinberg *et al.*, 1984).

(6) From (1), (2), and (5): the observed source sizes of F sources are ≈ 3.5 times the actual sizes in linear dimensions, and ≈ 10 times in area; for H sources these ratios are ≈ 1.7 in linear dimensions and ≈ 3 in area.

(7) Regarding the directivity of F sources, it is found that the radiation is typically confined to cones of half angle $\approx 45°$ at both meter and kilometer wavelengths (Dulk and Suzuki, 1980; Dulk, Steinberg, and Hoang, 1984). For H sources, the radiation is typically confined to cones of half angle $\gtrsim 80°$ at meter wavelengths (Dulk and Suzuki, 1980), and a poorly determined value $\approx 60°$ to $90°$ at kilometer wavelengths (Steinberg *et al.*, 1984).

(8) The degree of circular polarization of F sources is $\approx 35\%$ (range from 0 to 70%), and the polarization of H sources is $\approx 11\%$ (range from 0 to 25%) (Dulk and Suzuki, 1980). At kilometer wavelengths no systematic studies have been published, but the degree of circular polarization of F sources is probably low, $\lesssim 10\%$, for both F and H radiation (A. Lecacheux, private communication).

3.2. Type I bursts and storms (meter wavelengths)

(1) Observed heights are nearly identical to those of type III bursts, except that there is an apparent increase in height from the center to the limb (Suzuki, 1961; le Squeren, 1963).

(2) Source sizes are similar to those of type III bursts. Sometimes they are smaller and at the limit of resolution (< 2 arc min at 169 MHz), i.e., a factor ≈ 2 smaller than type III bursts (le Squeren, 1963; Kerdraon, 1973). Sometimes they are larger by a factor ≈ 2 (le Squeren, 1963).

(3) Source sizes increase hardly at all from the center to the limb (le Squeren, 1963).

(4) The directivity of individual type I bursts is high, with a half angle $\approx 15°$ at 169 MHz (Steinberg *et al.*, 1984). The directivity of the ensemble of bursts and the

continuum in storms is lower, with half angle $\approx 40°$ (Suzuki, 1961; le Squeren, 1963). Hence, different bursts must be directed into different parts of the larger cone.

(5) The flux density of storms near the limb is 0.05 to 0.1 that of storms near the center of the disk (le Squeren, 1963).

(6) The degree of circular polarization decreases on average from center to the limb (le Squeren, 1963). Storms with 100% polarization are generally within 45° of the disk center, and storms with 0% polarization are generally $> 45°$ from the disk center. The degree of polarization can take on an intermediate, but well defined value at intermediate longitudes (Zlobec, 1975).

3.3. Type III storms (meter/decameter wavelengths)

(1) At long meter and decameter wavelengths, the source heights, sizes, and directivities of type III storms have not been reported to differ greatly from those of normal type III bursts.

(2) The sense of polarization of type III storms is always the same as that of the accompanying type I storm, but the degree of polarization is usually lower to much lower. Most type III storms are identifiable as either F or H (because their degrees of polarization are so similar to those of normal F and H type III bursts), with H storms being the more common. Even for F storms, the degree of polarization of the type III portion is lower than that of the type I portion (Dulk, Suzuki, and Sheridan, 1984).

3.4. Type III storms (kilometer wavelengths)

(1) At kilometer wavelengths the source heights, sizes and directivities of type III storms probably differ somewhat from those of normal type III bursts. Near 1 AU the source heights correspond to $\omega \approx 2\omega_p$, while at 0.2 AU they correspond to $\omega \approx 3\omega_p$. The type III storms appear to start in regions that have a higher than average density, but also have a density gradient steeper than the r^{-2} dependence that is characteristic of the solar wind in general (Bougeret, Fainberg, and Stone, 1984b).

(2) Source heights apparently increase from center to limb by a factor of 2 (Bougeret, Fainberg, and Stone, 1984b).

(3) Judging from the fact that type III storms can be tracked for a few days (range 1 to 10) (Bougeret, Fainberg and Stone, 1984a), it appears that their directivity can be characterized by a cone of half angle $\approx 30°$ to $45°$.

3.5. Radiation from the quiet sun

(1) The brightness temperature T_B at the disk center should be $\approx 10^6$ K from ≈ 200 MHz to ≈ 50 MHz (above 200 MHz the level where $\tau \approx 1$ is in the (cooler) transition region while below ≈ 50 MHz, one may have $\tau < 1$ above the plasma level). However T_B is observed to decrease from (at best) 10^6 K at ≈ 100 MHz to 3 to 4×10^5 K at 30 MHz (Erickson *et al.*, 1977; Sheridan and McLean, 1985). Recently, Wang, Schmahl, and Kundu (1987) reported brightness temperatures as low as 1.5 to 2×10^5 K for quiet regions at 30 MHz. At these lower frequencies, most radiation originates near the plasma level where the plasma density is highest and the group speed

is lowest, so that it is the most subject to the constraints of the generalized étendue.

(2) The east-west diameter of the quiet Sun increases rapidly with wavelength, from about 1.4 R_0 at 80 MHz to 2.2 R_0 at 30 MHz (Sheridan and McLean, 1985). These correspond to levels with $\omega \approx 2\omega_p$ and $\omega \approx 3\omega_p$, respectively, as derived from the average coronal model of Saito (1970) or Baumbach-Allen.

(3) Scattering models (e.g., Aubier, Leblanc, and Boischot, 1971) have been used to account for some features of (1) and (2); this is discussed further below.

4. Estimated Brightness Temperatures

The brightness temperature of a source at a given λ is defined as the temperature that a black body would need to have to produce radiation with the observed flux density at that λ. The apparent brightness temperature is calculated from the observed flux density and the apparent size of the source. If the radiation comes from only a small fraction of the area of the apparent source, described by a 'filling factor' f, then the actual brightness temperature is larger than the apparent brightness temperature by the inverse f^{-1} of the filling factor. (If, as remarked in Section 2.2, the source is uniform and of low brightness rather than speckled, the factor f represents the ratio of apparent to true source brightness.)

4.1. Actual brightness temperatures of type III sources

The apparent brightness temperatures T_B for type III sources vary widely, from the lowest measurable value ($\approx 10^6$ K) to $\approx 10^{12}$ K for sources in the corona, and up to $\approx 10^{15}$ K for sources in the interplanetary medium (Suzuki and Dulk, 1985).

These estimates of T_B do not take the filling factor f of the source into account. Thus the values of T_B at the actual sources must be larger than these estimates by the inverse factor f^{-1}. The value of f may be estimated from (9). Assuming the refractive index at the apparent source and the filling factor at the actual source to be unity ($\mu' = f = 1$), (9) implies that the filling factor (f') at the apparent source is given by

$$f' = \mu^2(\delta A/\delta A')(\delta\Omega/\delta\Omega'). \tag{10}$$

For F sources the factor μ^2 is very small and, hence, the apparent value of T_B greatly underestimates the value of T_B at the actual source. A specific estimate of f' requires careful consideration of each of the factors in (10). An order of magnitude estimate, with $\Delta A' \sim 2$ to $5\Delta A$, $\Delta\Omega' \sim 1$, and $\Delta\Omega \sim 2\pi$, suggests f' between 10^{-3} and 10^{-2} for F sources, and between 10^{-1} and 1 for H sources.

The actual brightness temperature of a source is important in understanding the emission process involved. For example, incoherent gyro-synchrotron emission from particles with an energy ε has $T_B \lesssim \varepsilon/\kappa$, where κ is Boltzmann's constant, and plasma emission from Langmuir waves that are generated incoherently satisfies $T_B \lesssim mc^2/2\kappa$ (Robinson, 1978). An underestimate of the actual brightness temperature can, therefore, result in a misleading conclusion concerning the need for coherent emission processes. For example, one theory for type III bursts in the interplanetary medium relates the value of T_B directly to the effective temperature of the Langmuir waves (Melrose, Dulk, and Cairns, 1986; Melrose and Goldman, 1987) and the application of this theory needs to be revised in view of the higher values of T_B implied by the neglect of the filling factor.

4.2. Actual brightness temperatures of type I sources

Type I sources involve F emission in bursts and in a continuum, and these may or may not both occur in a given storm. The estimated T_B in the continuum is $\lesssim 10^{10}$ K, allowing the possibility of interpreting it without appealing to an instability to generate the required Langmuir waves (Melrose, 1980; Benz and Wentzel, 1981; Kerdraon and Mercier, 1983). However, the filling factor has not been taken into account in estimating T_B. An apparent $T_B = 10^{10}$ K implies an actual $T_B \gtrsim 10^{12}$ K for $f \lesssim 10^{-2}$, and the interpretation involving no instability is untenable. At least for the type I continuum, the implications of the generalized étendue rule out a possible theoretical interpretation that otherwise seems favorable.

4.3. The brightness temperatures of the quiet Sun

A surprising feature of radio observations of the quiet Sun is that the value of T_B is systematically less than 10^6 K (e.g., Sheridan and McLean, 1985), despite compelling evidence that the actual temperature of the coronal plasma is above 10^6 K. The argument here based on the generalized étendue offers a simple resolution of this long-standing inconsistency. The resolution is that the apparent source has a filling factor less than unity.

As pointed out in Section 3.2 above, the quiet Sun radiation appears to come from plasma levels well above that where it must be generated. By way of illustration, the evidence summarized in Section 3.5 indicates that the actual source of 30 MHz radiation is at $\approx 1.5\,R_0$ and that the observed (apparent) source is at $2.2\,R_0$; thus $\delta A'/\delta A = 2$. In addition, f' in (10) is the ratio of the observed brightness temperature, $\approx 1.7 \times 10^5$ K, to the electron temperature, $\gtrsim 10^6$ K (providing $\tau > 1$), implying $f' \lesssim \frac{1}{6}$. Further, assuming $\delta\Omega \approx \delta\Omega' \approx 2\pi$, we can use (10) to estimate the index of refraction μ at the actual source. We find $\mu \lesssim 0.3$. This implies that the radiation at 30 MHz does originate near the plasma level. Put another way, the low brightness temperatures observed at long meter wavelengths are consistent with their being a direct consequence of the low emissivity when $\mu < 1$ and the generalized étendue.

The explanation presented here for the low apparent brightness temperature of the quiet Sun is similar in part to that proposed by Aubier, Leblanc, and Boischot (1971). These authors based their argument on a ray-tracing model in which scattering near the plasma level increases the apparent size of the quiet Sun at decametric wavelengths and decreases the optical depth. The average brightness temperature is reduced by a factor equal to the ratio of the apparent area of the source to its actual area. Aubier, Leblanc, and Boischot were able to account for the quiet Sun size and the central brightness of $\approx 3.5 \times 10^5$ K then measured at 30 MHz. It is not clear whether the same explanation can account for the lower (by a factor ≈ 2) central brightness temperatures measured recently by Wang, Schmahl, and Kundu (1987). If enough additional scattering is added to decrease the optical depth and hence the central brightness by a factor of 2, the calculated size of the quiet Sun may be larger than is observed.

5. Discussion

The central argument in this paper involves the application of Liouville's theorem to the propagation of radio waves through the solar corona. Liouville's theorem implies that the extension in phase along the trajectory through phase space is conserved. For an optical system this may be expressed in terms of the conservation of the generalized étendue (1) along the ray path.

Observations of the apparent heights of sources of plasma emission seem to require that the apparent sources (the scatter images of the real sources) are higher in the corona than the plasma levels where the radiation must be generated. The transfer of the radiation from the actual to the apparent source may then be regarded as analogous to that in an optical system, and the generalized étendue applied to it. The most important implication of the generalized étendue is that the apparent sources must have a filling factor f which is less than unity, and very much less than unity for F sources, cf. Equation (10). The actual brightness temperatures T_B are then higher than the average, apparent values by the inverse of this filling factor. For F sources the value of T_B of the bright elements (if they exist) and of the true source are larger than the apparent, average values by a factor $\gtrsim 10^2$, and theories of types I, II, and III emission need to be reconsidered to take this into account. For the quiet Sun radiation, the inclusion of this filling factor seems to overcome a long-standing discrepancy between the values of the coronal temperature estimated from radio observations and by other means.

It should be emphasized that the arguments given here for the existence of a small filling factor are entirely theoretical, and it is highly desirable to find some way of confirming the inferred small values of f directly from observation. The small filling factor implies that the observed source should be either uniform and of low brightness or speckled, but, as mentioned in the Introduction, the detailed structure may be so small and/or complex as to be unresolvable by realistic instruments. Two arguments relevant to this point are as follows.

The first argument is due to M. Poquérusse (private communication), and concerns the broadening of a bundle of rays in phase space due to coronal scattering. For a source to appear spotty (i.e., localized in coordinate space) bundles of rays must remain localized in phase space. A speckled appearance results only if the radiation from the actual to the apparent source is confined locally to strands which separate but do not broaden. A spot then corresponds to the end of a strand. The point made by Poquérusse concerns scattering by small-scale irregularities. One of the conditions for the validity of geometric optics in a system of length L with irregularities of size l for radiation of wavelength λ is $l^2 \gg L\lambda$, e.g., Tatarski (1961). If scattering by sufficiently small-scale irregularities is important, then diffraction cannot be neglected, and this may smear out any spotty structure in the apparent source. Scattering by irregularities with $l < 10$ km could have this effect. The effect of diffraction is neglected both in the coronal scattering models, and in the approach adopted here; the consequences of this neglect need to be evaluated.

The other argument is due to J.-L. Steinberg (private communication), and concerns the expected size of a typical spot in the speckled source. The question is whether or not the spots should be resolvable. Fokker (1965) estimated that most of the angular broadening due to coronal scattering occurs more than $\approx 0.1\ R_0$ above the emission region. Two rays separated by, say, three inhomogeneity scale lengths (3 times 10 km) would be uncorrelated. Such rays would have an angular separation of 30/70000 radians. For a source at 2.5 R_0 the angular separation as observed from the Earth would be ≈ 0.5 arc sec. The expected bright spots should be smaller than this decorrelation separation. Such bright spots would be unobservable using current techniques. Hence, despite the roughness of the foregoing estimates, we conclude that the speckled nature of the source is likely to be unresolvable by current techniques.

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Appendix A. Propagation in a Spherical Corona

We treat propagation of rays through a spherically symmetric plasma using Hamiltonian optics. Rays are confined to a plane that passes through the origin, and we choose the axes such that a point on a ray is described by the radial distance r and polar angle θ. The Hamiltonian may be identified as $\hbar$ times the frequency expressed as a function of the plasma frequency $\omega_p(r)$ and of the conjugate momenta, which are expressed as $\hbar$ times corresponding wavenumber components. On omitting the factors $\hbar$, the Hamiltonian is

$$\Omega(r, k_r, k_\theta) = [\omega_p^2 + (k_r^2 + k_\theta^2/r^2)c^2]^{1/2}, \tag{A1}$$

where the r-dependence of ω_p is implied. The ray equations in Hamiltonian form are then

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial\Omega}{\partial k_r} = \frac{k_r c^2}{\Omega},$$

$$\frac{\mathrm{d}k_r}{\mathrm{d}t} = -\frac{\partial\Omega}{\partial r} = -\{\omega_p\, \mathrm{d}\omega_p/\mathrm{d}r - k_\theta^2 c^2/r^3\}/\Omega, \tag{A2}$$

$$\frac{d\theta}{dt} = \frac{\partial \Omega}{\partial k_\theta} = \frac{k_\theta c^2}{r^2 \Omega} ,$$

$$\frac{dk_\theta}{dt} = - \frac{\partial \Omega}{\partial \theta} = 0 .$$

The frequency Ω may now be identified as the wave frequency ω. The last of (A2) implies that k_θ is a constant, i.e., (denoting derivatives with respect to t by a dot)

$$k_\theta = \omega r^2 \dot{\theta}/c^2 = \text{constant.} \tag{A3}$$

Let ψ be the angle between the ray and the radial direction. Then one has

$$\tan \psi = k_\theta / r k_r \tag{A4}$$

and (A1), (A3), and (A4) imply

$$\mu \sin \psi = r \dot{\theta}/c , \tag{A5}$$

with

$$\mu = [1 - \omega_p^2/\omega^2]^{1/2} . \tag{A6}$$

It follows from (A3) and (A5) that for any two points P and P' along the ray path, the ray satisfies

$$\mu r \sin \psi = \mu' r' \sin \psi' . \tag{A7}$$

The ray trajectory may be found by integrating the equations between at fixed initial point P_0 and an arbitrary point P. One finds

$$\theta - \theta_0 = \mu_0 r_0 \sin \psi_0 \int_{r_0}^{r} dr\, r^{-2} \{ \mu^2 - (r_0^2/r^2) \mu_0^2 \sin^2 \psi_0 \}^{-1/2} , \tag{A8}$$

where μ is a function of r through its dependence on $\omega_p(r)$, cf. (A6). The integral in (A8) may be performed analytically for

$$\omega_p^2(r) = \omega_{p0}^2 r_0^2 / r^2 . \tag{A9}$$

One finds

$$\theta - \theta_0 = \mu_0 (r_0/R) \sin \psi_0 [\arccos(R/r) - \arccos(R/r_0)] , \tag{A10}$$

with

$$R = r_0 (1 - \mu_0^2 \cos^2 \psi_0)^{1/2} . \tag{A11}$$

An observer at r, θ viewing a source that lies on a spherical surface at r_0 can see rays originating from a range of values of θ_0, with rays at different θ_0 requiring different values of ψ_0 to pass through r, θ. The edge of the source that the observer can see corresponds to $\psi_0 = \pi/2$. For this ray at $r \gg r_0$ (A10) may be approximated by

$$\theta - \theta_0 \approx \mu_0 \arccos(r_0/r) \approx \mu_0[\pi/2 - r_0/r] . \tag{A12}$$

Then for $\mu_0 \ll 1$ it follows that the observer can see rays only from a range $\Delta\theta_0 \approx \mu_0 \pi/2$. This corresponds to a solid angle $\Delta\Omega_0 \approx \pi^3 \mu_0^2/4$, and to an area of the source at r_0 of

$$\Delta A_0 \approx \Delta\Omega_0 r_0^2 . \tag{A13}$$

The area ΔA_0 is the area of the source at r_0 from which rays can reach the observer. The extremum rays at $\psi_0 = \pi/2$ are at an angle $\psi \approx (r_0/r)\mu_0$ to the radial direction at r and, hence, appear to originate from an area

$$\Delta A = \pi\psi^2 r^2 \approx \pi\mu_0^2 r_0^2 . \tag{A14}$$

The areas ΔA and ΔA_0 differ only by a factor $4/\pi^2$.

An important feature of propagation in a spherically-symmetric coronal model is that the distant observer viewing an F source (i.e., $\mu_0 \ll 1$) can see rays originating only from a small area of a source, and cannot see that source at all when it is more than a few degrees away from the center of the Sun's disk. The linear dimensions of the source from which rays can reach the observer is $\Delta l \approx \pi\mu_0 r_0$, and the source is visible only if it is within an angle $\approx \Delta l/2r_0$ of the center, e.g., $< 9°$ for $\mu_0 < 0.1$. The brightness temperature of the (small) observed source is the true brightness temperature.

Appendix B: Derivation of the Generalized Étendue

Liouville's theorem implies that the extension in the 6-dimensional phase space is constant, i.e.,

$$\mathrm{d}^6\Gamma = \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{k}/(2\pi)^3 = \mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{k}'/(2\pi)^3 . \tag{B1}$$

The generalized étendue is a 4-dimensional form of this result with the components of $\mathbf{x}$ and $\mathbf{k}$ along a chosen axis (the z-axis here) eliminated. In Cartesian coordinates the standard form of the generalized étendue is (Welford and Winston, 1978)

$$\mu^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}l\,\mathrm{d}m = \mu'^2\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}l'\,\mathrm{d}m' , \tag{B2}$$

where l and m denote direction cosines. In spherical polar coordinates with angles θ and φ one has

$$l = \sin\theta\cos\varphi , \qquad m = \sin\theta\sin\varphi , \tag{B3}$$

and on writing

$$\mathrm{d}l\,\mathrm{d}m = \cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \cos\theta\,\mathrm{d}^2\Omega \tag{B4}$$

and $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}^2 A$, (B2) gives (1).

It is instructive to derive (1) directly from (B1). Consider rays propagating along a known ray path, and let s denote distance along this path. Rays emitted in time $\mathrm{d}t$ at one point P extend a length $\mathrm{d}s = v_g\,\mathrm{d}t$ along the ray path, where v_g is their group speed.

At another point P' these rays extend a distance $ds' = v_g' \, dt$. The group speed in an isotropic medium is given by

$$v_g = \partial\omega/\partial k \,. \tag{B5}$$

The wave frequency ω along the ray is a constant and, hence, one has

$$ds \, dk = ds' \, dk' \,. \tag{B6}$$

Now suppose (B1) is written in a spherical polar coordinate system. Let the element of volume be written $dz \, d^2A$, and let d^3k be written as $k^2 \, dk \, d^2\Omega$. Then (B1) becomes

$$dz \, d^2A k^2 \, dk \, d^2\Omega = dz' \, d^2A' k'^2 \, dk' \, d^2\Omega' \,. \tag{B7}$$

Now writing $k = \mu\omega/c$, $k' = \mu'\omega/c$, $dz = \cos\theta \, ds$, $dz' = \cos\theta' \, ds'$, (B7) with (B6) implies the generalized étendue in the form (1).

References

Aubier, M., Leblanc, Y., and Boischot, A.: 1971, *Astron. Astrophys.* **12**, 435.
Benz, A. O. and Wentzel, D. G.: 1981, *Astron. Astrophys.* **94**, 108.
Bougeret, J.-L. and Steinberg, J.-L.: 1977, *Solar Phys.* **61**, 777.
Bougeret, J.-L., Fainberg, J., and Stone, R. G.: 1984a, *Astron. Astrophys.* **136**, 255.
Bougeret, J.-L., Fainberg, J., and Stone, R. G.: 1984b, *Astron. Astrophys.* **141**, 17.
Dulk, G. A. and Suzuki, S.: 1980, *Astron. Astrophys.* **88**, 203.
Dulk, G. A., Melrose, D. B., and Suzuki, S.: 1979, *Proc. Astron. Soc. Australia* **3**, 375.
Dulk, G. A., Steinberg, J.-L., and Hoang, S.: 1984, *Astron. Astrophys.* **141**, 30.
Dulk, G. A., Suzuki, S., and Sheridan, K. V.: 1984, *Astron. Astrophys.* **130**, 39.
Duncan, R. A.: 1979, *Solar Phys.* **63**, 389.
Erickson, W. C., Gergely, T. E., Kundu, M. R., and Mahoney, M. J.: 1977, *Solar Phys.* **54**, 57.
Fokker, A. D.: 1965, *Bull. Astron. Inst. Neth.* **18**, 111.
Jaeger, J. C. and Westfold, K. C.: 1950, *Australian J. Sci. Res.* **A3**, 376.
Kerdraon, A.: 1973, *Astron. Astrophys.* **27**, 361.
Kerdraon, A., and Mercier, C.: 1983, *Astron. Astrophys.* **127**, 132.
Le Squeren, A.-M.: 1963, *Ann. Astrophys.* **26**, 97.
Melrose, D. B.: 1980, *Solar Phys.* **67**, 357.
Melrose, D. B. and Goldman, M. V.: 1987, *Solar Phys.* **107**, 329.
Melrose, D. B., Dulk, G. A., and Cairns, I. H.: 1986, *Astron. Astrophys.* **163**, 229.
Milne, E. A.: 1930, *Handbuch der Astrophysik* **3**, Pt. 1, p. 65. (Also in D. H. Menzel (ed.), *Selected Papers on the Transfer of Radiation*, Dover, 1966, p. 77.)
Riddle, A. C.: 1972a, *Proc. Astron. Soc. Australia* **2**, 98.
Riddle, A. C.: 1972b, *Proc. Astron. Soc. Australia* **2**, 148.
Riddle, A. C.: 1974, *Solar Phys.* **35**, 153.
Roberts, J. A.: 1959, *Australian J. Phys.* **11**, 327.
Robinson, R. D.: 1978, *Astrophys. J.* **222**, 696.
Saito, K.: 1970, *Ann. Tokyo Obs., Ser. 2* **12**, 53.
Sheridan, K. V. and McLean, D. J.: 1985, in D. J. McLean and N. R. Labrum (eds.), 'The Quiet Sun at Metre Wavelengths', *Solar Radiophysics*, Cambridge University Press, Cambridge, Ch. 17.
Steinberg, J.-L.: 1972, *Astron. Astrophys.* **18**, 382.
Steinberg, J.-L., Aubier-Giraud, M., Leblanc, Y., and Boischot, A.: 1971, *Astron. Astrophys.* **10**, 362.
Steinberg, J.-L., Dulk, G. A., Hoang, S., Lecacheux, A., and Aubier, M.: 1984, *Astron. Astrophys.* **140**, 39.
Steinberg, J.-L., Hoang, S., and Dulk, G. A.: 1985, *Astron. Astrophys.* **150**, 205.
Stewart, R. T.: 1972, *Proc. Astron. Soc. Australia* **2**, 100.

Suzuki, S.: 1961, *Ann. Tokyo Obs.* **7**, 75.
Suzuki, S. and Dulk, G. A.: 1985, in D. J. McLean and N. R. Labrum (eds.), 'Bursts of Type III and Type V', *Solar Radiophysics*, Cambridge University Press, Cambridge, Ch. 12.
Tatarski, V. I.: 1961, *Wave Propagation in a Turbulent Medium*, McGraw Hill, New York, p. 120.
Wang, Z., Schmahl, E. J., and Kundu, M. R.: 1987, *Solar Phys.* **111**, 49.
Welford, W. T. and Winston, R.: 1978, *The Optics of Nonimaging Concentrators*, Academic Press, New York, p. 151.
Zlobec, P.: 1975, *Solar Phys.* **43**, 453.