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Double-cyclotron Absorption: A Semiclassical Formulation

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Abstract

A general semiclassical formulation of the process where charged particles in a magnetised plasma simultaneously emit or absorb two gyromagnetic waves is presented. The kinetic equations, which describe the evolution of the wave number and of the particle distribution number due to these processes, are given. General expressions for the probability of double-cyclotron absorption are derived.

1. Introduction

In recent years a wide variety of wave-particle interactions have been discussed in an attempt to explain the acceleration and heating of ions in space and laboratory plasmas (e.g., Smith and Kaufman 1975; Ungstrup *et al.* 1979; Lysak *et al.* 1980; Chang and Coppi 1981; Ashour-Abdalla and Okuda 1984; Chang *et al.* 1986; Retterer *et al.* 1986). In this paper we present the formal equations describing the double-absorption process where ions in a magnetised plasma simultaneously absorb two gyromagnetic waves.

Double absorption has been proposed as a possible mechanism for the acceleration of oxygen (O^+) ions in the supra-auroral regions of the Earth's ionosphere, as a first step leading to the production of ion-conic and ion-bowl distributions (Temerin 1986). Temerin and Roth (1986) and Roth and Temerin (1986) derived an expression for the rate of acceleration of O^+ ions due to double-cyclotron absorption by considering the perturbation of the ion orbits due to the superposition of the electric fields of the two waves. The equations presented in this paper, whilst by no means restricted to this application, provide the basis for an alternative calculation of the acceleration rate of O^+ ions due to double-cyclotron absorption (Ball 1989, present issue p. 493).

In §2 we derive general kinetic equations which describe the evolution of the wave and particle distribution functions due to double-cyclotron absorption and emission. These equations are all expressed in terms of a quantity which may be called 'the probability of double absorption'. General expressions for this probability are derived in §3. The derivation given uses the fact that double-cyclotron absorption (and emission) is a crossed form of wave scattering in a magnetised plasma, a process for which a general semiclassical

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description has been discussed at some length by Melrose and Sy (1972*a*, 1972*b*).

2. The Kinetic Equations

The semiclassical formalism has been used extensively to describe wave-particle and wave-wave interactions, e.g., Melrose (1980*a*, chapter 5) and Melrose (1986, chapter 6). As implemented here the formalism involves using quantum mechanical notation to describe the waves and particles but all calculations are carried out classically. The advantage of introducing quantum mechanical notation and ideas is that it ensures that energy and momentum are conserved on a microscopic level (at each wave-particle event) so it is a simple matter to keep account of the effect of absorption and emission on the particles. In addition, quantum mechanical ideas allow us to relate absorption to emission on a microscopic level via the Einstein relations. As a result wave-particle interactions can be developed using a single-particle approach, in contrast with the collective approach that is necessary in any purely classical treatment. The notation used in this paper follows closely that of Melrose (1986, chapter 6).

In the semiclassical formalism the particles are described individually by their momentum $\mathbf{p}$ and collectively by their distribution function $f(\mathbf{p})$. Waves in a wave mode M are described by their dispersion relation $\omega = \omega_M(\mathbf{k})$ and polarisation vector $\mathbf{e}_M(\mathbf{k})$, and are regarded as collections of quanta with energy $\hbar|\omega_M(\mathbf{k})|$, momentum $\hbar\mathbf{k}$ and occupation number $N_M(\mathbf{k})$. The occupation number may be formally defined in terms of the total energy $W_M(\mathbf{k})$ of waves in a mode M with wave vector $\mathbf{k}$, enclosed in a system volume V:

$$N_M(\mathbf{k}) = \frac{W_M(\mathbf{k})}{\hbar|\omega_M(\mathbf{k})|V}\,. \tag{1}$$

Implicit in this description is the random phase approximation; the uncertainty principle implies that if the occupation number is specified then we have no information regarding the phase of the waves. Since the aim of this paper is to present equations describing double-cyclotron emission and absorption, we will discuss only simultaneous interactions between a single particle and two waves.

Consider first an isolated particle in an ambient magnetic field with no waves present. Let $w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p})$ be the probability per unit time that such a particle spontaneously emits wave quanta in the modes M and M' in the range $(d^3\mathbf{k}/(2\pi)^3)(d^3\mathbf{k}'/(2\pi)^3)$. Since we are interested in processes in a magnetised plasma we assume that the particle is an ion with gyrofrequency Ω_r and write

$$w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p}) = \sum_{s=-\infty}^{s=\infty} w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) \tag{2}$$

where $w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$ is the probability that emission occurs at a harmonic s of the gyrofrequency, and includes a Dirac δ-function of the form

$$\delta[\omega_M(\mathbf{k}) - k_\parallel v_\parallel + \omega_{M'}(\mathbf{k}') - k'_\parallel v_\parallel - s\Omega_r]\,. \tag{3}$$

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The subscript r is used on the ion gyrofrequency because the theory presented here includes relativistic effects. Thus $\Omega_r = |q|B/\gamma m$ where q is the ion charge, m is the rest mass of the ion, γ is the Lorentz factor and B is the magnitude of the ambient magnetic field $\mathbf{B}$. When emission occurs at a harmonic s the state of the ion changes from $\mathbf{p}$ to $\mathbf{p}-\Delta\mathbf{p}$ where $\Delta\mathbf{p}$ is the momentum carried by the wave quanta and has components $\Delta p_\perp = \hbar s\Omega_r/v_\perp$ and $\Delta p_\parallel = \hbar(k_\parallel + k'_\parallel)$. (Strictly, only the component $p_\parallel$ is continuous in the quantum case; $p_\perp$ is quantised as a simple harmonic oscillator, and becomes continuous only in the classical limit $\hbar \to 0$.) If we now include the effects of the wave populations and use the Einstein relations we find that the total probability of double emission at a harmonic s by a single particle is $w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)[1+N_M(\mathbf{k})][1+N_{M'}(\mathbf{k}')]$. The unit term which describes spontaneous double emission is a purely quantum term and is neglected in semiclassical theory. The term proportional to $N_M(\mathbf{k})$ describes emission stimulated by the wave population in mode M; similarly, the term proportional to $N_{M'}(\mathbf{k}')$ describes emission stimulated by waves in the mode M'. The term proportional to $N_M(\mathbf{k})N_{M'}(\mathbf{k}')$ describes emission stimulated by the presence of both wave populations. The inverse transition, where the state of the ion changes from $\mathbf{p}-\Delta\mathbf{p}$ to $\mathbf{p}$, corresponds to absorption. Of course absorption cannot occur unless the waves are present in both modes, and the probability of absorption of a wave at $\omega_M(\mathbf{k})$ must be proportional to the number of waves present, $N_M(\mathbf{k})$, so the probability of double absorption is proportional to $N_M(\mathbf{k})N_{M'}(\mathbf{k}')$. Considerations of equilibrium and the Einstein relations (Melrose 1980*a*, pp. 153,154) imply that the probability of double absorption is $w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)N_M(\mathbf{k})N_{M'}(\mathbf{k}')$. Finally we must sum over a set of particles and take the limit where the particle quantum number $\mathbf{p}$ becomes continuous. The net rate of double emission per unit volume is then

$$w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)[1+N_M(\mathbf{k})][1+N_{M'}(\mathbf{k}')]f(\mathbf{p}) \tag{4}$$

and the net rate of absorption is

$$w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)N_M(\mathbf{k})N_{M'}(\mathbf{k}')f(\mathbf{p}-\Delta\mathbf{p}) \tag{5}$$

each of which is to be operated on by an integral of the form $\int d^3\mathbf{p}$. (Note that $f(\mathbf{p})$ is normalised so that the particle number density n is given by $n = \int d^3\mathbf{p}f(\mathbf{p})$.)

It is apparent from the above expressions that $w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$ may equally well be referred to as "the probability for double emission" or "the probability for double absorption". We use these terms interchangeably from here on.

From expressions (4) and (5) we can obtain a kinetic equation for the waves which describes how the occupation numbers evolve. Each time a double emission occurs, $N_M(\mathbf{k})$ increases by unity and each time a double absorption takes place, $N_M(\mathbf{k})$ decreases by unity. Note also that for each $\mathbf{k}$ we have to consider the integral over all possible $\mathbf{k}'$. Hence we have

$$\frac{dN_M(\mathbf{k})}{dt} = \sum_{s=-\infty}^{s=\infty}\int d^3\mathbf{p}\int \frac{d^3\mathbf{k}'}{(2\pi)^3}\, w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)\{[1+N_M(\mathbf{k})]\times$$
$$\times[1+N_{M'}(\mathbf{k}')]f(\mathbf{p}) - N_M(\mathbf{k})N_{M'}(\mathbf{k}')f(\mathbf{p}-\Delta\mathbf{p})\}\,. \tag{6}$$

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In the classical limit we use the approximation

$$f(\mathbf{p}-\Delta\mathbf{p}) = f(\mathbf{p}) - \left(\Delta p_\perp \frac{\partial}{\partial p_\perp} + \Delta p_\parallel \frac{\partial}{\partial p_\parallel}\right) f(\mathbf{p}) \tag{7}$$

and then (6) becomes

$$\frac{dN_M(\mathbf{k})}{dt} = \sum_{s=-\infty}^{s=\infty} \int d^3\mathbf{p} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) \Big\{ [N_M(\mathbf{k}) + N_{M'}(\mathbf{k}')] f(\mathbf{p}) + \\ + \hbar N_M(\mathbf{k}) N_{M'}(\mathbf{k}') \left[\frac{s\Omega_r}{v_\perp} \frac{\partial f(\mathbf{p})}{\partial p_\perp} + (k_\parallel + k'_\parallel) \frac{\partial f(\mathbf{p})}{\partial p_\parallel} \right] \Big\} \tag{8}$$

where we have dropped the purely quantum term. The corresponding kinetic equation for waves in the mode M' is obtained from (8) by exchanging primed and unprimed quantities.

The evolution of the particle distribution function due to the double emission and absorption processes may be calculated in a similar fashion. Note that in this case we must consider the transitions $\mathbf{p}+\Delta\mathbf{p} \leftrightarrow \mathbf{p}$ as well as transitions $\mathbf{p} \leftrightarrow \mathbf{p}-\Delta\mathbf{p}$. The rate of change of $f(\mathbf{p})$ is thus determined by the increase due to double emission $\mathbf{p}+\Delta\mathbf{p} \to \mathbf{p}$ and double absorption $\mathbf{p}-\Delta\mathbf{p} \to \mathbf{p}$, and by the decrease due to double absorption $\mathbf{p} \to \mathbf{p}+\Delta\mathbf{p}$ and double emission $\mathbf{p} \to \mathbf{p}-\Delta\mathbf{p}$. The rates at which the second and fourth process occur are given by (5) and (4) respectively whilst the rates at which the first and third processes occur are obtained by replacing $\mathbf{p}$ by $\mathbf{p}+\Delta\mathbf{p}$ in (4) and (5) respectively; all four of these expressions are now to be operated on by the integrals $\int (d^3\mathbf{k}/(2\pi)^3)(d^3\mathbf{k}'/(2\pi)^3)$. Combining all these results we have

$$\frac{df(\mathbf{p})}{dt} = \sum_{s=-\infty}^{s=\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} \{ w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p}+\Delta\mathbf{p};s) \{ [1+N_M(\mathbf{k})] \times \\ \times [1+N_{M'}(\mathbf{k}')] f(\mathbf{p}+\Delta\mathbf{p}) - N_M(\mathbf{k}) N_{M'}(\mathbf{k}') f(\mathbf{p}) \} + \\ + w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) \{ N_M(\mathbf{k}) N_{M'}(\mathbf{k}') f(\mathbf{p}-\Delta\mathbf{p}) - \\ - [1+N_M(\mathbf{k})][1+N_{M'}(\mathbf{k}')] f(\mathbf{p}) \} \} . \tag{9}$$

The expansion of $f(\mathbf{p}\pm\Delta\mathbf{p})$ in $\Delta\mathbf{p}$ needs to be carried out to second order in this case as the first-order terms cancel. Thus one writes

$$f(\mathbf{p}\pm\Delta\mathbf{p}) = [1 \pm \Delta\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{p}} + (\Delta\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{p}})^2] f(\mathbf{p}) \tag{10}$$

and

$$w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p}+\Delta\mathbf{p};s) = (1 + \Delta\mathbf{p} \cdot \frac{\partial}{\partial \mathbf{p}}) w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) . \tag{11}$$

When (9) is expanded in this way all the terms independent of Δp (and hence independent of $\hbar$) cancel. In the classical limit, $\hbar \to 0$ and $N_M(\mathbf{k}) \to \infty$ in such a way that $\hbar N$ (the classical action variable) remains finite. Thus we retain

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only terms proportional to either $\hbar N$ or $\hbar^2 N^2$. The resulting semiclassical kinetic equation for $f(\mathbf{p})$ may be written as a generalised diffusion equation in momentum space. The details of this procedure are omitted, as is the transformation to cylindrical co-ordinates. The resulting kinetic equation for the particle distribution can then be written in the form

$$\frac{df(\mathbf{p})}{dt} = \frac{\partial}{\partial p_\parallel}\left\{A_\parallel f(\mathbf{p}) + \left[D_{\parallel\parallel}\frac{\partial}{\partial p_\parallel} + D_{\parallel\perp}\frac{\partial}{\partial p_\perp}\right]f(\mathbf{p})\right\} + $$
$$+ \frac{1}{p_\perp}\frac{\partial}{\partial p_\perp}\left\{p_\perp\left[A_\perp f(\mathbf{p}) + \left(D_{\perp\parallel}\frac{\partial}{\partial p_\parallel} + D_{\perp\perp}\frac{\partial}{\partial p_\perp}\right)f(\mathbf{p})\right]\right\} \tag{12}$$

with

$$\begin{pmatrix} A_\parallel \\ A_\perp \end{pmatrix} = \sum_{s=-\infty}^{s=\infty}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}[N_M(\mathbf{k}) + N_{M'}(\mathbf{k}')]\times$$
$$\times\begin{pmatrix} \hbar(k_\parallel + k'_\parallel) \\ \hbar s\Omega_r/v_\perp \end{pmatrix} w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)\,, \tag{13}$$

and

$$\begin{pmatrix} D_{\parallel\parallel} \\ D_{\parallel\perp} = D_{\perp\parallel} \\ D_{\perp\perp} \end{pmatrix} = \sum_{s=-\infty}^{s=\infty}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3} N_M(\mathbf{k})N_{M'}(\mathbf{k}')\times$$
$$\times\begin{pmatrix} [\hbar(k_\parallel + k'_\parallel)]^2 \\ \hbar(k_\parallel + k'_\parallel)\hbar s\Omega_r/v_\perp \\ (\hbar s\Omega_r/v_\perp)^2 \end{pmatrix} w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)\,. \tag{14}$$

The term in (12) of direct relevance to the postulated acceleration of oxygen ions, and hence the production of ion conics via double-cyclotron absorption, is that involving $D_{\perp\perp}$. This point is discussed in some detail by Ball (1989).

3. Absorption Probability: General Expressions

The kinetic equations derived in §2 describe the evolution of the wave and particle populations due to double absorption and emission in terms of the absorption probability $w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$. In this section we derive the general expression for $w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$ from the probability for wave scattering in a magnetised plasma. Note that $w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$ was defined as the probability of double emission, and then related to absorption by the Einstein relations. In the following derivation it is the probability of double emission which is calculated explicitly.

A general semiclassical treatment of the scattering of gyromagnetic waves in a magnetised plasma was developed by Melrose and Sy (1972*a*). They derived an expression for the 'scattering probability' which we will denote by $\hat{w}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$; this quantity is the probability per unit time that a particle with momentum $\mathbf{p}$ will scatter a wave in mode M' with wave vector $\mathbf{k}'$, into

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a wave in mode M with wave vector $\mathbf{k}$, via the harmonic s of the particle's gyrofrequency. The results of Melrose and Sy (1972*a*) may be written in the form

$$\hat{w}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)=\frac{2\pi q^4}{m^2\epsilon_0^2}\frac{|\hat{a}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)|^2}{|\omega_M(\mathbf{k})\omega_{M'}(\mathbf{k}')|}R_M(\mathbf{k})R_{M'}(\mathbf{k}')\times$$
$$\times\delta[\omega_M(\mathbf{k})-k_\parallel v_\parallel-\omega_{M'}(\mathbf{k}')+k'_\parallel v_\parallel-s\Omega_r] \tag{15}$$

with

$$\hat{a}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)=\hat{a}^{TS}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)+\hat{a}^{NL}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)\,, \tag{16}$$

and

$$\hat{a}^{TS}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)=e^*_{Mi}(\mathbf{k})e_{M'j}(\mathbf{k}')\hat{a}^{TS}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s)\,,$$

$$\hat{a}^{NL}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)=e^*_{Mi}(\mathbf{k})e_{M'j}(\mathbf{k}')\hat{a}^{NL}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s)\,. \tag{17}$$

Quantities in (15) which refer to the scattering ions are the charge q, rest mass m, relativistic gyrofrequency Ω_r, and velocity $\mathbf{v}$ with components $v_\parallel$, $v_\perp$ parallel and perpendicular to $\mathbf{B}$. Quantities in (15) which refer to the waves are $R_M(\mathbf{k})$, the ratio of electric energy density to total energy density of waves in mode M, and $k_\parallel$, $k_\perp$ the parallel and perpendicular components of $\mathbf{k}$. In (17) $\mathbf{e}_M(\mathbf{k})$ is the polarisation vector for waves in mode M (e.g., Melrose 1980*a*, pp. 42–45) and the subscripts i and j refer to components in a cartesian co-ordinate system where the magnetic field is along the 3-axis.

The form of (16) reflects the method used by Melrose and Sy (1972*a*) who solved the inhomogeneous wave equation for the power radiated in the scattered waves, with the source current identified as the sum of two terms. The first contribution to the source current, which gives rise to the *TS* (Thomson scattering) term of (16), is the current density due to the perturbed motion of the scattering charge in the fields of the unscattered wave. The second contribution to the source current, which gives rise to the *NL* (nonlinear scattering) term of (16), is the current density due to the nonlinear response of the plasma to the fields of the unscattered wave and the shielding fields due to the unperturbed motion of the scattering charge. These two processes via which a particle in a plasma can scatter waves were discussed in some detail by Tsytovich (1970) and it is from that work that the name 'nonlinear scattering' arises. This nomenclature may be confusing since Thomson scattering is itself a nonlinear process. Hereafter the terms 'nonlinear scattering' and 'nonlinear contribution' refer specifically to the process which gives rise to the second term of (16) and to the second part of (17).

Still following Melrose and Sy (1972*a*) and correcting a sign error in their equation (5), the Thomson scattering term of (17) has the form

$$\hat{a}^{TS}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s)=\sum_{t'=-\infty}^{\infty}\hat{\alpha}_{ij}(s+t',\mathbf{k};t',\mathbf{k}';\mathbf{p})\,, \tag{18}$$

with

$$\hat{\alpha}_{ij}(t,\mathbf{k};t',\mathbf{k}';\mathbf{p})=\frac{\exp[-i\epsilon(t\psi-t'\psi')]}{\gamma\omega_t\omega'_{t'}}\{\omega_t\omega'_{t'}J_t(z)J_{t'}(z')\tau_{ij}+$$
$$+\omega_tJ_t(z)V^*_j(\mathbf{k}',\mathbf{p};t')k_l\tau_{il}+\omega'_{t'}J_{t'}(z')V_i(\mathbf{k},\mathbf{p};t)k'_l\tau_{lj}+$$
$$+[k_lk'_n\tau_{ln}-\omega_M(\mathbf{k})\omega_{M'}(\mathbf{k}')/c^2]V_i(\mathbf{k},\mathbf{p};t)V^*_j(\mathbf{k}',\mathbf{p};t')\} \tag{19}$$

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where $\epsilon = q/|q|$, $\mathbf{k} = (k_\perp \cos\psi, k_\perp \sin\psi, k_\|)$, $\omega_t = \omega - t\Omega_r - k_\| v_\|$, $z = k_\perp v_\perp/\Omega_r$ and the primed quantities are defined analogously. Here $J_t(z)$ is the Bessel function of order t. The vector $\mathbf{V}(\mathbf{k},\mathbf{p};t)$ is the velocity function for the spiralling ion and is given by

$$\mathbf{V}(\mathbf{k},\mathbf{p};t) = \{v_\perp[e^{i\epsilon\psi}J_{t-1}(z) + e^{-i\epsilon\psi}J_{t+1}(z)]/2\,,$$

$$-i\epsilon v_\perp[e^{i\epsilon\psi}J_{t-1}(z) - e^{-i\epsilon\psi}J_{t+1}(z)]/2\,, v_\| J_t(z)\}\,, \qquad (20)$$

and the asterisk denotes complex conjugation. The quantity $\tau_{ij}(\omega)$ is given by

$$\tau_{ij}(\omega) = \begin{pmatrix} \omega^2/(\omega^2-\Omega_r^2) & i\epsilon\omega\Omega_r/(\omega^2-\Omega_r^2) & 0 \\ -i\epsilon\omega\Omega_r/(\omega^2-\Omega_r^2) & \omega^2/(\omega^2-\Omega_r^2) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (21)$$

In (19) we write $\tau_{ij} = \tau_{ij}(\omega_t) = \tau_{ij}(\omega'_{t'})$, where the equality $\omega_t = \omega'_{t'}$ arises from the δ-function in (15) with s replaced by $t-t'$. Note that although the tensor τ_{ij} is usually associated with the cold plasma approximation, no such approximation has been made in the derivation of (19). This expression is the probability for scattering by a single particle and has not been averaged over any particle distribution.

The 'nonlinear' contribution to the scattering probability has the form (Melrose and Sy 1972*a*):

$$\hat{a}_{ij}^{NL}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = \frac{2m}{q\epsilon_0}\,\frac{\exp[-i\epsilon s(\psi+\psi')]}{[\omega_M(\mathbf{k})-\omega_{M'}(\mathbf{k}')]^2}\,\alpha_{ijl}(k,k',k-k')\,\frac{\lambda_{lm}(k-k')}{\Lambda(k-k')}\times$$

$$\times V_m(\mathbf{k}-\mathbf{k}',\mathbf{p};s) \qquad (22)$$

where we have included a phase factor and used the shorthand notation $\omega,\mathbf{k} \to k$. The inclusion of the phase factor in (22) is important because the relative phase of $\hat{a}_{MM'}^{TS}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$ and $\hat{a}_{MM'}^{NL}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$ is needed to evaluate the crossed term $2\hat{a}_{MM'}^{TS}(\mathbf{k},\mathbf{k}',\mathbf{p};s)\hat{a}_{MM'}^{NL}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$ of $|\hat{a}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)|^2$ which appears in equation (15). The phase factor is often neglected in applications where it can be argued *a priori* that only the Thomson scattering term or the 'nonlinear' term will contribute.

The quantities in equation (22) which have not already been defined describe the collective response of the plasma. Further, α_{ijl} is the quadratic nonlinear response tensor and describes the second-order (lowest-order nonlinear) response of the plasma to an applied field (e.g., Melrose 1986, pp. 11, 81–82); note that α_{ijl} is not related to the 2-index quantity $\hat{\alpha}_{ij}$ which appears in equations (18) and (19). The quantity $\Lambda(k)$ is the determinant of the tensor $\Lambda_{ij}(k)$ and $\lambda_{ij}(k)$ is the cofactor of the tensor element $\Lambda_{ji}(k)$. These last two quantities are related by

$$\Lambda_{ij}(k)\lambda_{jl}(k) = \Lambda(k)\delta_{il} \qquad (23)$$

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where δ_{il} is the Kronecker delta. Finally, the tensor $\Lambda_{ij}(k)$ may be defined in terms of the dielectric tensor $\kappa_{ij}(k)$,

$$\Lambda_{ij}(k) = (k_i k_j - k^2 \delta_{ij})c^2/\omega^2 + \kappa_{ij}(k)\,. \tag{24}$$

The dielectric tensor describes the first-order (linear) response of the plasma to an applied field (e.g., Melrose 1986, p. 11).

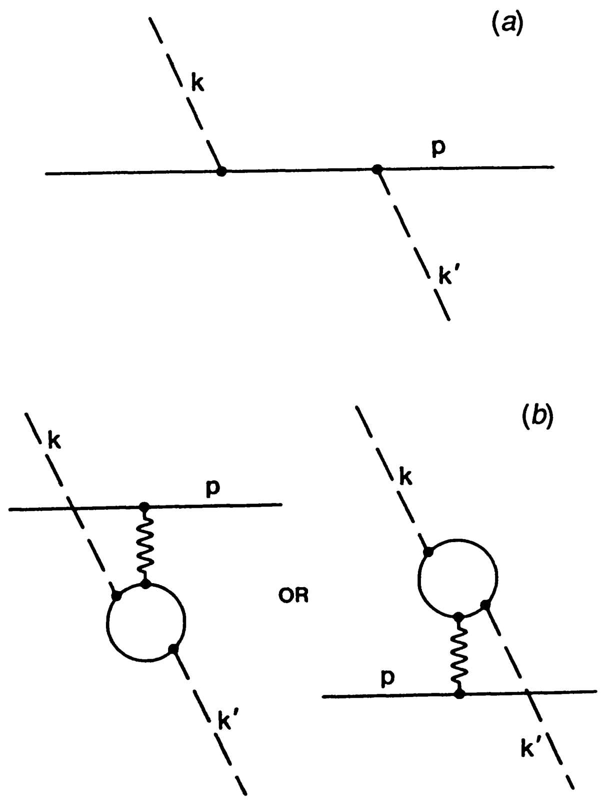

Fig. 1. Diagrammatic representations of wave scattering. Fig. 1*a* represents Thomson scattering and Fig. 1*b* represents 'nonlinear' scattering.

Recalling that the primed quantities refer to the unscattered or incident wave, the scattering processes for which we have set out the equations above may be represented diagrammatically. Fig. 1*a* represents the Thomson scattering contribution and Fig. 1*b* represents the 'nonlinear' contribution. These diagrams have been constructed according to the following rules (Melrose 1980*b*, p. 165):

(1) A particle is represented by a solid horizontal line.
(2) A dashed line represents a 'real' photon. A photon in the initial state is represented by a dashed line extending from the bottom of the diagram to a dot. A photon in the final state is represented by a dashed line extending from a dot to the top of the diagram.

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(3) A squiggle represents an internal or 'virtual' photon and always starts and ends in a dot.
(4) A circle with n dots represents an $(n-1)$-fold nonlinear response.

Note that here 'photon' means 'wave quantum'.

The process of double emission is 'crossed' relative to the process of scattering, in the sense discussed by Melrose (1986, p. 87). If 'scattering' describes a transition from some initial state I to final state F, then 'double emission' describes a transition from a state I' to a state F', where (I',F') may be obtained from (I,F) by transferring the wave in mode M' from the initial state I to the final F. Thus there are contributions to the probability of double emission from the 'crossed' Thomson scattering process and from the 'crossed' nonlinear scattering process. These processes may be represented by diagrams constructed according to the rules given above. Fig. 2*a* represents the Thomson-scattering-like (hereafter abbreviated to TS-like) double emission process, and Fig. 2*b* represents the 'nonlinear' double emission process. In this diagrammatic description, the 'crossed' relation refers to the fact that the diagram for TS-like double emission (Fig. 2*a*) may be obtained from the diagram for Thomson scattering (Fig. 1*a*) by taking the wave line for k' 'across' the ion line, i.e. by transferring the wave line for k' from the initial state (below the ion line) to the final state (above). The diagram for 'nonlinear' double emission (Fig. 2*b*) may be obtained from the diagram for 'nonlinear' scattering (Fig. 1*b*) in exactly the same way.

It follows from this 'crossing symmetry' that the probability for double emission may be obtained from the probability for scattering, equation (15)

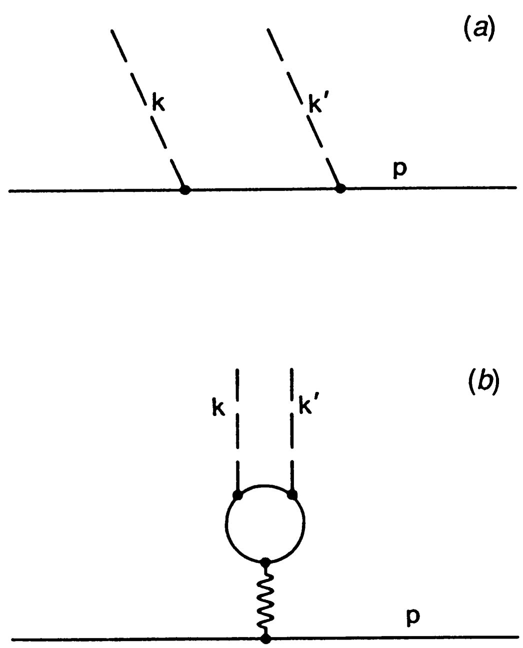

Fig. 2. Diagrammatic representations of double-cyclotron emission. Fig. 2*a* represents the Thomson-scattering-like process and Fig. 2*b* represents 'nonlinear' double emission.

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et seq., by making the replacement $\mathbf{k}' \to -\mathbf{k}'$ according to the conventions (Melrose 1980*b*, p. 171; Melrose 1986, p. 83)

$$\omega_{M'}(-\mathbf{k}') \to -\omega_{M'}(\mathbf{k}'),\ \ \mathbf{e}_{M'}(-\mathbf{k}') \to \mathbf{e}^*_{M'}(\mathbf{k}'),\ \ R_{M'}(-\mathbf{k}') = R_{M'}(\mathbf{k}')\,. \tag{25}$$

So far we have used a hat to denote all quantities which refer specifically to scattering. From here on we use the same symbols, without the hat, for the related quantities which refer specifically to double emission. The probability of double emission is thus

$$w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = \hat{w}_{MM'}(\mathbf{k},-\mathbf{k}',\mathbf{p};s)\,. \tag{26}$$

Using equation (15) and rewriting $|\hat{a}_{MM'}(\mathbf{k},-\mathbf{k}',\mathbf{p};s)|^2$ as $|a_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)|^2$ we have

$$w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = \frac{2\pi q^4}{m^2\epsilon_0^2}\frac{|a_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)|^2}{|\omega_M(\mathbf{k})\omega_{M'}(\mathbf{k}')|}R_M(\mathbf{k})R_{M'}(\mathbf{k}')\times$$
$$\times\,\delta[\omega_M(\mathbf{k}) - k_\parallel v_\parallel + \omega_{M'}(\mathbf{k}') - k'_\parallel v_\parallel - s\Omega_r]\,. \tag{27}$$

Continuing in this way, we have from (16)

$$a_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = a^{TS}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) + a^{NL}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) \tag{28}$$

with

$$a^{TS}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = e^*_{Mi}(\mathbf{k})e^*_{M'j}(\mathbf{k}')a^{TS}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s)\,,$$
$$a^{NL}_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = e^*_{Mi}(\mathbf{k})e^*_{M'j}(\mathbf{k}')a^{NL}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s) \tag{29}$$

and

$$a^{TS}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = \hat{a}^{TS}_{ij}(\mathbf{k},-\mathbf{k}',\mathbf{p};s)\,,$$
$$a^{NL}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = \hat{a}^{NL}_{ij}(\mathbf{k},-\mathbf{k}',\mathbf{p};s)\,. \tag{30}$$

To obtain explicit expressions for the quantities in (30) the change of sign of $\mathbf{k}'$ is effected by making the replacements

$$k'_\parallel \to -k'_\parallel;\quad \psi' \to \psi' + \pi \tag{31}$$

in the equations for $\hat{a}^{TS}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$ and $\hat{a}^{NL}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$.

Consider first the Thomson scattering contribution. Equations (18) and (30) imply

$$a^{TS}_{ij}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = \sum_{t'=-\infty}^{\infty}\hat{\alpha}_{ij}(s+t',\mathbf{k};t',-\mathbf{k}';\mathbf{p})\,. \tag{32}$$

Now making the replacements (31) we find

$$\omega'_{t'} = \omega_{M'}(\mathbf{k}') - t'\Omega_r - k'_\parallel v_\parallel \;\to\; -\omega_{M'}(\mathbf{k}') - t'\Omega_r + k'_\parallel v_\parallel \tag{33}$$

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so if we arbitrarily relabel t' as $-t'$ we have $\omega'_{t'} \to -\omega'_{t'}$. Relabelling t' as $-t'$ in (32) gives

$$a_{ij}^{TS}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = \sum_{t'=-\infty}^{\infty} \hat{\alpha}_{ij}(s-t',\mathbf{k};-t',-\mathbf{k}';\mathbf{p})\,. \tag{34}$$

Rewriting (34) in the form of (27–30) we obtain

$$a_{ij}^{TS}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = \sum_{t'=-\infty}^{\infty} \alpha_{ij}(s-t',\mathbf{k};t',\mathbf{k}';\mathbf{p}) \tag{35}$$

where

$$\alpha_{ij}(s-t',\mathbf{k};t',\mathbf{k}';\mathbf{p}) = \hat{\alpha}_{ij}(s-t',\mathbf{k};-t',-\mathbf{k}';\mathbf{p})\,. \tag{36}$$

Using the standard identity $J_{-t}(z) = (-1)^t J_t(z)$ and (20) one finds

$$\mathbf{V}^*(-\mathbf{k}',\mathbf{p};-t) = (-1)^t\,\mathbf{V}(\mathbf{k}',\mathbf{p};t)\,. \tag{37}$$

Next we have the quantity $\tau_{ij}(\omega)$ which is to be evaluated at $\omega = \omega_t$ or $\omega = \omega'_{t'}$. In the case of Thomson scattering the resonance condition is $\omega_t - \omega'_{t'} = 0$ and the two possible arguments are equal. For the case of double emission the argument of the δ-function in (27), with $t = s - t'$ in (35) implies that the resonance condition is $\omega_t + \omega'_{t'} = 0$ so the possible arguments now have opposite signs. Since

$$\tau_{ij}(-\omega) = \tau_{ij}^*(\omega) = \tau_{ji}(\omega) \tag{38}$$

it matters whether we write $\tau_{ij}(\omega_t)$ or $\tau_{ij}(\omega'_{t'})$ in (19) before making the replacement $\omega'_{t'} \to -\omega'_{t'}$. A calculation based on the covariant formulation, which is omitted here, indicates that τ_{ij} is to be evaluated at $\omega'_{t'}$. Now (36) and (19), with (37) and (38), imply

$$\begin{aligned}\alpha_{ij}(s-t',\mathbf{k};t',\mathbf{k}';\mathbf{p}) = {} & \frac{\exp[-i\epsilon(t\psi + t'\psi')]}{\gamma\omega_t\omega'_{t'}}\{\omega_t\omega'_{t'}J_t(z)J_{t'}(z')\tau_{ij}^*(\omega'_{t'}) - \\ & - \omega_t J_t(z)V_j(\mathbf{k}',\mathbf{p};t')k_l\tau_{il}^*(\omega'_{t'}) - \omega'_{t'}J_{t'}(z')V_i(\mathbf{k},\mathbf{p};t)k'_l\tau_{lj}^*(\omega'_{t'}) + \\ & + [k_l k'_n\tau_{ln}^*(\omega'_{t'}) - \omega_M(\mathbf{k})\omega_{M'}(\mathbf{k}')/c^2]V_i(\mathbf{k},\mathbf{p};t)V_j(\mathbf{k}',\mathbf{p};t')\}\end{aligned} \tag{39}$$

with $t = s - t'$.

The 'nonlinear' term follows simply from (31) and (22):

$$\begin{aligned}a_{ij}^{NL}(\mathbf{k},\mathbf{k}',\mathbf{p};s) = {} & \frac{2m(-1)^s}{q\epsilon_0}\,\frac{\exp[-i\epsilon s(\psi + \psi')]}{[\omega_M(\mathbf{k}) + \omega_{M'}(\mathbf{k}')]^2}\,\alpha_{ijl}(k,-k',k+k')\times \\ & \times \frac{\lambda_{lm}(k+k')}{\Lambda(k+k')}\,V_m(\mathbf{k}+\mathbf{k}',\mathbf{p};s)\,.\end{aligned} \tag{40}$$

Finally, we repeat that although we have considered explicitly the probability of double-cyclotron *emission* in this section, it was shown in §2 that this

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quantity may equally well be referred to as the probability of double-cyclotron *absorption.*

4. Summary

In this paper we have presented the equations which describe, in the context of a general semiclassical theory, the processes of double-cyclotron absorption and emission. The equations describing the evolution of the distribution function of the absorbing ions, and of the occupation numbers of the absorbed and emitted waves, have been presented. We have then presented a general expression for the pivotal quantity, the absorption probability $w_{MM'}(\mathbf{k},\mathbf{k}',\mathbf{p};s)$, which has been derived from the scattering probability of Melrose and Sy (1972*a*) using a crossing symmetry relation. All quantum effects have been neglected.

The relevant final equations are as follows. The evolution of the wave distribution, described semiclassically by the occupation number $N_M(\mathbf{k})$, is given by (8). The evolution of the particle distribution function $f(\mathbf{p})$ is described by (12) to (14). Finally, the 'absorption probability' is given explicitly by (27) to (29), (35), (39) and (40). These equations constitute a general, semiclassical formulation of the processes of simultaneous emission and absorption of two gyromagnetic waves by ions in a magnetised plasma.

Reduction of these equations to a form suitable for treatment of the processes discussed in the Introduction requires a careful consideration of various terms in the absorption probability. This is discussed in detail by Ball (1989).

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