QUASI-LINEAR RELAXATION OF ELECTRONS INTERACTING WITH AN INHOMOGENEOUS DISTRIBUTION OF LANGMUIR WAVES

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Abstract. Quasi-linear theory, describing the diffusion of electrons in velocity space due to resonant interaction with Langmuir waves, is generalized to treat the case where the waves are distributed inhomogeneously (in 'clumps'). The method used is a generalization of an approach developed by Morales and Lee (1974) to treat the interaction of electrons with a distribution of solitons. It is shown that quasi-linear theory, specifically the diffusion of electrons in velocity space due to resonant interaction with Langmuir waves, applies irrespective of how the waves are distributed in space, provided that an electron has multiple encounters with clumps of Langmuir waves, and that the evolution of the distribution of electrons is considered only on a time-scale long compared with the time between such encounters. This generalization of quasi-linear theory is of relevance to type III solar radio bursts, where the Langmuir waves are known to be distributed inhomogeneously, and yet the electron distribution is consistent with that expected from a balance between ballistic effects and quasi-linear relaxation.

1. Introduction

Quasi-linear theory describes the effect of resonant particle-wave interactions on a distribution of particles under the assumptions that the waves are of small amplitude with random phases, and that the level of the wave turbulence is uniform in space and slowly varying in time. The effect of the waves on the particles is then to cause them to diffuse in velocity or momentum space. The essential 'quasi-linear assumption' is that the effect of the waves on an individual particle may be treated as a perturbation; the actual orbit of the particle must deviate only slightly from its unperturbed orbit in the absence of the waves. In principle quasi-linear theory should be applicable when the distribution of waves is inhomogeneous, specifically, when the waves exist as a collection of isolated 'clumps', each clump being a localized region where the waves have been excited. A quasi-linear theory describing the interaction of fast particles with such an inhomogeneous distribution of waves is developed in this paper.

The original motivation for the investigation reported here arose from the theory of type III solar radio bursts, as discussed below. For type III bursts one is concerned with the effect of Langmuir waves on fast but nonrelativistic electrons, and it is this case that is considered here. The procedure adopted is a generalization of that used by Morales and Lee (1974), who discussed the effect of a distribution of solitons on a distribution

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of fast electrons. This procedure involves describing the effect of the interaction in terms of a Fokker–Planck equation.

A one-dimensional form of quasi-linear theory was applied to type III solar radio events in the interplanetary plasma by Grognard (1975, 1982), who found that the theory provides a good explanation for the form of the observed distribution of electrons. Type III events involve a stream of electrons propagating away from the Sun and generating Langmuir waves through a bump-in-tail instability. Partial conversion of the energy in the Langmuir waves into transverse waves (near the fundamental or second harmonic of the plasma frequency) causes the observed radio emission. For specific events, *in situ* data are available on the electrons, the Langmuir waves, the radio emission and the local plasma parameters in the source (Lin *et al.*, 1981). Grognard (1975, 1982) developed a numerical code that treats the growth of the Langmuir waves, and the outward propagation of the electrons including the effects of quasi-linear relaxation. The specific conclusion of interest here is as follows: Grognard showed that if the observed distribution of electrons is taken as input into the numerical code, then the output for the distribution of electrons after the transients die away has the same form as the input. Thus the observed distribution is a self-consistent one that balances the effects of ballistic propagation, which distorts the initial spectrum due to faster electrons outpacing slower electrons, and of quasi-linear relaxation, which tends to smooth the distribution. This result is somewhat surprising because the conditions for the quasi-linear theory used to be valid are not satisfied. In particular the Langmuir waves are not distributed uniformly in space but are confined to localized clumps (Gurnett and Anderson, 1977; Gurnett *et al.*, 1978; Lin *et al.*, 1981). Thus it would appear that, at least in this case, quasi-linear theory applies even when the waves are distributed in a very inhomogeneous way. It is shown here that the average effect of many clumps of Langmuir waves on the electrons is equivalent to the effect of a homogeneous distribution of waves.

The essential ideas are as follows. When the waves are not uniformly distributed, but are confined to many separate localized clumps, an individual particle is affected by the waves only when it is inside a clump. On a time that is long compared with the characteristic time between a particle leaving one clump and entering another, the particle experiences a sequence of perturbations as it passes through a sequence of clumps. Each such perturbation depends on the size of the clump and the level of the wave turbulence in it. A theory for the effect on the distribution of particles involves a calculation of the perturbative effect of an individual clump, together with a statistical description of the clumps to enable one to determine the average effect of many such perturbations. The calculation of the effect of an individual clump is described in Section 2, and models for clumps are discussed in Section 3.

2. Diffusion Equation

In this section a Fokker–Planck equation is derived for the interaction of fast particles with an idealized model for an inhomogeneous distribution of waves. The waves are assumed to be longitudinal (Langmuir waves), and the particles are assumed nonrelativistic.

2.1. Model for the Wave Field

For longitudinal waves the electric field may be written in the form

$$\mathbf{E}(t, \mathbf{x}) = -\operatorname{grad} \phi(t, \mathbf{x}), \tag{1}$$

where $\phi(t, \mathbf{x})$ is the electric potential. A specific model for a clump of waves is assumed here. This form is

$$\phi(t, \mathbf{x}) = A g(\mathbf{x}) \cos(\omega_0 t - \mathbf{k}_0 \cdot \mathbf{x} - \theta). \tag{2}$$

In (2), ω_0, $\mathbf{k}_0$ describes a plane carrier wave with an initial phase θ and an amplitude A, and $g(\mathbf{x})$ describes the envelope of the wave, including its spatial variations. One may regard the model (2) as describing three loosely defined cases depending on the product of the wave number k_0 times the characteristic distance over which $g(\mathbf{x})$ changes: a 'soliton' corresponds to this product being zero ($k_0 = 0$), a 'wave packet' corresponds to this product being of order unity, and a 'clump' corresponds to this product being large.

For a single clump, $g(\mathbf{x})$ is a spatially localized function. One could also regard $g(\mathbf{x})$ as describing a distribution of clumps, but this is not done here. The fact that $g(\mathbf{x})$ does not depend on time implies that the shape of the clump (or the distribution of the clumps if $g(\mathbf{x})$ is used to describe such a distribution) does not change with time.

The Fourier transform $\Phi(\omega, \mathbf{k})$ of $\phi(t, \mathbf{x})$ is defined by

$$\Phi(\omega, \mathbf{k}) = \int \mathrm{d}t\, \mathrm{d}^3\mathbf{x}\, \phi(t, \mathbf{x}) \exp[i(\omega t - \mathbf{k} \cdot \mathbf{x})]. \tag{3}$$

On inserting the form (2) one obtains

$$\Phi(\omega, \mathbf{k}) = \tfrac{1}{2} A [2\pi\delta(\omega + \omega_0) e^{i\theta} G(\mathbf{k} + \mathbf{k}_0) + 2\pi\delta(\omega - \omega_0) e^{-i\theta} G(\mathbf{k} - \mathbf{k}_0)], \tag{4}$$

where $G(\mathbf{k})$ is the spatial Fourier transform of $g(\mathbf{x})$. The Fourier transform of the electric field (1) is then given by

$$\mathbf{E}(\omega, \mathbf{k}) = -i\mathbf{k}\Phi(\omega, \mathbf{k}). \tag{5}$$

2.2. Perturbations in the Orbit of a Fast Particle

The equation of motion for a nonrelativistic particle with mass m and charge q in an electric field is

$$m \frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathbf{X}(t) = q\mathbf{E}(t, \mathbf{X}(t)), \tag{6}$$

where $\mathbf{x} = \mathbf{X}(t)$ describes the orbit of the particle and $\mathbf{v} = \mathrm{d}\mathbf{X}(t)/\mathrm{d}t$ is its instantaneous velocity. A perturbative solution may be found by expanding in powers of the amplitude of the electric field. The expansion of the orbit and of the instantaneous velocity may be written in the form

$$\mathbf{X}(t) = \mathbf{x}_0 + \mathbf{v}_0 t + \mathbf{x}^{(1)}(t) + \cdots , \tag{7}$$

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{v}^{(1)}(t) + \mathbf{v}^{(2)}(t) + \cdots , \tag{8}$$

where $\mathbf{x}_0$ is an arbitrary initial position, and $\mathbf{v}_0$ is the initial velocity. The following perturbations are required:

$$\mathbf{v}^{(1)}(t) = \frac{q}{m} \int_{-\infty}^{t} \mathrm{d}t' \, \mathbf{E}(t', \mathbf{x}_0 + \mathbf{v}_0 t') , \tag{9}$$

$$\mathbf{X}^{(1)}(t) = \frac{q}{m} \int_{-\infty}^{t} \mathrm{d}t' \int_{\infty}^{t'} \mathrm{d}t'' \, \mathbf{E}(t'', \mathbf{x}_0 + \mathbf{v}_0 t'') , \tag{10}$$

$$\mathbf{v}^{(2)}(t) = \frac{q}{m} \int_{-\infty}^{t} \mathrm{d}t' \, \mathbf{x}^{(1)}(t') \frac{\partial}{\partial \mathbf{x}_0} \mathbf{E}(t', \mathbf{x}_0 + \mathbf{v}_0 t') . \tag{11}$$

On inserting the expression for the electric field implied by (1) and (2) and expressing it in terms of its Fourier transform, the t'-integral in (9) may be performed. The first-order perturbation in the velocity reduces to

$$\mathbf{v}^{(1)}(t) = \frac{q}{m} \int \frac{\mathrm{d}\omega \, \mathrm{d}^3\mathbf{k}}{(2\pi)^4} \exp[i\mathbf{k}\cdot\mathbf{x}_0 - i(\omega - \mathbf{k}\cdot\mathbf{v}_0)t] \frac{\mathbf{k}\Phi(\omega, \mathbf{k})}{(\omega - \mathbf{k}\cdot\mathbf{v}_0) + i0} . \tag{12}$$

The singular integral in (12) is to be evaluated using the Landau prescription, together with the Plemelj formula

$$\frac{1}{(\omega - \mathbf{k}\cdot\mathbf{v}_0 + i0)} = \wp \frac{1}{(\omega - \mathbf{k}\cdot\mathbf{v}_0)} - i\pi\delta(\omega - \mathbf{k}\cdot\mathbf{v}_0) . \tag{13}$$

where $\wp$ denotes the Cauchy principal value. The first-order perturbation (10) in the orbit gives

$$\mathbf{X}^{(1)}(t) = \frac{q}{m} \int \frac{\mathrm{d}\omega \, \mathrm{d}^3\mathbf{k}}{(2\pi)^4} \exp[i\mathbf{k}\cdot\mathbf{x}_0 - i(\omega - \mathbf{k}\cdot\mathbf{v}_0)t] \frac{i\mathbf{k}\Phi(\omega, \mathbf{k})}{(\omega - \mathbf{k}\cdot\mathbf{v}_0 + i0)^2} . \tag{14}$$

The second-order perturbation (11) in the velocity then reduces to

$$\begin{aligned} \mathbf{v}^{(2)}(t) = {} & -\frac{q^2}{m^2} \int \frac{\mathrm{d}\omega \, \mathrm{d}^3\mathbf{k}}{(2\pi)^4} \int \frac{\mathrm{d}\omega' \, \mathrm{d}^3\mathbf{k}'}{(2\pi)^4} \times \\ & \times \exp[i(\mathbf{k} + \mathbf{k}')\cdot\mathbf{x}_0] \exp[-i(\omega - \mathbf{k}\cdot\mathbf{v}_0)t - i(\omega' - \mathbf{k}'\cdot\mathbf{v}_0)t] \times \\ & \times \frac{\mathbf{k}\Phi(\omega, \mathbf{k})\mathbf{k}\cdot\mathbf{k}'\Phi(\omega', \mathbf{k}')}{(\omega - \mathbf{k}\cdot\mathbf{v}_0 + \omega' - \mathbf{k}'\cdot\mathbf{v}_0 + i0)(\omega' - \mathbf{k}'\cdot\mathbf{v}_0 + i0)^2} . \end{aligned} \tag{15}$$

2.3. Fokker–Planck coefficients

The Fokker–Planck equation for the evolution of a distribution $f(\mathbf{v}, t)$ of particles in velocity space is

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) = -\frac{\partial}{\partial v_i}\left(\left\langle\frac{\mathrm{d}v_i}{\mathrm{d}t}\right\rangle f(\mathbf{v}, t)\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i v_j}\left(\left\langle\frac{\mathrm{d}v_i v_j}{\mathrm{d}t}\right\rangle f(\mathbf{v}, t)\right). \tag{16}$$

The Fokker–Planck coefficients $\langle \mathrm{d}v_i/\mathrm{d}t\rangle$ and $\langle \mathrm{d}v_i v_j/\mathrm{d}t\rangle$ may be derived in two steps. The first step is the calculation of the phase averages of the product $v_i^{(1)}(t)v_j^{(1)}(t)$ and $v_i^{(2)}(t)$. These describe the changes resulting from the interaction of one particle with one clump. The second step is the introduction of a statistical theory that enables one to calculate the frequency with which such interactions occur. The simplest statistical assumption is that there is a characteristic time τ between encounters of a given particle with clumps; the Fokker–Planck coefficients are then found by dividing these phase averages by τ.

The first step in the calculation of the coefficient $\langle \mathrm{d}v_i v_j/\mathrm{d}t\rangle$ is the averaging of the product $v_i^{(1)}(t)v_j^{(1)}(t)$ over the phase θ in (4). Let this average be denoted by an overline. After some rearrangements, the resulting average reduces to

$$\overline{v_i^{(1)}(t)v_j^{(1)}(t)} = \frac{q^2A^2}{4m^2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\,k_ik_j' \times$$
$$\times\left\{\frac{\exp[-i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}_0+\mathbf{v}_0t)]}{(\omega_0-\mathbf{k}\cdot\mathbf{v}_0-i0)(\omega_0-\mathbf{k}'\cdot\mathbf{v}_0+i0)}\,G(\mathbf{k}_0-\mathbf{k})G^*(\mathbf{k}_0-\mathbf{k}')+\right.$$
$$\left.+\frac{\exp[i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}_0+\mathbf{v}_0t)]}{(\omega_0-\mathbf{k}\cdot\mathbf{v}_0+i0)(\omega_0-\mathbf{k}'\cdot\mathbf{v}_0-i0)}\,G^*(\mathbf{k}_0-\mathbf{k})G(\mathbf{k}_0-\mathbf{k}')\right\}. \tag{17}$$

The only parts that contribute to the final result are the semi-residues from the resonant terms in (17), which is evaluated using (13). It is convenient to change the variables of integration from $\mathbf{k}$ and $\mathbf{k}'$ to $\mathbf{K}$ and $\Delta\mathbf{k}$ defined by

$$\mathbf{K} := \mathbf{k}_0 - \tfrac{1}{2}(\mathbf{k}+\mathbf{k}'), \qquad \Delta\mathbf{k} := \mathbf{k}-\mathbf{k}'. \tag{18}$$

The resulting expression is

$$\overline{v_i^{(1)}(t)v_j^{(1)}(t)} = \frac{q^2A^4}{2m^2}\int\frac{\mathrm{d}^3\mathbf{K}}{(2\pi)^3}\int\frac{\mathrm{d}^3\Delta\mathbf{k}}{(2\pi)^3}\,(\mathbf{k}_0-\mathbf{K}+\tfrac{1}{2}\Delta\mathbf{k})_i(\mathbf{k}_0-\mathbf{K}-\tfrac{1}{2}\Delta\mathbf{k})_j \times$$
$$\times\,\pi\delta(\omega_0-\mathbf{k}_0\cdot\mathbf{v}_0-\mathbf{K}\cdot\mathbf{v}_0)\,\pi\delta(\Delta\mathbf{k}\cdot\mathbf{v}_0)\times$$
$$\times\,\{\exp[-i\Delta\mathbf{k}\cdot\mathbf{x}_0]G(\mathbf{K}-\tfrac{1}{2}\Delta\mathbf{k})G^*(\mathbf{K}+\tfrac{1}{2}\Delta\mathbf{k})+$$
$$+\exp[i\Delta\mathbf{k}\cdot\mathbf{x}_0]G^*(\mathbf{K}-\tfrac{1}{2}\Delta\mathbf{k})G^*(\mathbf{K}+\tfrac{1}{2}\Delta\mathbf{k})\}. \tag{19}$$

The derivation of the phase average of $v_i^{(2)}(t)$ requires more care because a second-order singularity $(\sim(\omega-\mathbf{k}\cdot\mathbf{v}_0)^{-2})$ appears in evaluating $\mathbf{X}^{(1)}(t)$. The derivation is discussed in the Appendix, where it is argued that in the cases of interest here the result may be approximated by

$$\overline{v_i^{(2)}(t)} = \frac{1}{2} \frac{\partial}{\partial v_j} \overline{v_i^{(1)}(t) v_j^{(1)}(t)}, \tag{20}$$

with $\overline{v_i^{(1)}(t) v_j^{(1)}(t)}$ given by (19).

Rather than formulate a detailed statistical theory for the clumps, we use a simple collisional model. Let the mean free path between encounters with lumps be λ_0. Then the time between encounters with clumps is $\tau = v_0/\lambda_0$. The Fokker–Planck coefficients are then

$$\left\langle \frac{\mathrm{d}v_i v_j}{\mathrm{d}t} \right\rangle = \frac{v}{\lambda_0} \overline{v_i^{(1)}(t) v_j^{(1)}(t)}, \qquad \left\langle \frac{\mathrm{d}v_i}{\mathrm{d}t} \right\rangle = \frac{v}{\lambda_0} \overline{v_i^{(2)}(t)}, \tag{21}$$

where the subscript 0 on v_0 is now omitted.

2.4. Explicit form of the diffusion equation

With the approximate relation (20) and the form (21) for the Fokker–Planck coefficients, the Fokker–Planck equation (16) reduces to the diffusion equation

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) = -\frac{\partial}{\partial v_i} (A_i(\mathbf{v}) f(\mathbf{v}, t)) + \frac{\partial}{\partial v_i} \left(D_{ij}(\mathbf{v}) \frac{\partial}{\partial v_j} f(\mathbf{v}, t) \right), \tag{22}$$

with

$$D_{ij}(\mathbf{v}) = \frac{1}{2} \left\langle \frac{\mathrm{d}v_i v_j}{\mathrm{d}t} \right\rangle, \qquad A_i(\mathbf{v}) = -\frac{v_j}{v} D_{ij}(\mathbf{v}). \tag{23}$$

The term involving $D_{ij}(\mathbf{v})$ in (22) describes diffusion in velocity space, and the term involving $A_i(\mathbf{v})$ describes a systematic slowing down. The latter term is not present in conventional quasi-linear theory. There is a systematic slowing down associated with spontaneous emission, e.g., Melrose (1980), but this is unrelated to the systematic term in (22). The systematic term in (22) arises from the assumption that the time between interactions with clumps is proportional to the speed of the particle. In conventional quasi-linear theory the interaction time is independent of the speed of the particle, as discussed in the Appendix. The systematic term is not important from a semi-quantitative viewpoint, and it is neglected below. (If the systematic term is retained, the two terms in (22) may be written as a single diffusive term by making appropriate choices of a modified distribution function and of an independent variable replacing $\mathbf{v}$.)

3. Special Cases

The general expressions (21) with (19) and (20) for the Fokker–Planck coefficients are rather cumbersome, and it is appropriate to introduce simplifying assumptions.

Considerable simplification occurs if a one-dimensional model is assumed. Then the expressions derived above reduce to a generalization (for $k_0 \neq 0$) of the results derived by Morales and Lee (1974). Another relevant approximation is the large-k_0 limit. This corresponds to the wave number k_0 of the carrier wave being large compared with the typical wave numbers in $G(\mathbf{k})$. In other words, the large-k_0 limit corresponds to the size of the clump being large compared with a wavelength.

3.1. One-dimensional model

A one-dimensional model involves assuming that $\mathbf{v}_0$ and $\mathbf{k}_0$ are parallel and that $g(\mathbf{x}) = g(z)$ depends only on the coordinate along their common direction, assumed to be the z-axis. Then $G(\mathbf{k})$ is of the form

$$G(\mathbf{k}) = 2\pi\delta(k_x)2\pi\delta(k_y)G(k_z)\,, \tag{24}$$

where $G(k_z)$ is the Fourier transform (in one dimension) of $g(z)$.

The integrals in (19) may be performed trivially in the one-dimensional case. The diffusion coefficient in one dimension reduces to

$$D(v) = \frac{q^2A^2\omega_0^2}{2m^2v^3\lambda_0}\,|G(k_0 - \omega_0/v)|^2\,. \tag{25}$$

The case considered by Morales and Lee (1974) corresponds to $k_0 = 0$ in (25). The generalization to a non-zero k_0 affects only the wave number k of the Fourier components of the shape function $g(z)$ that contribute to the resonance, changing from $k = \omega_0/v$ in the case considered by Morales and Lee to the more general case $k = k_0 - \omega_0/v$ in (25).

3.2. The large-k_0 approximation

The case of most interest here is for k_0 much larger than the inverse of the characteristic dimensions of the clump, that is, for a clump whose size is much greater than a wavelength. In this limit (23) with (19) simplifies by approximating the tensorial dependence in (19) by $k_{0i}k_{0j}$, which factor may then be taken outside the integral. This gives

$$D_{ij}(\mathbf{v}) \approx k_{0i}k_{0j}\,\frac{q^2A^4v}{4m^2\lambda_0}\int\frac{\mathrm{d}^3\mathbf{K}}{(2\pi)^3}\int\frac{\mathrm{d}^3\Delta\mathbf{k}}{(2\pi)^3}\,\pi\delta(\omega_0 - \mathbf{k}_0\cdot\mathbf{v}_0 - \mathbf{K}\cdot\mathbf{v}_0)\pi\delta(\Delta\mathbf{k}\cdot\mathbf{v}_0)\,\times$$
$$\times\,\{\exp[-i\Delta\mathbf{k}\cdot\mathbf{x}_0]G(\mathbf{K} - \tfrac{1}{2}\Delta\mathbf{k})G^*(\mathbf{K} + \tfrac{1}{2}\Delta\mathbf{k}) +$$
$$+\,\exp[i\Delta\mathbf{k}\cdot\mathbf{x}_0]G^*(\mathbf{K} - \tfrac{1}{2}\Delta\mathbf{k})G^*(\mathbf{K} + \tfrac{1}{2}\Delta\mathbf{k})\}\,. \tag{26}$$

The expression (26) is the starting point for the discussion in the next section.

4. Models for Clumps

The shape of a clump is described by $g(\mathbf{x})$ in (2) or by its Fourier transform $G(\mathbf{k})$ in (4) and subsequent equations. Explicit evaluation of the Fokker–Planck coefficients (21) with (19) and (20) requires an assumed form for this shape.

4.1. Gaussian clumps

Suppose that the clump has a Gaussian profile, with different characteristic lengths $L_\parallel$ and $L_\perp$ along and perpendicular to $\mathbf{k}_0$, respectively. This corresponds to

$$g(\mathbf{x}) = \frac{1}{(2\pi)^{3/2} L_\perp^2 L_\parallel} \exp\left[- \frac{(x - x_c)^2 + (y - y_c)^2}{2L_\perp^2} - \frac{(z - z_c)^2}{2L_\parallel^2} \right], \tag{27}$$

where $\mathbf{x}_c$ denotes the center of the clump, where $\mathbf{k}_0$ is along the z-axis, and where the normalization condition is

$$\int d^3\mathbf{x}\, g(\mathbf{x}) = 1 . \tag{28}$$

Then one has

$$G(\mathbf{k}) = \exp[- i\mathbf{k}\cdot\mathbf{x}_c] \exp[-\tfrac{1}{2}k_\perp^2 L_\perp^2 - \tfrac{1}{2}k_z^2 L_\parallel^2] , \tag{29}$$

with $k_\perp^2 = k_x^2 + k_y^2$.

In this case the diffusion coefficient (26) in the large-k_0 approximation reduces to

$$D_{ij}(\mathbf{v}) \approx k_{0i}k_{0j} \frac{q^2 A^2}{16\pi^2 m^2 v \lambda_0 L_\perp^4} \exp\left[- \left(\frac{\omega_0 - \mathbf{k}_0\cdot\mathbf{v}}{v}\right)^2 L_\parallel^2 \right] \times$$
$$\times \exp\left[- \frac{(\mathbf{x}_0 - \mathbf{x}_c)_\perp^2}{L_\perp^2} \right]. \tag{30}$$

The factor A^2 may be eliminated in favor of the energy $\mathscr{E}_c$ in the waves. Assuming that the energy density in the waves is twice the mean electrical energy, (1) with (25) in the large-k_0 approximation implies

$$\mathscr{E}_c = \frac{\varepsilon_0 A^2 k_0^2}{16\pi^{3/2} L_\perp^2 L_\parallel} . \tag{31}$$

Then (30) gives, for electrons ($q = -e$, $m = m_e$),

$$D_{ij}(\mathbf{v}) = \frac{k_{0i}k_{0j}}{k_0^2} \frac{\omega_p^2 \mathscr{E}_c}{n_e m_e v} \frac{\pi^{1/2} L_\parallel}{\lambda_0 L_\perp^2} \exp\left[- \left(\frac{\omega_0 - \mathbf{k}_0\cdot\mathbf{v}}{v}\right)^2 L_\parallel^2 \right] \times$$
$$\times \exp\left[- \frac{(\mathbf{x}_0 - \mathbf{x}_c)_\perp^2}{L_\perp^2} \right]. \tag{32}$$

4.2. Arbitrarily shaped clumps

Before proceeding with an analysis based on (32) let us note that a form analogous to (30) may be derived for a clump of arbitrary shape. Although is not necessary to do so, it is convenient for expressing the result to assume that $g(\mathbf{x})$ factorizes:

$$g(\mathbf{x}) = g_z(z - z_c) g_\perp(\mathbf{x}_\perp - \mathbf{x}_{\perp c}) . \tag{33}$$

Let the Fourier transforms of the two factors be $G_z(k_z)$ and $G_\perp(\mathbf{k}_\perp)$, respectively. Then in place of (30) one finds

$$D_{ij}(\mathbf{v}) = \frac{k_{0i} k_{0j}}{k_0^2} \frac{q^2 A^2 k_0^2}{4m^2 v \lambda_0} \, |G_z([\omega_0 - \mathbf{k}_0 \cdot \mathbf{v}]/v) g_\perp(\mathbf{x}_0 - \mathbf{x}_c)_\perp)|^2 . \tag{34}$$

The interpretation of the spatial dependence in (34) is particularly simple in the case of sharply defined clumps. Suppose that $g_\perp$ is non-zero inside the clump and drops to zero at the edge of the clump. Then the spatial dependence of $D_{ij}(\mathbf{v})$ is the same as that of the clump; in particular it is non-zero for particles that pass through the clump and drops to zero for a particle at the edge of the clump or outside the clump.

For an arbitrarily large clump (along the z-axis) the Fourier components are arbitrarily small; that is, $G_z(k_z)$ is sharply peaked around $k_z = 0$. In the limit where $G_z(k_z)$ approaches a δ-function, the only particles that are affected are those satisfying the resonance condition $\omega_0 - \mathbf{k}_0 \cdot \mathbf{v} = 0$. The finite length (along the z-axis) of the clump broadens the resonance by introducing a spread in wave numbers of order the inverse of this length.

4.3. A DISTRIBUTION OF CLUMPS

Consider a distribution of clumps with N clumps in a volume with cross sectional area A_0 and length λ_0 along the z-axis. We have already included λ_0 in estimating the frequency at which a particle encounters clumps. The average effect of the interaction between an individual particle and many clumps may be found by averaging (32) over the cross sectional area A_0. This average involves the integral

$$\frac{1}{A_0} \int \mathrm{d}^2\mathbf{x}_{0\perp} \, \exp\left[- \frac{(\mathbf{x}_0 - \mathbf{x}_c)_\perp^2}{L_\perp^2} \right] = \frac{\pi L_\perp^2}{A_0} . \tag{35}$$

The average diffusion coefficient $\overline{D}_{ij}$ in encounters with many clumps then reduces to

$$\overline{D}_{ij}(\mathbf{v}) = \frac{k_{0i} k_{0j}}{k_0^2} \frac{\pi^{3/2} \omega_p^2}{n_e m_e v} \frac{N \mathscr{E}_c L_\parallel}{\lambda_0 A_0} \exp\left[- \left(\frac{\omega_0 - \mathbf{k}_0 \cdot \mathbf{v}}{v} \right)^2 L_\parallel^2 \right] . \tag{36}$$

The ratio $N \mathscr{E}_c / \lambda_0 A_0$ is the average energy density in the waves, that is the sum of the wave energy in each clump divided by the total volume.

We wish to show that (36) reproduces the familiar one-dimensional quasi-linear diffusion coefficient in an appropriate limit. This form is

$$D(v) = \frac{\pi \omega_p v}{n_e m_e} \, W(v) , \tag{37}$$

where $W(v)\,dv$ is the energy density in the waves in the range dv of phase speeds. In the present model, the energy (31) may be written in the form

$$\mathscr{E}_c = \int dv\,\mathscr{E}_c(v)\,, \qquad \mathscr{E}_c(v) = \frac{\varepsilon_0 A^2 k_0^2 \omega_0}{16\pi^2 L_\perp^2 v^2}\exp[-(\omega_0/v - k_0)^2 L_\parallel^2]\,. \tag{38}$$

The mean energy density $\overline{W}(v)$ per unit phase speed in the waves may be defined by

$$\int dv\,\overline{W}(v) = N\mathscr{E}_c/\lambda_0 A_0\,. \tag{39}$$

Then (36) reduces to

$$\overline{D}_{ij}(\mathbf{v}) = \frac{k_{0i}k_{0j}}{k_0^2}\,\frac{\pi\omega_p^2 v}{n_e m_e \omega_0}\,\overline{W}(v)\,. \tag{40}$$

For Langmuir waves we may approximate ω_0 by ω_p, which approximation has already been made in the derivation of (37).

Thus we find that the quasi-linear diffusion coefficient for an inhomogeneous distribution of Langmuir waves has the same form as for a homogeneous distribution of Langmuir waves. One simply replaces the (uniform) energy density in the waves by the average energy density, where the average is over all the clumps. A condition for this result to apply is that the time-scale over which the quasi-linear relaxation is considered to be long enough for a particle to have passed through many clumps.

Although this result has been derived for the Gaussian case, it is straightforward to derive it for the more general case of an arbitrarily shaped clump. Also, the result has been derived for the particular model (1) for the waves, and this is for a monochromatic carrier wave. For a clump with an intrinsic spread in wave vectors, one may replace the monochromatic carrier wave by a sum of such waves, repeat the analysis for each component in the sum, and hence deduce that the final result follows from (36) by summing over all the carrier waves with an appropriate weighting for each. It then follows that (40) applies in the more general case in which each clump involves waves with a range of wave vectors.

4.4. The uniform clump

If the envelope function $g(z)$ is uniform in some finite region of z, and zero elsewhere, the Fourier transform is a sinc function and the diffusion coefficient is not very different from that given by (32). However, for this envelope the equation of motion for an electron may be solved analytically, so that the use of the perturbation equations (9)–(11) is unnecessary. This has been considered by Tanaka (1987), who applied his results to the radio-frequency heating of a laboratory Tokamak plasma of toroidal shape. A single large-amplitude coherent wave packet is launched by an antenna at the lower-hybrid frequency, and electrons make successive passes through the packet by circulating around the major radius R of the Tokamak, so that the mean free path between encounters is $2\pi R$. The successive orbits of the electrons are decorrelated due to the Coulomb collisions they experience during an orbit. The amplitude E_0 of the electric field of the wave packet is contained in the parameter $\alpha = k_0 q E_0 / m\omega_0^2$, so that the perturbed velocity and position $v^{(1)}$ and $X^{(1)}$ are of order α, and $v^{(2)}$ is of order α^2. When the amplitude of the packet becomes large, particle trapping by the wave potential occurs, and the bounce frequency in this potential is $\sqrt{\alpha}$. This trapping is not taken into account by the quasi-linear theory, since it assumes that the parameter $\sqrt{\alpha}\,\tau_0$ is small, where τ_0 is the transit time of a particle through the packet; $\sqrt{\alpha}\,\tau_0$ being small is the condition for convergence of the perturbation calculation of the particle trajectory.

Using the perturbation approach with the uniform envelope, and integrating the equations of motion directly, Tanaka (1987) found the usual quasi-linear diffusion equation, just as we do here in the general case using Fourier transforms. Tanaka also found the exact nonlinear particle trajectory in the uniform envelope for arbitrary values of $\sqrt{\alpha}\,\tau_0$, and calculated the corresponding diffusion coefficient, finding that the perturbation treatment gives close agreement with the exact result for $\sqrt{\alpha}\,\tau_0 \lesssim 1$. For parameters typical of type III solar radio bursts, with electric field amplitude 40 μV m^{-1}, the bounce time is ≈ 500 s, compared with a transit time of $\approx 10^{-2}$ s for a 100 km wide clump, so that nonlinear effects are negligible.

5. Discussion and Conclusions

We have shown that quasi-linear theory, specifically the diffusion of electrons in velocity space due to resonant interaction with Langmuir waves, applies irrespective of how the waves are distributed in space. The diffusion coefficient is given by the expression which applies in the case of a uniform distribution of waves, but with the spectral energy density of the waves in the homogeneous case replaced by the spatial average of the spectral energy density in the inhomogeneous case. A *proviso* is that the evolution of the electron spectrum be considered only over times long compared with the time between encounters of a typical electron with clumps of waves.

This result justifies the use of quasi-linear theory by Grognard (1975, 1982) in discussing type III solar radio events in the interplanetary medium, where the Langmuir waves are known to be inhomogeneously distributed. Grognard's results indicate that the observed electron spectrum is that resulting from a balance between ballistic effects, tending to steepen the electron distribution due to faster electrons outpacing slower electrons, and quasi-linear diffusion tending to flatten the spectrum. The fact that the waves are distributed very inhomogeneously is not important.

Our generalization of quasi-linear theory to allow the waves to be distributed inhomogeneously has been made under several restrictive assumptions. The particles have been assumed nonrelativistic, the waves to be longitudinal, and the distribution of waves to consist of time-independent clumps on a monochromatic carrier wave. With the possible exception of the time independence of the clumps, none of these assumptions appear to be essential in deriving the generalization. We conjecture that a general treatment will show that quasi-linear theory may be generalized to apply to the average effect of any resonant interaction between particles and waves, where the average is over variations in both the particle and wave distributions in space and time, provided that these variations are on small spatial or temporal scales compared with the variations implied by the average quasi-linear theory.

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Appendix. The Second-Order Change in the Velocity

On inserting (4) and (14) in (11), one obtains

$$\frac{\mathrm{d}\overline{v_i^{(2)}(t)}}{\mathrm{d}t} = \frac{q^2A^2}{4m^2}\frac{\partial}{\partial\omega_0}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\,k_i\mathbf{k}\cdot\mathbf{k}' \times$$
$$\times\left\{\frac{\exp[-i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}_0+\mathbf{v}_0t)]}{i(\omega_0-\mathbf{k}\cdot\mathbf{v}_0-i0)}\,G(\mathbf{k}_0-\mathbf{k})G^*(\mathbf{k}_0-\mathbf{k}')+\text{c.c.}\right\}, \tag{A1}$$

where 'c.c.' refers to the complex conjugate. To see how this expression is to be integrated over time to find $\overline{\mathbf{v}^{(2)}(t)}$, consider the following derivation of $\overline{\mathbf{v}^{(1)}(t)\mathbf{v}^{(1)}(t)}$. Using (6) and (12), one has

$$\frac{\mathrm{d}\overline{v_i^{(1)}(t)v_j^{(1)}(t)}}{\mathrm{d}t} = \frac{q}{m}\left(\overline{E_i(t,\mathbf{x}_0+\mathbf{v}_0t)v_j^{(1)}(t)}+\overline{v_i^{(1)}(t)E_j(t,\mathbf{x}_0+\mathbf{v}_0t)}\right) =$$
$$= \frac{q^2A^2}{4m^2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\,k_ik_j' \times$$
$$\times\left\{\exp[-i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}_0+\mathbf{v}_0t)]\,G(\mathbf{k}_0-\mathbf{k})G^*(\mathbf{k}_0-\mathbf{k}')\times\right.$$
$$\left.\times\left[\frac{1}{i(\omega_0-\mathbf{k}\cdot\mathbf{v}_0-i0)}+\frac{1}{-i(\omega_0-\mathbf{k}'\cdot\mathbf{v}_0+i0)}\right]+\text{c.c.}\right\}. \tag{A2}$$

On integrating (A2) over time and retaining only the semi-residues of the denominators, the result must reproduce (17). Comparison leads to the identification

$$\pi\delta(\omega_0-\mathbf{k}\cdot\mathbf{v}_0)\pi\delta(\mathbf{k}\cdot\mathbf{v}_0-\mathbf{k}'\cdot\mathbf{v}_0)\to\tfrac{1}{2}\pi\delta(\omega_0-\mathbf{k}\cdot\mathbf{v}_0)\pi\delta(\omega_0-\mathbf{k}'\cdot\mathbf{v}_0)$$
$$\to\pi\delta(\omega_0-\mathbf{k}_0\cdot\mathbf{v}_0-\mathbf{K}\cdot\mathbf{v}_0)\pi\delta(\Delta\mathbf{k}\cdot\mathbf{v}_0),$$

where the final expression involves the variables introduced in (18). Using this identification and symmetrizing over the variables of integration $\mathbf{k}$ and $\mathbf{k}'$, the time-integral of

(A1) gives

$$\overline{v_i^{(2)}(t)} = -\frac{1}{2}\frac{q^2A^2}{4m^2}\frac{\partial}{\partial\omega_0}\int\frac{\mathrm{d}^3\mathbf{K}}{(2\pi)^3}\int\frac{\mathrm{d}^3\Delta\mathbf{k}'}{(2\pi)^3}(\mathbf{k}_0+\mathbf{K})_i[(\mathbf{k}_0+\mathbf{K})^2-(\Delta\mathbf{k})^2]\times$$
$$\times\,\pi\delta(\omega_0-\mathbf{k}_0\cdot\mathbf{v}_0-\mathbf{K}\cdot\mathbf{v}_0)\pi\delta(\Delta\mathbf{k}\cdot\mathbf{v}_0)\times$$
$$\times\,(\exp[-i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}_0)]G(\mathbf{k}_0-\mathbf{k})G^*(\mathbf{k}_0-\mathbf{k}')+\text{c.c.})\,. \tag{A3}$$

Comparison of (A3) with (19) shows that one has

$$\overline{v_i^{(2)}(t)} = \frac{\partial}{\partial v_i}\,\overline{v_j^{(1)}(t)v_j^{(1)}(t)} + \overline{\delta v_i^{(2)}(t)}\,, \tag{A4}$$

with

$$\overline{\delta v_i^{(2)}(t)} = \frac{1}{2}\frac{q^2A^2}{4m^2}\int\frac{\mathrm{d}^3\mathbf{K}}{(2\pi)^3}\int\frac{\mathrm{d}^3\Delta\mathbf{k}'}{(2\pi)^3}[(\mathbf{k}_0+\mathbf{K})^2-(\Delta\mathbf{k})^2]\times$$
$$\times\,\pi\delta(\omega_0-\mathbf{k}_0\cdot\mathbf{v}_0-\mathbf{K}\cdot\mathbf{v}_0)\,\frac{\partial}{\partial v_i}\,\pi\delta(\Delta\mathbf{k}\cdot\mathbf{v}_0)\times$$
$$\times\,(\exp[-i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}_0)]G(\mathbf{k}_0-\mathbf{k})G^*(\mathbf{k}_0-\mathbf{k}')+\text{c.c.})\,. \tag{A5}$$

The identity

$$\frac{\partial}{\partial v_i}\,\overline{v_j^{(1)}(t)v_j^{(1)}(t)} = \frac{\partial}{\partial v_j}\,\overline{v_i^{(1)}(t)v_j^{(1)}(t)} \tag{A6}$$

is implied by the fact that the resonant part of $\mathbf{v}^{(1)}$ satisfies

$$\frac{\partial}{\partial v_i}\,v_j^{(1)} = \frac{\partial}{\partial v_j}\,v_i^{(1)}\,.$$

The approximate relation

$$\overline{v_i^{(2)}(t)} \approx \frac{\partial}{\partial v_j}\,\overline{v_i^{(1)}(t)v_j^{(1)}(t)} \tag{A7}$$

then follows provided that the term (A5) may be neglected.

The term (A5) is strictly zero in the one-dimensional case due to the δ-function implying that the component of $\Delta\mathbf{k}$ along $\mathbf{v}_0$ is zero. In the three-dimensional case (A5) is not strictly zero in general, but it is a small correction whenever the wave number k_0 is large compared with the effective value of $|\Delta\mathbf{k}|$, which is of order the inverse of the characteristic size of the clump. Thus the term (A5) is small or zero in the cases of interest here.

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