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ON "HIGH FLYERS" IN FERMI ACCELERATION

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ABSTRACT

The suggestion by Bryant, Powell, & Perry that "high flyers", which result from a statistical excess of head-on versus overtaking collisions in Fermi acceleration, can account for acceleration of cosmic rays to $\sim 10^{20}$ eV is discussed critically. It is shown that the statistical approach of Bryant et al., with some corrections, is equivalent to diffusive Fermi acceleration. The "high flyers" result from diffusion in energy, which complements systematic acceleration by redistributing the particles in energy space, and does not lead to more effective acceleration to higher energies as claimed.

Subject headings: acceleration of particles — cosmic rays — diffusion

1. INTRODUCTION

Bryant, Powell, & Perry (1992) pointed out the possible importance of "high flyers" in Fermi acceleration (Fermi 1949, 1954). In Fermi's original treatment of collisions between cosmic rays and magnetized, interstellar clouds moving with typical speed βc, first-order (in β) energy changes cancel and the net acceleration is attributed to second-order changes and a bias in favor of head-on collisions. (We use "collision" to refer to the reflection of a cosmic ray by a cloud.) Bryant et al. (1992) argued that there must be a distribution of "high flyers" which experience a purely statistical excess of head-on (energy-increasing) collisions versus overtaking (energy-decreasing) collisions. They argued that the "high flyers" are accelerated much more effectively, and suggested that they might account for acceleration to the highest energies observed ($E \sim 10^{20}$ eV), cf. also comments by Zank (1992).

The procedure used by Bryant et al. (1992) to treat the acceleration is a statistical one, which they related to the theory of coin tossing. They omitted systematic effects and considered the probability for N collisions of there being an excess x that are head-on. Here is is shown that when the systematic effects are included, the procedure used by Bryant et al. (1992) produces results that are equivalent to those of a standard one-dimensional treatment of Fermi acceleration with diffusion in energy included (e.g., Davis 1956). The "high flyers" correspond to the diffusive high-energy tail of the distribution of accelerated particles.

The claim by Bryant et al. (1992) that the "high flyers" are accelerated more effectively than in second-order Fermi acceleration is misleading. This claim suggests either that the acceleration of the "high flyers" is intrinsically different in some way from second-order Fermi acceleration or that the acceleration of the "high flyers" occurs faster (or more favorably in some other sense) than has been recognized hitherto. Our argument here is that all effects associated with the "high flyers" are included in diffusive Fermi acceleration, and that no new physical effects or new astrophysical implications can result from the alternative approach adopted by Bryant et al. (1992).

In § 2 we modify a standard treatment of Fermi acceleration to a one-dimensional case, appropriate for comparison with the approach of Bryant et al. (1992). In § 3 we show that, with minor modifications, the procedure used by Bryant et al. (1992) reproduces the results of our one-dimensional treatment of Fermi acceleration. However, the power-law index derived by Bryant et al. (1992) does not reduce to known results for the stationary spectrum when stochastic Fermi acceleration is balanced by energy-independent escape or loss (e.g., Kardashev 1962; Burn 1975; Melrose 1980, p. 114), even when these are modified to the one-dimensional case; this is due to the neglect by Bryant et al. (1992) of the bias favoring head-on collisions. In § 4 we discuss the application to galactic cosmic rays.

2. FERMI ACCELERATION

Fermi acceleration is based on the energy change of a cosmic ray on reflection from a moving cloud. In the rest frame of the cloud there is no energy change, and the energy change in the laboratory frame may be obtained by a Lorentz transformation from that rest frame. In a one-dimensional version of a standard treatment of Fermi acceleration (e.g., Melrose 1980, p. 67) the energy changes for collisions with clouds moving with speed $V \ll c$ are by factors

$$\Delta E_{\pm} = E_{\pm} - E_0 \,, \quad E_{\pm} = E_0 \frac{(1 \pm 2Vv/c^2 + V^2/c^2)}{(1 - V^2/c^2)} \tag{1}$$

for head-on and overtaking collisions, respectively. In equations (1), E_0 is the initial energy of the particle and v is its initial velocity. The rates at which these collisions occur are

$$\nu_{\pm} = \frac{v \pm V}{L} \tag{2}$$

for $v > V$, where L is the characteristic distance between the clouds. The evolution of the number of particles per unit energy, $N(E, t)$, may be described by the Fokker-Planck equation

$$\frac{\partial N(E, t)}{\partial t} = -\frac{\partial}{\partial E}\left[\left\langle \frac{dE}{dt} \right\rangle N(E, t)\right] + \frac{1}{2}\frac{\partial^2}{\partial E^2}\left[\left\langle \frac{dE^2}{dt} \right\rangle N(E, t)\right] + Q(E, t) - \frac{N(E, t)}{T_E} \,, \tag{3}$$

where the final two terms describe injection and energy-

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independent escape or loss. The Fokker-Planck coefficients are determined by

$$\left\langle\frac{dE}{dt}\right\rangle=\sum_{\pm}\nu_{\pm}\Delta E_{\pm}\,,\quad\left\langle\frac{dE^2}{dt}\right\rangle=\sum_{\pm}\nu_{\pm}(\Delta E_{\pm})^2\,. \qquad (4)$$

To lowest order in V/v, equations (1) and (2) in equation (4) give

$$\left\langle\frac{dE}{dt}\right\rangle=\frac{8V^2}{Lc^2}Ev\,,\quad\left\langle\frac{dE^2}{dt}\right\rangle=\frac{8V^2}{Lc^2}\frac{E^2v^3}{c^2}\,, \qquad (5)$$

and the acceleration time T_A is identified by writing $1/T_A = 8V^2/Lc$. (Note that the factor 8 in the expression for $1/T_A$ applies to the one-dimensional treatment, and is replaced by 4/3 in a three-dimensional treatment.)

As shown by Tverskoi (1967), Fermi acceleration can be described by a diffusion equation in momentum space. In one dimension the momentum diffusion equation is

$$\frac{\partial f(p, t)}{\partial t}=\frac{4V^2}{L}\frac{\partial}{\partial p}\left[\frac{p^2}{v}\frac{\partial}{\partial p}f(p, t)\right], \qquad (6)$$

with $p = Ev/c^2$ and $f(p, t) \propto vN(E, t)$. The result (6) follows directly from the foregoing equations.

For comparison with the work of Bryant et al. (1992), we require the solution of equation (3) for an initial injection at $E = E_0$. If one replaces $Q(E, t)$ by $\delta(E - E_0)\delta(t)$, then the solution of equation (3), specifically,

$$N(E, E_0, t)=\frac{1}{E_0}\left(\frac{E}{E_0}\right)^{-\gamma}g(E/E_0, t)\,, \qquad (7)$$

$$\gamma=\frac{1}{2}\left[1+\left(1+\frac{8T_A}{T_E}\right)^{1/2}\right], \qquad (8)$$

where

$$g(E/E_0, t)=\frac{1}{(2\pi t/T_A)^{1/2}}\exp\{-[\ln(E/E_0)-(t/2T_A)\times(1+8T_A/T_E)^{1/2}]^2/(2t/T_A)\}\,. \qquad (9)$$

may be interpreted as the Green's function for equation (3). The solution for an arbitrary injection function, $Q(E, t)$, is then (e.g., Davis 1956; Kardashev 1962):

$$N(E, t)=\int_0^\infty dE'\int_0^t dt'\,\frac{1}{E'}\left(\frac{E}{E'}\right)^{-\gamma}Q(E', t')g(E/E', t-t')\,. \qquad (10)$$

The asymptotic solution for a constant injection, $Q(E, t) = Q_0\,\delta(E - E_0)$, follows from (10) by setting $t \to \infty$ and using $\int_0^\infty dt\, g(E, t) = 1$. This leads to the power-law spectrum, $N(E, E_0, \infty) \propto (E/E_0)^{-\gamma}$, with γ given by equation (8), as may be derived directly from equation (3) with the terms $\partial N(E, t)/\partial t$ and $Q(E, t)$ omitted. Bryant et al. (1992) obtained a power-law index, γ, that differs from equation (8) due to their neglect of the probability bias, as discussed below.

The asymptotic solution derived here is the one-dimensional version of a result that is well-known in three dimensions (e.g., Davis 1956; Kardashev 1962; Burn 1975; Melrose 1980, p. 114). We note that for pure diffusion in momentum space in three dimensions the appropriate power-law index, replacing that given by equation (8) for one dimension, is (cf. Kardashev 1962; Burn 1975)

$$\gamma=\frac{1}{2}\left[-1+\left(9+\frac{16T_A}{T_E}\right)^{1/2}\right]. \qquad (11)$$

3. STATISTICAL APPROACH

Bryant et al. (1992) adopted a statistical approach in treating the "high flyers". To facilitate comparison with Bryant et al. (1992) we write $V = \beta c/2$. After N collisions, with an excess of x head-on (energy-increasing) versus overtaking (energy-decreasing) collisions, the energy of a particle with initial energy E_0 is

$$E=E_0(1+\beta+\tfrac{1}{2}\beta^2)^{(N+x)/2}(1-\beta+\tfrac{1}{2}\beta^2)^{(N-x)/2}\approx E_0e^{\beta x}\,, \qquad (12)$$

where equation (1) is used and where the approximate form applies for $\beta \ll 1$. Note that the exponent βx is correct to second-order in β. Thus, when the second-order terms are included, as in equation (12), the assumption $\beta^2 N \ll 2$ made by Bryant et al. (1992) at this point is unnecessary. From equation (2) the probabilities of energy-increasing and energy-decreasing collisions are $(1 \pm \beta/2)/2$, respectively. The probability of an excess x of energy-increasing collisions then, in a total of N collisions, is

$$P(x\,|\,N)=\frac{N!}{[(N+x)/2]!\,[(N-x)/2]!}\times\left\{\frac{[1+\beta/2]}{2}\right\}^{(N+x)/2}\left\{\frac{[1-\beta/2]}{2}\right\}^{(N-x)/2}\approx\frac{2}{(2\pi N)^{1/2}}\exp\left\{-\frac{[x-\beta N/2]^2}{2N}\right\}, \qquad (13)$$

where the approximate form applies for $N \gg 1$, $\beta \ll 1$, $x/N \ll 1$. The probability (13) includes a bias, with $x = \beta N/2$ being the average excess expected due to the higher probability of head-on versus overtaking collisions. The bias implies systematic acceleration with $E = E_0 \exp[\beta^2 N/2]$, where equation (12) is used; the factor 1/2 must be present in a one-dimensional treatment that corresponds to diffusion in p, as must be the case according to Tverskoi (1967).

Bryant et al. (1992) defined a probability $P(x, N_0)$ which includes Fermi's probability of escape: $P(x, N_0)$ is defined by multiplying the probability (13) by $N_0^{-1}\exp(-N/N_0)$, where $N_0 = T_E/T_S$ is the number of collisions in the mean escape (or loss) time, T_E, and integrating over N. They then identified the energy spectrum as $N(E, E_0, t) \propto P(x, N_0)dx/dE$, with x and E related by equation (12). Using equations (12) and (13) we find

$$N(E, E_0, t)\propto P(x, N_0)\frac{dx}{dE}=\frac{2}{\beta E(2\pi N)^{1/2}}\frac{\exp(-N/N_0)}{N_0}\times\exp\left\{-\frac{[\ln(E/E_0)-\beta^2N/2]^2}{2\beta^2N}\right\}, \qquad (14)$$

with $N = t/T_S$, $N_0 = T_E/T_S$. Apart from notation, equation (14) is equivalent to (7)–(9) with $T_A = T_S/\beta^2$, establishing the equivalence of the two approaches. Thus equation (14) may be interpreted either as the solution of equation (3) for an initial injection $Q(E, t) \propto \delta(E - E_0)\delta(t)$, or as the probability per unit energy that a particle has an energy E after N collisions.

The equivalence of equations (14) and (7)–(9) implies that the solution for a constant injection spectrum $Q(E, t) \propto \delta(E - E_0)$ balanced by losses is obtained as in equation (10), that is, it follows from equation (14) by integrating over N, in accord with the procedure adopted by Bryant et al. (1992). However, Bryant et al. (1992) omitted the bias in equation (14), and as a consequence the value of the spectral index, γ, that they obtained does not agree with the value (8), nor with the value

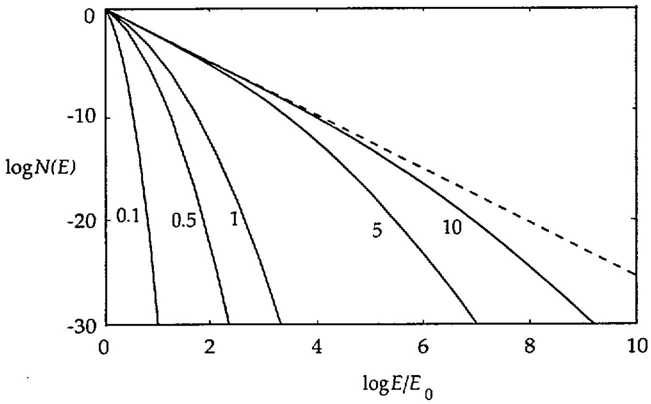

FIG. 1.—Energy spectrum for a constant injection turned on at an initial time, given by integrating eqn. (14) from 0 to N, is plotted for $N_0 \beta^2 = 1/2$ and for the values of N/N_0 given by the labels on the curves. The dashed line is the asymptotic power-law solution with index, from (8), $\gamma = [1 + (1 + 8/\beta^2 N_0)^{1/2}]/2$.

$\gamma = (1 + 2/\beta^2 N_0)^{1/2}$ obtained if the factor of $\frac{1}{2}$ in the one-dimensional version of systematic acceleration is omitted.

As an aside it is interesting to note that the systematic term in the Fokker-Planck equation (cf. eq. [4]) has two equal contributions, from the second-order energy changes in equation (1) and from the probability bias in equation (2), and yet the systematic acceleration in the statistical approach arises entirely from the bias in equation (13), with the second order energy changes canceling in equation (12). This may be attributed to the fact that in the Fokker-Planck approach all changes are made relative to a given initial energy, whereas in the statistical approach the change is made relative to the current energy which is continuously updated.

4. DISCUSSION

Bryant et al. (1992) disputed the accepted conclusion that Fermi acceleration is too slow to account for the highest energy cosmic rays ($\sim 10^{20}$ eV), suggesting that this conclusion results from retaining only the systematic terms in the acceleration. They argued that "high flyers" in the wings of the statistical distribution are accelerated much more effectively. Our analysis does not support this suggestion. We argue (a) that the statistical approach of § 3 is equivalent to the Fokker-Planck approach of § 2, and (b) that the "high flyers" correspond to the diffusive high-energy tail. Moreover, in the Fokker-Planck approach it is clear that the acceleration is due to the systematic term in equation (3) and not to the diffusive term in equation (3), so that, contrary to the claim by Bryant et al. (1992), the "high flyers" can play no important role in determining the effectiveness of the acceleration.

The suggestion that the long-standing problem of the acceleration of cosmic rays with $E \sim 10^{20}$ eV (e.g., Hillas 1984) might be solved by appealing to the "high flyers" is misleading in that it suggests that acceleration can occur faster than the characteristic time determined by systematic acceleration. We now argue that in the discussion of acceleration to $E \sim 10^{20}$ eV by Bryant et al. (1992) the high energies are achieved by long acceleration times, and not by anomalously rapid acceleration. Consider the asymptotic spectrum, $N(E, E_0, \infty)$, for a constant injection at $E = E_0$, found by setting $t \to \infty$ in equations (7)–(9), or by integrating equation (14) over N $(0 \leq N < \infty)$. We illustrate the approach to this asymptotic solution in Figure 1. Clearly, given an infinite time the asymptotic power-law spectrum can be set up to an arbitrarily high energy. Following Bryant et al. (1992), consider the number of collisions required for an increase in energy by 10 decades, denoted N_{10}. For the asymptotic spectrum to bet set up over about 10 decades in energy requires $\exp[N_{10} \beta^2/2] = 10^{10}$, or $N_{10} \approx 46/\beta^2$. With $N_0 \beta^2 = 1/2$ assumed by Bryant et al. (1992) this implies $N_{10}/N_0 \approx 92$, that is, acceleration over 10 decades requires that the particles reside in the galaxy for nearly 100 times longer than the mean escape time. It is this long residence time and not the rapid acceleration ("high flying") that allows the high energies to be reached. Plausible cloud speeds give β in the range 10^{-3}–10^{-4}, and a collision time of order $T_S \approx 300$ yr is appropriate for clouds a distance ≈ 100 pc apart. Then $N_{10} \approx 46/\beta^2$ corresponds to a time $N_{10} T_S \approx 10^{10}$–$10^{12}$ yr, of order the Hubble time, for the acceleration to be effective. Extreme values of other parameters, such as the magnetic field in the cloud, are required for the acceleration to operate to such high energies. However, we do not wish to emphasize the practical arguments that implausible values of the relevant parameters are required for Fermi acceleration to be effective for Galactic cosmic rays, especially those of highest energy. The point emphasized here is that an appeal to the "high flyers" adds nothing new to the argument, because they are already included in standard treatments of Fermi acceleration. More specifically we argue that an appeal to the "high flyers" is misleading because acceleration to higher energies occurs not because the acceleration of a few "high flyers" is more rapid, as Bryant et al. (1992) appear to claim, but because the statistical probability of escape or loss allows a favored few particles to have a much longer than average residence time, thereby allowing the acceleration to operate for a much longer time than for an average particle.

The basic result of the present discussion is that the "high flyers" in a statistical approach are just the diffusive high-energy tail in the conventional stochastic form of Fermi acceleration. While not helping to resolve the problem of the acceleration of the highest energy cosmic rays, the statistical approach introduced by Bryant et al. (1992) is a useful alternative to the standard Fokker-Planck treatment of Fermi acceleration, and it provides new insights into the details of the process.

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