

# Uniqueness of the Alfvén singularity in a relativistic pulsar wind

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## ABSTRACT

A splitting of the Alfvén surface in an axisymmetric wind due to relativistic effects, suggested by Okamoto & Ardavan, does not occur. It is shown how spurious singularities can arise in the theory of relativistic winds, and it is argued that only one singularity is genuine. This singularity is shown to correspond to a matching between the poloidal flow velocity and the poloidal component of the phase velocity of Alfvén waves, provided that the effect of the non-zero electric field and charge density is taken into account in deriving the dispersion relation of the Alfvén waves.

**Key words:** MHD – relativity – stars: mass-loss – pulsars: general.

## 1 INTRODUCTION

In a non-relativistic, axisymmetric, stellar wind, the Alfvén singularity occurs where the poloidal flow velocity,  $v_p$ , is equal to the poloidal Alfvén velocity,  $v_{Ap} = B_p / \sqrt{4\pi\rho}$  (cf. Weber & Davis 1967; Mestel 1968), where  $B_p$  is the poloidal magnetic field and  $\rho$  is the density. It has been argued that, in a wind that is relativistic, axisymmetric, gravitation-free and cold, for example in a pulsar wind, the conventional Alfvén singularity is split into an *Alfvén singularity*, which is near the light cylinder, and a *pure Alfvén singularity*, which is further away from the light cylinder (Okamoto 1978; Ardavan 1979, and references therein). The pure Alfvén singularity occurs at  $v_p = v_{Ap}/\gamma$ , where  $\gamma$  is the Lorentz factor of the flow, and  $v_{Ap} = B_p / \sqrt{4\pi\rho_0}$  ( $\rho_0$  is the proper density) acts like the ‘Alfvén’ speed similar to the non-relativistic case. The Alfvén singularity, which Michel (1969) referred to as the Dicke–Alfvén point, occurs at  $v_p = (v_{Ap}/\gamma)\sqrt{1 - \alpha^2\varpi^2/c^2}$ , where  $\alpha$  (cf. Section 2) may be interpreted as the angular speed of rotation of the star, and  $\varpi$  is the radial cylindrical coordinate. Okamoto (1978) suggested that ‘the Alfvén cylinder has bifurcated from the pure Alfvénic cylinder by a relativistic effect.’ In addition to these two singularities, Ardavan (1979) argued for the existence of a third singularity at  $v_p = (v_{Ap}/\gamma)\sqrt{1 - \alpha\varpi v_t/c^2}$ , where  $v_t$  is the toroidal component of the wind velocity. However, the suggestion that the Alfvén singularity is split by relativistic effects is not supported by other authors (e.g. Michel 1969; Kennel, Fujimura & Okamoto 1983; Camenzind 1986). In particular, Camenzind’s (1986) analysis implies that there is only one singularity – the Alfvén singularity. Here we present two

complementary arguments that there is indeed only one Alfvén singularity in the presence of relativistic effects, and we conclude that the pure Alfvén singularity is spurious and of no physical significance.

In Section 2, we briefly summarize the arguments leading to the suggested bifurcation of the singularity. In Section 3, we show how some mathematical treatments of a relativistic wind can lead to spurious singularities, and also that the pure Alfvén singularity is spurious. In Section 4, we argue that there is no splitting of the Alfvén wave mode due to relativistic effects, and we show that the condition that the poloidal flow velocity match the poloidal Alfvén velocity defines the same Alfvén surface as the condition for the Alfvén singularity

## 2 SUGGESTED ALFVÉN SINGULARITIES

The original argument for the splitting of the Alfvén surface due to relativistic effects is based on singularities that appear in the wind equations. In summarizing the argument here, we follow Okamoto (1978), cf. also the summaries by Mestel (1968, 1972).

By combining Maxwell’s equations and the hydrodynamical equations, the properties of an axisymmetric, steady-state wind may be expressed in terms of four integrals which are constant along a poloidal field line. Separating the flow velocity,  $\mathbf{v}$ , and the magnetic field,  $\mathbf{B}$ , into poloidal and toroidal components, one finds

$$v_p = \kappa B_p, \quad (1)$$

$$v_t = \alpha\varpi + \kappa B_t, \quad (2)$$

$$\kappa\rho = \eta, \quad (3)$$

where  $\kappa$  is a scalar function of position, and  $\alpha$  and  $\eta$  are two such constants of integration. In the wind theory, one can

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assume that the wind originates from the stellar surface with a negligible initial poloidal velocity, so that to a good approximation  $\alpha$  is equal to the stellar surface angular velocity. The other two constants,  $\beta$  and  $\mu$ , arise from the angular momentum and energy integrals, respectively:

$$-\frac{\varpi B_t}{4\pi} + \rho\kappa\varpi\gamma v_t = -\frac{\beta}{4\pi}, \quad (4)$$

$$\gamma \left( 1 - \frac{\alpha\varpi v_t}{c^2} \right) = \mu, \quad (5)$$

where  $\rho = \rho_0 \gamma$  relates the density to the proper density.

Okamoto (1987) noted that equations (1)–(3) and the torque integral (4) can be written in a form similar to their non-relativistic counterparts (cf. Mestel 1968):

$$\varpi B_t = \frac{\beta + 4\pi\rho\gamma\kappa\alpha\varpi^2}{1 - 4\pi\rho\gamma\kappa^2}, \quad (6)$$

$$\varpi v_t = \frac{\alpha\varpi^2 + \beta\kappa}{1 - 4\pi\rho\gamma\kappa^2}. \quad (7)$$

Okamoto then argued that (6) and (7) imply the existence of the ‘pure Alfvénic singularity’ at  $4\pi\rho\gamma\kappa^2 = 1$ . He also argued that the energy equation (5), which reduces to

$$\gamma = \mu \frac{[1 - (1 - \alpha\beta/4\pi\eta\mu c^2)4\pi\rho\gamma\kappa^2]}{[1 - 4\pi\rho\gamma\kappa^2 - \alpha^2\varpi^2/c^2]}, \quad (8)$$

implies the existence of the ‘Alfvénic singularity’ at  $4\pi\rho\gamma\kappa^2 = 1 - \alpha^2\varpi^2/c^2$ .

Before discussing the suggested relativistic splitting of the Alfvénic surface, it is appropriate to comment briefly on the non-relativistic formulation. In a non-relativistic wind theory (cf. Mestel 1992),  $\gamma$  is put equal to unity in (4), and in the energy integral (5); then, because the leading terms in an expansion in  $1/c^2$  cancel, one retains the next-order terms  $\gamma - 1 = (v_p^2 + v_t^2)/2c^2 + \dots$ , with  $(\mu - 1)c^2$  remaining finite in the limit  $c \rightarrow \infty$ . The standard procedure involves first decoupling the toroidal and poloidal dynamics. Equations (6) and (7) depend only on  $\rho$  (monotonically decreasing outwards in any reasonable model) and  $\varpi$ . Satisfaction of the finiteness of (4) then fixes  $\beta$ . In the energy integral (5), which normally includes thermal and gravitational terms as well as the centrifugal slingshot term, as in the stellar wind theory, there are a singularity and two critical points. These correspond to where  $v_p$  equals, respectively, the *poloidal* Alfvénic speed – the Alfvén singularity – and the slow and fast magnetosonic speeds (Belcher & MacGregor 1976). (The slow magnetosonic point is absent for a cold wind, as in the present paper.) However, it is only the condition of smooth flow through the *magnetosonic* points that actually constrains the flow, enabling all the parameters to be fixed. The condition of non-singular behaviour at the Alfvénic point is automatically satisfied (Sakurai 1985), once the finiteness of  $v_t$  and  $B_t$  is ensured by the appropriate choice of  $\beta$ . There is no splitting of the singularity in the non-relativistic case:  $\gamma \equiv 1$  and  $c \rightarrow \infty$  imply that the pure Alfvén singularity and the Alfvén singularity coincide.

### 3 SINGULARITY, GENUINE OR SPURIOUS?

In a physical theory such as a wind theory, a singularity (mathematically, a removable singularity) is said to arise when the solution for a physical quantity  $f$  is of the form

$$f = \frac{A(x, y, a_i)}{B(x, y, a_i)}, \quad (9)$$

and  $B(x, y, a_i)$  vanishes, where  $A$  and  $B$  are arbitrary functions of  $x$ ,  $y$  and  $a_i$ . Here  $x$  and  $y$  are independent variables, and the  $a_i$  ( $i = 1, 2, \dots, n$ ) are  $n$  parameters corresponding to, for example, the various integral constants of MHD pulsar wind equations. A physical quantity must be finite, so that one requires  $A(x, y, a_i) = 0$  simultaneously, with  $B(x, y, a_i) = 0$ . Let  $x = x_c$ ,  $y = y_c$  be the solution of  $B(x, y, a_i) = 0$ ,  $A(x, y, a_i) = 0$  for  $x$  and  $y$  as functions of the  $a_i$ . Assuming that one of  $x$  or  $y$  is related to distance, this solution determines not only the location of the singular point (or the singular surface in general), but also the values of the physical variables at this surface in terms of the parameters  $a_i$ . Even though the value of  $f$  in (9) is not fixed at  $x_c$  and  $y_c$ , the equations  $A = 0$  and  $B = 0$  provide a constraint on the parameters  $a_i$ . In other words, the requirement that the solution pass through the singular point implies that the  $a_i$  are related to the values of the physical variables at the singular point.

We refer to a singularity, at  $(x_c, y_c)$ , that appears in the mathematical form (9) as *genuine* if  $A(x, y, a_i)$  and  $B(x, y, a_i)$  have no common factor that vanishes at  $(x_c, y_c)$ , and provided that  $B(x, y, a_i)$  has a simple zero there. We call a singularity *spurious* when there is a common factor that vanishes at  $(x_c, y_c)$  in (9). Such a common factor does not affect the regularity of  $f$  itself at  $(x_c, y_c)$ , which is a normal point of  $f$ .

In the discussion of singularities in a relativistic wind, the  $a_i$  correspond to the four constants  $\alpha$ ,  $\beta$ ,  $\eta$  and  $\mu$ . We choose as our independent variables

$$\begin{aligned} x &= \alpha^2\varpi^2/c^2, \\ y &= 4\pi\rho\gamma\kappa^2. \end{aligned} \quad (10)$$

All other variables can be expressed in terms of these independent variables. At the Alfvén singularity the simultaneous vanishing of the denominator and numerator in (8) determines the values  $x = x_A$  and  $y = y_A$  in terms of the four constants:

$$y_A = 1 - x_A = \frac{1}{1 - \alpha\beta/4\pi\eta\mu c^2}. \quad (11)$$

Thus this singularity is genuine. The parameters  $\alpha$ ,  $\beta$ ,  $\eta$  and  $\mu$  are therefore constrained, because they are related to the physical value  $y_A$  at  $x_A$ .

According to Okamoto (1978), the pure Alfvén singularity occurs where the denominators on the right-hand sides of (6) and (7) vanish, that is, at  $y = 1$ . To show that the pure Alfvén singularity is spurious, we use (4), (5) and (8) to re-express the numerators in (6) and (7) in terms of our independent variables. One finds

$$\beta + 4\pi\rho\gamma\kappa\alpha\omega^2 = \beta \frac{(1-x-y_{\text{A}})(1-y)}{(1-y_{\text{A}})(1-x-y)},$$

$$\alpha\omega^2 + \beta\kappa = \frac{c^2}{\alpha} \frac{[y_{\text{A}}(x+y)-y](1-y)}{y_{\text{A}}-y}. \quad (12)$$

Both numerators are proportional to  $1-y$ , and these factors cancel the denominators  $1-y$  in (6) and (7). Thus the pure Alfvén singularity is spurious. In fact, both  $\omega B_{\text{t}}$  and  $\omega v_{\text{t}}$  have genuine singularities, but these are at the Alfvén point and are regularized by (11).

An additional argument that the pure Alfvén singularity is of no physical significance follows by assuming the contrary, that is, assuming the pure Alfvén singularity to be real, and exploring the consequences of this assumption. On regularizing (6) or (7) at  $y=1$ , or at the pure Alfvén point, one obtains  $-\beta/4\pi = \gamma_{\text{PA}}\eta\alpha\omega_{\text{PA}}^2$ . Unlike the regularization of (8) at the Alfvén singularity, this equation does not determine  $\beta$  in terms of other integral constants, because  $\gamma_{\text{PA}}$  is unknown, and an additional equation is required to fix  $\beta$  uniquely. One might expect (8) to provide such an additional equation, but on substituting  $y=1$  into (8) one obtains the same equation,  $-\beta/4\pi = \gamma_{\text{PA}}\eta\alpha\omega_{\text{PA}}^2$ . It follows that the assumption that the pure Alfvén singularity is real neither defines a unique surface in the wind, nor provides a unique relation between the integral constants. In this sense, the pure Alfvén point is of no physical significance.

Ardavan (1979) also argued for a critical point at  $1-y-\alpha\omega v_{\text{t}}/c^2=0$  by considering the relation between the parameters  $\kappa \propto y/\gamma$ ,  $\kappa\gamma \propto y$  and  $\gamma$ . As the foregoing results show, there is only one genuine singularity in  $\gamma$ , which is at the Alfvén point, and hence, whatever the significance of Ardavan's critical point, it cannot be a genuine singularity in  $\gamma$ . In the Appendix, we show how an infinite class of spurious singularities can be generated, and also, as illustrative examples, how spurious singularities can be generated at the pure Alfvén singularity and Ardavan's critical point, starting from the expression (8) for  $\gamma$ .

#### 4 WAVE-SINGULARITY ANALOGY

The Alfvén surface in a non-relativistic wind can be interpreted as the surface at which inwardly propagating Alfvén waves are stationary with respect to the star, corresponding to the poloidal component of the Alfvén velocity balancing the poloidal component of the wind velocity. We show here that in the relativistic case this condition defines a unique 'Alfvén surface', and does not imply the 'pure Alfvén surface'. In treating the waves' properties, the plasma is assumed to be locally homogeneous, which is valid for wavelengths much smaller than the characteristic distance,  $\sim \omega$ , over which the properties of the wind change. One is then free to consider the waves in an inertial frame in which the plasma is locally at rest; this is referred to as the local rest frame of the plasma.

The dispersion relation for Alfvén waves in a relativistic wind may be derived in two ways. The direct approach is to derive the wave equation by linearizing the MHD equations for the relativistic wind, and to seek plane-wave solutions of the resulting equations. In the relativistic generalization, it is important to include two additional ingredients. One is the

displacement current: it is well known that inclusion of the displacement current in the local rest frame of a plasma leads to the Alfvén speed,  $v_{\text{A}} = B/(4\pi\rho)^{1/2}$ , being replaced by  $v_0 = v_{\text{A}}/(1+v_{\text{A}}^2/c^2)^{1/2}$  in the dispersion relations for the MHD modes. The other ingredient is an additional force term,  $\rho_e \mathbf{E}$ , which appears in the equation of fluid motion, with the electric field,  $\mathbf{E} = -\mathbf{v} \times \mathbf{B}/c$ , and the charge density,  $\rho_e = \nabla \cdot \mathbf{E}/4\pi$ , both being non-zero in the wind. (In a non-relativistic wind, both  $\mathbf{E}$  and  $\rho_e$  are non-zero, but they contribute only a relativistic correction to the dispersion relation.) The derivation of the dispersion relations for the Alfvén and magneto-sonic modes is then straightforward. The alternative way of deriving the dispersion relations is to start from the dispersion relations in the local rest frame of the plasma, including the effect of the displacement current as outlined above, and to make a Lorentz transformation to the laboratory frame, which is moving with velocity  $-\mathbf{v}$  relative to the local rest frame of the plasma. These two approaches lead to the same dispersion relations. Here we outline the latter approach.

Let quantities in the local rest frame of the plasma be denoted by primes, and quantities in the laboratory frame be unprimed. The dispersion relation for Alfvén waves in the local rest frame can be written in the form

$$\omega'^2 \left( 1 + \frac{B'^2}{4\pi\rho_0 c^2} \right) = \frac{(\mathbf{k}' \cdot \mathbf{B}')^2}{4\pi\rho_0}, \quad (13)$$

where  $\rho_0$  may be interpreted as the proper density of the plasma. The Lorentz transformation leads to the following relations between quantities in the two frames:

$$\omega' = \gamma(\omega - \mathbf{k} \cdot \mathbf{v}),$$

$$B'^2 = B^2 - \frac{|\mathbf{v} \times \mathbf{B}|^2}{c^2},$$

$$\mathbf{k}' \cdot \mathbf{B}' = \frac{\mathbf{k} \cdot \mathbf{B}}{\gamma} - \gamma(\omega - \mathbf{k} \cdot \mathbf{v}) \frac{\mathbf{v} \cdot \mathbf{B}}{c^2}. \quad (14)$$

The dispersion relation for Alfvén waves in the laboratory frame then reduces to

$$\gamma^2(\omega - \mathbf{k} \cdot \mathbf{v})^2 \left[ 1 + \frac{1}{4\pi\rho_0 c^2} \left( B^2 - \frac{|\mathbf{v} \times \mathbf{B}|^2}{c^2} \right) \right]$$

$$= \frac{1}{4\pi\rho_0} \left[ \frac{\mathbf{k} \cdot \mathbf{B}}{\gamma} - \gamma(\omega - \mathbf{k} \cdot \mathbf{v}) \frac{\mathbf{v} \cdot \mathbf{B}}{c^2} \right]^2. \quad (15)$$

The condition for a standing wave, defining the Alfvén surface, is that the poloidal component of the flow velocity match the poloidal component of the phase velocity of Alfvén waves. This corresponds to  $\omega=0$ ,  $\mathbf{k} \cdot \mathbf{v} = kv_{\text{p}}$ ,  $\mathbf{k} \cdot \mathbf{B} = kB_{\text{p}}$  in (15). On writing  $B^2 = B_{\text{p}}^2 + B_{\text{t}}^2$ ,  $\mathbf{v} \cdot \mathbf{B} = v_{\text{p}}B_{\text{p}} + v_{\text{t}}B_{\text{t}}$  and so on in (15) and, using (3), after some algebra one finds that this condition requires  $4\pi\rho\gamma\kappa^2 = 1 - \alpha^2\omega^2/c^2$ . It follows that the condition for a standing Alfvén wave in the flow reproduces the condition for the Alfvén singularity, as expected. Note that there is only

one Alfvén mode and only one Alfvén surface defined by this standing-wave condition.

## 5 DISCUSSION AND CONCLUSION

We present two arguments that there is only one Alfvén singularity, and that it is not split by relativistic effects.

The first argument involves three steps. The first step is to define a singularity to be (a) genuine when a physical quantity is expressed in terms of two independent variables ( $x = \alpha^2 \omega^2 / c^2$ ,  $y = 4\pi\rho\gamma\kappa^2$ ) and the denominator has a zero at a point where the numerator is not identically zero, and (b) spurious when the numerator and denominator have a common factor so that the vanishing of the denominator only appears to give a singularity. The second step is to show that the Alfvén singularity at  $v_p = (v_{Ap}/\gamma)\sqrt{1 - \alpha^2\omega^2/c^2}$  is genuine. The third step is to show that the ‘pure Alfvén singularity’ at  $v_p = v_{Ap}/\gamma$  is spurious. The argument is that, after (8) is regularized at the Alfvén singularity, the value of  $\beta$  so determined implies that the numerators in (6) and (7) vanish at  $y=1$ , so that the apparent singularity from the vanishing of the denominators at  $y=1$  does not occur because the factors  $(1-y)$  in the numerators and denominators cancel. A complementary argument against any relevance for the pure Alfvén singularity involves assuming, on the contrary, the singularity to be real and requiring that (6) or (7) be non-singular at  $y=1$ . However, regularization of (6) or (7), even when combined with (8) at  $y=1$ , does not determine  $\beta$  uniquely in terms of other integral constants, and nor does it determine a unique surface in the wind. In this sense, the pure Alfvén singular surface does not exist, and there is no pure Alfvén singularity.

The second argument involves determining the dispersion relation for Alfvén waves in a relativistic wind, which requires the inclusion of the displacement current and the effects of the non-zero electric field and charge density in the wind. We show that the Alfvén surface corresponds to the surface where the poloidal flow velocity,  $v_p$ , matches the poloidal component of the phase velocity of Alfvén waves,  $(v_{Ap}/\gamma)\sqrt{1 - \alpha^2\omega^2/c^2}$ .

The derivation of the dispersion relation for Alfvén waves given here provides an interpretation of the factor  $\sqrt{1 - \alpha^2\omega^2/c^2}$  that appears in the expression for the Alfvén surface. The Alfvén speed evaluated in the local rest frame of the plasma is  $v_A = B'/\sqrt{4\pi\rho_0}$ , where  $B'$  is the magnitude of the magnetic field in this frame and the proper density,  $\rho_0$ , is the actual density in this frame. In the Lorentz transformation from the local rest frame of the plasma to the laboratory frame,  $v_A$  is treated as a parameter, whose numerical value is the same in all frames. The magnetic field  $B$  and electric field  $E$  in the laboratory frame are related to  $B'$  by  $B'^2 = B^2 - E^2$ , which gives  $B'^2 = B_t^2 + B_p^2(1 - \alpha^2\omega^2/c^2)$ . In the non-relativistic case (here  $\alpha^2\omega^2/c^2 \rightarrow 0$ ) only the poloidal component,  $B_p$ , appears in the dispersion relation for Alfvén waves propagating in the poloidal direction, and in the relativistic generalization this is replaced by an effective poloidal component  $B_p\sqrt{1 - \alpha^2\omega^2/c^2}$ , found by evaluating  $B'$  for  $B_t = 0$ . Thus the factor  $\sqrt{1 - \alpha^2\omega^2/c^2}$  may be attributed to the appropriate magnetic field in the definition of the Alfvén speed being the Lorentz invariant quantity  $B'$ , rather than the actual magnetic field in the laboratory frame.

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## REFERENCES

- Ardavan H., 1979, MNRAS, 189, 397  
 Belcher J. W., MacGregor K. B., 1976, ApJ, 210, 409  
 Camenzind M., 1986, A&A, 162, 32  
 Kennel C. F., Fujimura F. S., Okamoto I., 1983, J. Astrophys. Geophys. Fluid Dyn., 26, 147  
 Mestel L., 1968, MNRAS, 138, 359  
 Mestel L., 1992, Proc. Astron. Soc. Aust., 10, 3  
 Michel F. C., 1969, ApJ, 158, 727  
 Okamoto I., 1978, MNRAS, 185, 69  
 Sakurai T., 1985, A&A, 152, 121  
 Weber E. J., Davis L., 1967, ApJ, 148, 217

## APPENDIX A

Spurious singularities can arise when the numerator on the right-hand side of (9) is a function of  $f$ . To explore the effects of such a dependence, we note that for an arbitrary function  $C(x, y, a_i)$  one has

$$\frac{A(x, y, a_i) + C(x, y, a_i)f}{B(x, y, a_i) + C(x, y, a_i)} = \frac{A(x, y, a_i)}{B(x, y, a_i)}. \quad (\text{A1})$$

Using (A1), one may rewrite (9) in the form

$$f = \frac{D(x, y, f, a_i)}{E(x, y, a_i)},$$

$$D(x, y, f, a_i) = A(x, y, a_i) + C(x, y, a_i)f,$$

$$E(x, y, a_i) = B(x, y, a_i) + C(x, y, a_i). \quad (\text{A2})$$

Since  $C(x, y, a_i)$  is an arbitrary function, by choosing different functions  $C(x, y, a_i)$  and setting  $E(x, y, a_i) = 0$ , one can apparently generate an infinite number of singularities. For a specific choice of  $C(x, y, a_i)$ , let  $E(x, y, a_i) = 0$  imply that a singularity occurs at the point  $x = x_1 \neq x_c$ ,  $y = y_1 \neq y_c$ . Now consider the value of  $D(x, y, f, a_i)$  at this point. Combining the two expressions (9) and (A2) for  $f$ , one finds

$$D(x, y, f, a_i) = \frac{A(a, y, a_i)E(x, y, a_i)}{B(x, y, a_i)} = \frac{A(x, y, a_i)E(x, y, a_i)}{E(x, y, a_i) - C(x, y, a_i)}. \quad (\text{A3})$$

It follows that in (A3), with  $C(x, y, a_i) \neq 0$ , the numerator  $D(x, y, f, a_i)$  vanishes whenever the denominator  $E(x, y, a_i)$  vanishes. The numerator and denominator share a common factor, namely  $E(x, y, a_i)$  itself, and it follows that all these new singularities are spurious.

Two relevant examples involve generating spurious singularities at the pure Alfvén point (at  $1 - y = 0$ ) and at Ardavan’s additional critical point (at  $1 - y - \alpha\omega v_i/c^2 = 0$ ) by starting from (8) and using (A1) and (A2). To rearrange (8) in a form such that the denominator is the same as in (6)

and (7) involves choosing  $f=\gamma$  and  $C=\alpha^2\omega^2/c^2$  in (A2). The numerator can be rearranged to give

$$\gamma = \frac{\mu(1-y/y_A) + \gamma x}{1-y}. \quad (\text{A4})$$

By construction, the pure Alfvén singularity at  $y=1$  is spurious. In the same way, by choosing  $f=\gamma$  and  $C=$

$\alpha^2\omega^2/c^2 - \alpha\omega v_t/c^2$  in (A2), and rearranging the resulting numerator using (5), one derives an expression

$$\gamma = \frac{\mu(2-y/y_A) - \gamma(1-x)}{1-y - \alpha\omega v_t/c^2}. \quad (\text{A5})$$

Again by construction the singularity at  $1-y - \alpha\omega v_t/c^2 = 0$  is spurious. Similarly, one may generate spurious singularities starting from (6) and (7).