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Diffusive Shock Acceleration by Multiple Shock Fronts with Differing Properties

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Abstract: The effect of diffusive shock acceleration on a distribution of particles is explored for multiple shocks, taking into account adiabatic expansion between the shocks. Specifically, the spectral index is calculated numerically for two cases: a sequence of identical shocks, and a sequence of pairs of shocks with alternating shock strength. How these two cases evolve to the asymptotic limit is examined, and it is shown that the evolution of the paired-shock case can be described by a sequence of identical shocks with shock strength equal to the mean of the two.

1. Introduction

Diffusive shock acceleration is the favoured mechanism for the acceleration of galactic cosmic rays and of relativistic electrons in some, and maybe most, synchrotron sources (see Blandford and Eichler 1987 for a recent review). A feature of the mechanism is that it produces a power-law momentum distribution, $f(p) \propto p^{-b}$, corresponding to a synchrotron frequency spectrum $I_\nu \propto \nu^{-\alpha}$, with $b = 2\alpha + 3$. The flattest spectrum that can be produced by a single shock has $b = 4$ and thus $\alpha = 0{\cdot}5$. Flatter spectra are inferred for some synchrotron sources, and these cannot be explained in terms of acceleration by a single shock.

Acceleration by a sequence of identical shocks, with fresh injection at each shock and decompression between the shocks, does produce a spectrum that is flatter than can be produced by a single shock; specifically, the distribution approaches $f(p) \propto p^{-3}$ after an arbitrarily large number of shocks (e.g. White 1985; Achterberg 1990a, 1990b; Schneider 1993; Melrose and Pope 1993). In this paper we investigate (a) how this asymptotic distribution is approached as a function of the number identical shocks and of the strength of the shocks, and (b) the effect of replacing a sequence of identical shocks by a sequence of pairs of shocks. The latter is intended as a first step in generalising to more general sequences of shocks.

Specific assumptions are made, in that we consider a sequence of one-dimensional shocks, where the only propagation effects between shocks are adiabatic losses due to the expansion of the magnetic field. Energy losses (or gains) and diffusive effects that result in particle escape to outside the acceleration region are neglected. Schneider (1993) discussed the effects of both energy and particle losses in some detail, with cosmic ray propagation in mind. He found that in a case without energy losses, particle losses cause the spectrum to steepen. Energy losses proportional to p^2 (i.e. synchrotron emission or the inverse Compton effect) lead to an exponential cutoff, where the losses dominate the acceleration process. A 'hump' in the spectra develops around the momentum where the loss timescale is equal to the acceleration timescale, and a significant fraction of the cosmic ray energy is contained in a small interval about this momentum. The larger the escape time for the particles, the larger the hump, as there are more particles at higher momentum to cool down into this hump. For protons and nuclei, nuclear reactions and ionisation losses produce a black-body cutoff at an energy determined by the propagation time for the cosmic rays, i.e. the time that they have been propagating through the ISM since their initial acceleration (Berezinskiĭ *et al.* 1990). Such effects would need to be included in our model to enable detailed comparison with physical systems.

2. The Asymptotic Distribution

Single Shock Theory

The theory of diffusive shock acceleration implies the following form for the distribution function, $f_+(p)$, of energetic ions or electrons downstream of the shock, in terms of the distribution function, $f_-(p)$, upstream:

$$f_+(p) = bp^{-b} \int_0^p \mathrm{d}p'\, p'^{(b-1)} f_-(p')\,,$$

$$b = \frac{3r}{r-1}\,, \quad r = \frac{u_-}{u_+}\,, \tag{1}$$

where u_- is the upstream flow velocity and u_+ the downstream flow velocity, both measured in the shock rest frame. Thus if the shock propagates through a region where the electron distribution is $f_-(p)$, then after the shock has passed the distribution is $f_+(p)$. After the shock a decompression occurs. If the compression ratio at the shock is written as $r = R^{-3}$, then during decompression $p \to pR$. Then Liouville's theorem implies that the distribution function $f'_+(p)$ after decompression is given by $f'_+(p) = f_+(p/R)$. Hence the distribution function far downstream of the shock is:

$$f'_+(p) = b(p/R)^{-b} \int_0^{p/R} \mathrm{d}p'\, p'^{b-1} f_-(p')\,. \tag{2}$$

Here we assume that two qualitatively different sources of particles are injected into the acceleration process at the shock, an upstream distribution

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$f_-(p)$, and a source of fresh injection $\phi_0(p)$. The fresh injection is a distribution of suprathermal particles created at the shock from the thermal distribution. We take the form of this distribution to be $\phi_0(p) = C(r)\delta(p-p_0)$, where p_0 is a constant and $C(r)$ is the injection efficiency function. We arbitrarily choose $C(r) = (r-1)/3$, which implies that this fresh injection disappears when the shock does, i.e. when $r \to 1$, and is more efficient for stronger shocks. With this form of injection, the flattest possible spectrum that can be produced by this mechanism, namely $f(p) \propto p^{-4}$, $I_\nu \propto \nu^{-0\cdot5}$, results from the strongest possible shock, corresponding to $r = 4$.

Multiple Shocks

We now consider a sequence of shocks, where the number of shocks is arbitrarily large. At each shock a new distribution of particles is injected and accelerated, and the particles injected at earlier shocks are accelerated further. Adiabatic decompression occurs after each shock. Summing over an infinite number of identical shocks gives $f_\infty(p) \propto p^{-3}$, which is the distribution found by White (1985), who used $N(p) \propto p^2 f(p)$, Achterberg (1990a, 1990b), Schneider (1993) and Melrose and Pope (1993). Note that the asymptotic distribution is independent of b, that is, any sequence of identical shocks produces a p^{-3} power law.

3. Evolution of the Distribution Function

Identical Shocks

In an earlier paper (Melrose and Pope 1993), we reported results of numerical calculations for acceleration by a sequence of identical shocks. We showed how the distribution function of a fixed group of particles injected at the first shock, and reaccelerated at subsequent shocks, evolves for a limited number (five) of shocks. The lower cutoff momentum p_0 is shifted each time according to the scaling $p_0 \to Rp_0$ due to the adiabatic decompression. The peak of the distribution gets broader and moves to smaller p with each subsequent reacceleration and adiabatic decompression. We showed how the power law $f(p) \propto p^{-4}$, which is produced by acceleration at a single strong shock, evolves towards $f(p) \propto p^{-3}$ with subsequent reaccelerations. Here, we extend the number of shocks and the momentum range considered.

After a large but finite number of shocks, the distribution function $f(p)$ is a sum of power laws resulting from the injection and subsequent reacceleration at each shock. Thus after n shocks, there are components that have been accelerated n times, $n-1$ times, and so on. The steepest component is that resulting from fresh injection at the last shock, which produces a p^{-b} power law. Adiabatic decompression after each shock means particles lose momentum, and the spectrum at low momentum ($p > p_0$) flattens to p^{-3} due to this effect combined with reacceleration. The spectrum is steeper at higher momentum, but because it is the sum of many reaccelerated components, the spectrum is everywhere flatter than the steepest component p^{-b}. Figure 1 shows how the spectral index evolves towards the asymptotic limit for a sequence of $r = 4$ shocks. The spectrum hardens with each reacceleration and fresh injection, steepening slowly with increasing momentum. Figure 2 shows how this build-up to a p^{-3} law happens more slowly for a sequence of $r = 3$ shocks [curve (b)] than for a sequence of $r = 4$ shocks [curve (a)]. Here we plot the characteristic spectral index over the momentum range $10p_0$–$50p_0$ against the number of shocks. The slower build-up for the weaker shocks (smaller r) may be attributed to the fact that weaker shocks produce steeper spectra with fewer particles at higher momenta. Therefore, more reaccelerations are required to accelerate the particles to higher momenta and thus to flatten the spectrum to p^{-3}.

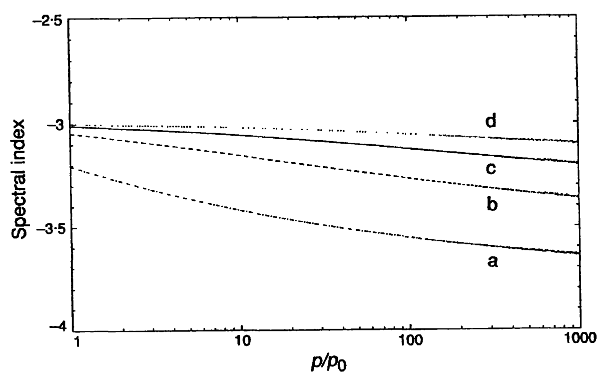

Figure 1—Spectral index for fresh injection and reacceleration at a sequence of $r = 4$ shocks for (a) 5, (b) 10, (c) 15 and (d) 20 shocks. The evolution towards the $\lambda = 3$, $f(p) \propto p^{-\lambda}$ limit is shown, as the spectral index hardens with each successive reacceleration.

In an attempt to find an approximate expression for the spectral index resulting from acceleration at a sequence of identical shocks, we derived an analytic expression as follows. The distribution function $f_n(p,p_0)$ after n shocks is obtained by summing over all the contributions from injection and reacceleration at all the shocks. We then obtain the spectral index by taking $\partial[\ln f_n(p,p_0)]/\partial \ln p$,

$$\frac{\partial[\ln f_n(p,p_0)]}{\partial \ln p} = \frac{\partial \ln\left\{\sum_{i=1}^{n} b^i (p/p_0R^i)^{-b}[\ln(p/p_0R^i)]^{i-1}\right\}}{\partial \ln p}. \quad (3)$$

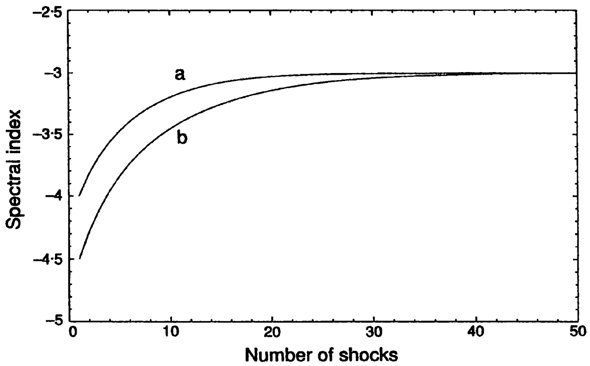

Figure 2—Characteristic spectral index $\{\log[(f(p_2)] - \log[f(p_1)]\}/[\log(p_2) - \log(p_1)]$, with $p_1 = 10p_0$, $p_2 = 50p_0$, versus number of shocks for (a) $r = 4$, (b) $r = 3$.

It seems reasonable to expect that a useful approximation for the spectral index could be obtained by keeping a small number of terms in the sum in equation (3). Figure 3 shows the effect of keeping just the last term (i.e. $i = n$), the last two terms, and so on. It is clear from Figure 3 that we must keep all n terms in equation (3), that is, no simplifying approximation can be made.

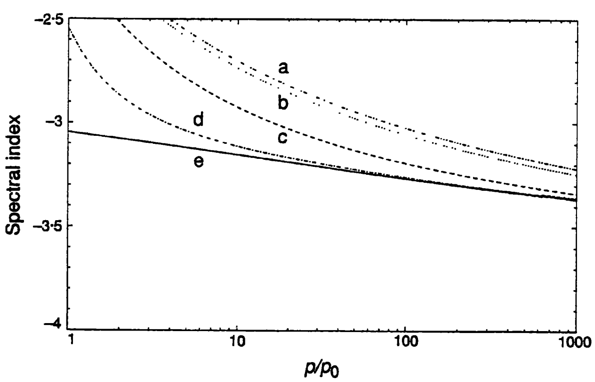

Figure 3—Approximations to the spectral index from equation (3) for a sequence of 10 shocks with $r = 4$: (a) $i = 10$ (i.e. only the last term in the series); (b) $i = 9$ (i.e. the last two terms); (c) $i = 6$; (d) $i = 2$ (i.e. all terms except fresh injection); and (e) the numerical result for 10 shocks.

Non-identical Shocks

So far only sequences of identical shocks have been considered. As a first step in exploring the more general case, we now consider a sequence of 'pairs' of shocks with shock strengths r_1 and r_2 respectively. That is, we consider a sequence of shocks with strengths $r_1, r_2, r_1, r_2, \ldots$

Figure 4 shows the result of fresh injection and reacceleration for a sequence of ten shocks, with $r_1 = 4$ and $r_2 = 3$. The build-up to the asymptotic limit is slower than for a sequence of $r = 4$ shocks, but more rapid than that for a sequence of $r = 3$ shocks. The behaviour at high momenta for 'pairs' in either order (i.e. starting with r_1 or r_2) can be characterised by a sequence of shocks with $r = 3{\cdot}5$, the mean shock strength. This is shown after ten shocks in Figure 5, where we see how at high momenta the sequence that ends with an $r = 3$ shock is well approximated by the $r = 3{\cdot}5$ curve. In general, for any number of shocks, the $r = 3{\cdot}5$ fit is better for the sequence of paired shocks that ends with an $r = 3$ shock. The evolution of the spectral index with the number of shocks is shown in Figure 6. Note that, apart from the wiggles, the paired-shock case is well described by a sequence of identical shocks with shock strength equal to the mean of the paired-shock strengths. The wiggles in the characteristic spectral index plot, the difference in the spectral index of the paired-shock cases shown in Figure 5, and the differences in the spectral index in Figure 4, are all due to the effect of the last shock in the sequence. If the last shock is an $r = 3$ shock then fewer electrons are injected (the injection efficiency is less for weaker shocks), reacceleration is less efficient, and thus the spectrum is steeper at low momenta. If the last shock has a strength of $r = 4$, then the spectrum is flatter at low momenta.

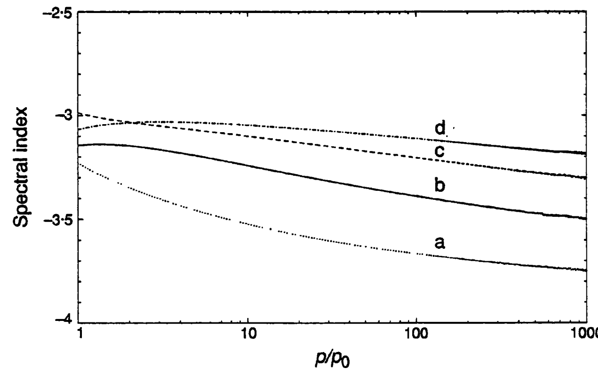

Figure 4—Spectral index for fresh injection and reacceleration at a sequence of pairs of shocks with $r_1 = 4, r_2 = 3$ for (a) 5, (b) 10, (c) 15 and (d) 20 shocks.

An Analytical Solution of the Paired-shock Problem

Two attempts have been made to obtain an analytical expression for acceleration at a sequence of paired shocks. Using equation (2) directly and iterating quickly produces a large number of terms, and no simple expression. An alternative involves generalising a method due to Achterberg (1990b).

The method of Achterberg can be summarised as follows:

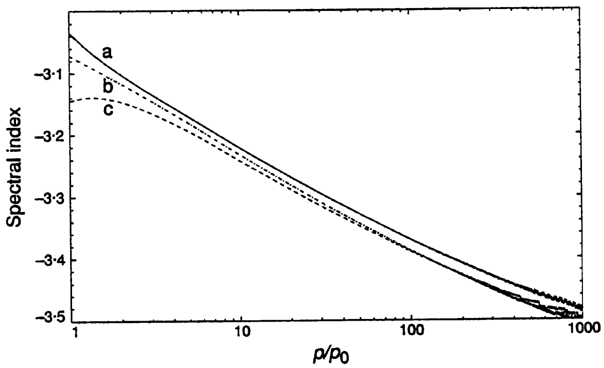

Figure 5—Spectral index for fresh injection and reacceleration at a sequence of 10 shocks for (a) $r_1 = 3, r_2 = 4$; (b) $r = 3{\cdot}5$; and (c) $r_1 = 4, r_2 = 3$. The $r = 3{\cdot}5$ case is a reasonable approximation to both curves (a) and (c).

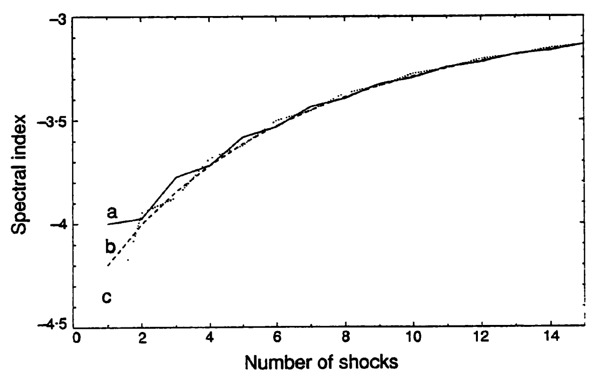

Figure 6—Characteristic spectral index from $p = 10p_0$ to $p = 50p_0$ versus number of shocks for (a) $r_1 = 4, r_2 = 3$; (b) $r = 3{\cdot}5$; and (c) $r_1 = 3, r_2 = 4$. The $r = 3{\cdot}5$ case matches the evolution of the paired cases remarkably well.

(a) The transport equation for the energy flux, $\Sigma(p) = uF(p) = up^2 f(p)$, is solved across a single shock by using Green's function $G(p/p_0)$, where u is the flow velocity and p_0 is the injection momentum.

(b) A transmission function $T(p/p')$ is defined, which describes the propagation of particles between shocks, taking into account particle and energy losses (and gains).

(c) One can sum the effect of an infinite sequence of identical shocks on the particle distribution by defining a 'renormalised Green function' $G_r(p/p_0)$.

(d) An integral equation for $G_r(p/p_0)$ is obtained but is not readily solved. By a special choice of $T(p/p')$, the integral equation can be Mellin-transformed (Morse and Feshbach 1953) to reduce it to an algebraic equation for the Mellin-transformed Green function $g_r(\xi)$.

(e) To obtain $G_r(p/p_0)$, we invert the Mellin transform. The solution is a sum of power laws, where the exponents ξ_m are determined by the poles of the integral, given by the roots of a function $N(\xi) = N_1(\xi) = s+1-\xi-sU_m(\xi)$, where $s = 3/(r-1)$ and $U_m(\xi)$ is the Mellin transform of the transmission function, $U_m(\xi) = r^{(1-\xi)/3}$. The solutions are $\xi = 1$ for $p > p_0$ [i.e. $F(p) \propto p^{-1}, f(p) \propto p^{-3}$], and $\xi = -2$ for $p < p_0$ [i.e. $F(p) \propto p^2, f(p) \propto$ constant].

A similar technique can be applied to the paired-shock case, but here the equation for $N(\xi)$ is more complicated. We find $N(\xi) = N_2(\xi) = A(\xi)/B(\xi)$, where $A(\xi) = (s_1+1-\xi)(s_2+1-\xi) - (s_1 s_2)U_{m1}(\xi)U_{m2}(\xi)$ and $B(\xi) = s_1+1-\xi+s_1U_{m1}(\xi)$ and the subscripts refer to shocks 1 and 2. Here $U_{mi}(\xi) = r_i^{(1-\xi)/3}$ is the transmission function downstream of shock i. Solutions of $A(\xi) = 0$ give the exponents ξ_m as before. The roots of $B(\xi) = 0$ occur where $G_r(p/p_0)$ vanishes, and in general do not çoincide with the roots of $A(\xi)$.

In addition to the two solutions for ξ_m obtained in the identical-shock case, a third solution is obtained, which we have determined numerically. For $r_1 = 4$, $r_2 = 3$ this solution is $\xi_m = 2{\cdot}87$, corresponding to $f(p) \propto p^{-4{\cdot}87}$. In the limit $r_1 \to r_2$, $A(\xi) \to N_1(\xi)B(\xi)$, and thus $N_2(\xi) \to N_1(\xi)$, and the third solution vanishes.

The interpretation of this third solution is unclear. We speculate that it may be related to the oscillation in the spectral index that we observe for low momentum ($p > p_0$), which occurs between different shocks, but we have been unable to substantiate this speculation.

4. Discussion and Conclusions

We have investigated how the asymptotic spectrum $f(p) \propto p^{-3}$ forms for a sequence of identical one-dimensional shocks, and compared its formation for different shock strengths r. We find that a sequence of weak shocks evolves more slowly to the p^{-3} limit, that is, the p^{-3} law holds over a smaller momentum range than for a sequence of the same number of stronger shocks. This is because a single weak shock produces a steeper spectrum than a strong shock, and so more reaccelerations at weak shocks are required to harden the spectrum to p^{-3}. Thus an observed spectrum which is proportional to p^{-3} over some momentum range may be the result of injection and reacceleration either by a few stronger shocks, or by a larger number of weaker shocks. We also consider a sequence of 'pairs' of shocks with shock strengths r_1 and r_2. The build-up to the asymptotic limit for paired shocks can be characterised by a sequence of identical shocks with a shock strength equal to the mean shock strength of the pairs.

In both cases, the only propagation effects between shocks are adiabatic losses due to the expansion of the magnetic field. Energy losses (or gains) and particle escape due to diffusive effects are ignored. This is done so that the effects of the shocks themselves can be studied in detail. For electrons, particle escape steepens the spectra, and energy losses introduce an exponential cutoff and a hump in the spectra where the energy-loss and acceleration timescales are equal (Schneider 1993). For protons and nuclei, nuclear reactions and ionisation losses result in a black-body cutoff at an energy determined by the propagation time for the cosmic rays (Berezinskiǐ *et al.* 1990). Realistic propagation models would include these mechanisms for applications to physical systems.

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