MODE COUPLING DUE TO TWISTING OF MAGNETIC FIELD LINES

D. B. MELROSE, P. A. ROBINSON and T. M. FELETTO
School of Physics, University of Sydney, NSW 2006, Australia

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Abstract. A new approach to the theory of mode coupling in an inhomogeneous, birefringent medium is used to treat mode coupling in a magnetized plasma with a twisted magnetic field. The twist introduces a resonance, corresponding to the rate of twisting being equal to the rate of generalized Faraday rotation. When this resonance occurs it introduces a new regime of strong mode coupling. The implications of this regime are discussed in connection with the long-standing problem in solar radiophysics that mode coupling appears to be stronger than theory implies, but no obvious resolution of the problem is found.

1. Introduction

The polarization of electromagnetic radiation propagating through a homogeneous birefringent medium changes periodically with distance along the ray path. If the natural modes are circularly polarized, the effect on incident linearly polarized radiation is to cause Faraday rotation of the plane of linearly polarized radiation, and incident circularly polarized radiation is unaffected. If the natural modes are linearly polarized, the effect on incident circularly polarized radiation is to cause it to become elliptical with the axial ratio of the ellipse varying periodically. The general case, where the natural modes are elliptically polarized and the radiation has an arbitrary polarization that is different from either natural mode is referred to here as generalized Faraday rotation. A helpful description is based on the Poincaré sphere (Ulrich and Simon, 1979). On the surface of the unit sphere, a polarization is represented by a point, which we call the polarization point, and the two natural modes by two points on opposite sides of the sphere, thereby defining an axis, which we call the modal axis, through the sphere. Generalized Faraday rotation corresponds to rotation of the polarization point about the modal axis at constant latitude relative to that axis.

Mode-coupling in an inhomogeneous, magnetized plasma results from a change in the properties of the natural modes along the ray path, taken here to be the z-axis. In this paper we are concerned with mode coupling at a so-called QT region where the angle, θ, between the magnetic field and the ray path passes through $\theta = \pi/2$ (e.g., Budden, 1960; Zheleznyakov, 1970, p. 350; Melrose, 1980, p. 300). ('QT' stands for 'quasi-transverse' in an outdated use of language, and corresponds to quasi-perpendicular propagation in modern usage.) At a QT region, the polarization of each natural mode changes from nearly circular through linear to nearly circular in the opposite sense. The problem addressed concerns the effect on incident circularly polarized radiation: does it follow the modes and so change its

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sense of circular polarization ('weak' mode coupling), or does it ignore the change in the properties of the natural modes and so retain its initial sense of circular polarization ('strong' mode coupling)? Note the counter-intuitive implications of the terminology: when mode coupling is weak the polarization changes, and when it is strong the polarization is unaffected. Note also that mode coupling refers to coupling enabling radiation to be transferred *between* the modes, and not to a coupling between the modes themselves. The modes are defined as those of an equivalent locally homogeneous medium, so that their properties change due to spatial gradients in the plasma parameters when the inhomogeneity is taken into account. In weak mode coupling, the radiation remains 'stuck' in the one mode, and its polarization changes to follow the changes in the polarization of that mode as the plasma parameters change. In strong mode coupling the radiation ignores the changes in the properties of the natural modes and is freely transferred between them.

Mode coupling is relevant to a long-standing problem in solar radiophysics concerning the interpretation of the polarization of radio emission from magnetically bipolar regions. Observationally, the distinguishing feature of such emission is that the opposite footpoints of the bipolar region are opposite circularly polarized (e.g., Piddington and Minnett, 1951; McLean and Sheridan, 1972; Suzuki and Sheridan, 1980; White, Thejappa, and Kundu, 1992). In terms of the Stokes parameters (I, Q, U, V) the two footpoints have opposite signs of V. This is predicted by simple theory in which the emission occurs in one of the natural modes of the plasma. There are two familiar types of emission from bipolar regions. At lower frequencies, $\ll 1$ GHz, type I storm emission due to plasma emission is polarized in the sense of the o-mode of magneto-ionic theory, and at higher frequencies, $\gtrsim 1$ GHz, gyrosynchrotron emission favors the x-mode. The long-standing problem concerns possible reversals of the sign of V along the line of sight at QT regions. Application of a conventional form of mode-coupling theory to this problem is well known in the case where there is no twist in the magnetic field lines. Mode coupling is necessarily strong at sufficiently high frequencies, where the effect of the plasma becomes negligible and necessarily weak at sufficiently low frequencies where the properties of the two modes differ significantly. It is convenient to define a characteristic frequency, $\omega_t \sim (n_e B^3 L)^{1/4}$, where n_e is the electron density and B is the magnetic field in the QT region, and $L = |\mathrm{d}H/\mathrm{d}z|^{-1}$ is the characteristic thickness of the QT region. Then for $\omega \gg \omega_t$ mode coupling is strong, and for $\omega \ll \omega_t$ mode coupling is weak. The problem that arises is that the observations imply that mode coupling is strong under conditions where simple theory suggests that the wave frequency is well below this transition frequency.

For emission from the two feet of a bipolar region a ray from the more distant (from the observer) footpoint passes through one more QT region than a ray from the nearer footpoint (e.g., Piddington and Minnett, 1951). If mode coupling is weak, then V changes sign at each QT region, and the additional QT region encountered by the ray from the more distant footpoint would cause the sign of V

to be the same from both footpoints, contrary to the observed opposite polarizations. Simple estimates of the characteristic frequency ω_t for bipolar regions give several gigahertz (e.g., Cohen, 1960; Takakura, 1961; Zheleznyakov, 1970, p. 367; Melrose, 1980, p. 302). However, observations show opposite signs of V at bipolar regions at much lower frequencies. Suggested ways in which this inconsistency might be resolved involve seeking ways to explain a lower ω_t, such as a weak B in the QT region, e.g., in a neutral sheet (Zheleznyakov and Zlotnik, 1977), or a thin QT region due to local large gradients. In this paper, we examine the effect of a twist in the magnetic field on mode coupling. We use a treatment of mode coupling based on integration of the transfer equation for polarized radiation. It was shown elsewhere (Melrose and Robinson, 1994) that this treatment reproduces the known result of a more conventional treatment for an untwisted field (Zheleznyakov and Zlotnik, 1963), establishing the equivalence of the two theories in this case.

In Section 2 we summarize the theory of the transfer of polarized radiation through a birefringent medium and how it is used to treat mode coupling at a QT region for an untwisted field. In Section 3 we show how the effect of a twist may be included. In Section 4 we present numerical results which show that there is a hitherto unrecognized region of strong mode coupling when the rate per unit length of the twisting of the field lines exceeds the rate of rotation of the plane of polarization due to Faraday rotation. In Section 5 we discuss the conditions under which this effect might be important in the solar corona.

2. Generalized Faraday Rotation

The transfer of polarized radiation may be described in terms of the Stokes parameters, which may be written as the *Stokes vector* S_A, $A = I - V$, with $S_I = I$, $S_Q = Q$, $S_U = U$, $S_V = V$. In a non-dissipative medium, the total intensity, I, is a constant along the ray, and the polarized part of the radiation may be described by the transfer equation (e.g., Melrose and McPhedran, 1991, p. 187),

$$\frac{\mathrm{d}S_A}{\mathrm{d}z} = \rho_{AB} S_B \,, \tag{1}$$

where z denotes distance along the ray path, and where the sum over the repeated index $B = Q, U, V$ is implied. The matrix ρ_{AB} has the form

$$\rho_{AB} = \begin{pmatrix} 0 & -\rho_V & \rho_U \\ \rho_V & 0 & -\rho_Q \\ -\rho_U & \rho_Q & 0 \end{pmatrix} . \tag{2}$$

Explicit expressions for the parameters in (2) are determined by the properties of the natural modes in the medium. Assuming that these properties are given by the magnetoionic theory, with collisions neglected, one has

$$\rho_Q = \frac{\omega_p^2 \Omega_e^2}{2c\omega(\omega^2 - \Omega_e^2)} \sin^2\theta \cos(2\phi) ,$$
$$\rho_U = \frac{\omega_p^2 \Omega_e^2}{2c\omega(\omega^2 - \Omega_e^2)} \sin^2\theta \sin(2\phi) , \tag{3}$$
$$\rho_V = \frac{\omega_p^2 \Omega_e^2}{c(\omega^2 - \Omega_e^2)} \cos\theta ,$$

where ω_p is the plasma frequency, Ω_e is the electron cyclotron frequency, and θ, ϕ are the polar angles of the ray relative to the magnetic field. In this paper we assume $\omega^2 \gg \Omega_e^2$, and approximate the denominators in (3) by neglecting Ω_e^2 in comparison with ω^2.

Equation (2) implies that the quantity $Q^2 + U^2 + V^2$ is a constant, and for simplicity is set equal to unity in the following. It is sometimes convenient to write

$$Q = \cos(2\chi)\cos(2\psi) , \qquad U = \cos(2\chi)\sin(2\psi) , \qquad V = \sin(2\chi) . \tag{4}$$

On the Poincaré sphere, the polarization is then represented by a polarization point at latitude 2χ and longitude 2ϕ. Right and left circular polarizations are represented by the north ($V = 1$) and south ($V = -1$) poles, respectively, and linear polarizations by a point on the equator ($\chi = 0$), with the plane of polarization related to ψ. The natural modes may also be represented by two diametrically opposite points on the Poincaré sphere. If one of these points is represented by latitude $2\chi_0$ and longitude $2\psi_0$, then one has

$$\rho_Q : \rho_U : \rho_V = \cos(2\chi_0)\cos(2\psi_0) : \cos(2\chi_0)\sin(2\psi_0) : \sin(2\chi_0) . \tag{5}$$

The modal axis passes through the point $2\chi_0$, $2\psi_0$ and its conjugate point. As mentioned above, the Poincaré sphere permits a geometric interpretation of generalized Faraday rotation. Equation (2) implies that the polarization point rotates about the modal axis, following a circle at constant latitude relative to this axis. There are two familiar cases. One is Faraday rotation itself, which corresponds to circularly polarized natural modes ($2\chi_0 = \pi/2$) so that the modal axis is the same as the original polar axis of the sphere. The other is for a uniaxial crystal, for which the natural modes are linearly polarized. In this case, the modal axis is in the equatorial plane. In a quarter-wave plate, the face of the crystal is such that the modal axis passes through $2\psi = 0$, with the incident radiation linearly polarized orthogonal to this axis, so that the polarization point is initially at $2\chi = 0$, $2\psi = \pi$; this point moves around a great circle that passes through both poles, converting the linear into circular polarization in one quarter of a revolution.

2.1. Mode coupling at a QT region

Consider how the modal axis changes as a QT region is approached from one side ($z < 0$). The point corresponding to one of the natural modes moves away from the relevant pole of the sphere, crosses the equator at the center of the QT region ($z = 0$) and approaches the opposite pole of the sphere far on the other side of the QT region ($z \gg +L$). The instantaneous motion of the polarization point is around a circle at constant latitude relative to this changing modal axis. In general, this leads to a complicated path that can only be determined by detailed calculation. However, the cases of strong and weak mode coupling can be understood relatively simply. The strength of the mode coupling is determined by the relative speeds (variation as a function of z) of the modal axis and of the polarization point as it rotates about the instantaneous modal axis.

Consider what happens to right circularly polarized radiation incident on one side of the QT region. Far on one side of the QT region, the natural modes are very nearly circularly polarized, and the modal axis is effectively identical to the polar axis of the Poincaré sphere. There is no change in polarization until the modal axis deviates away from the polar axis, and then generalized Faraday rotation occurs, with the polarization point rotating about the moving modal axis as the QT region is approached. In the case of strong mode coupling the modal axis moves faster, due to the change in orientation of the magnetic field, than the polarization point moves due to generalized Faraday rotation. As the modal axis moves away from the axis of the sphere, the rotation of the polarization point around the modal axis is so slow in comparison that it does not move significantly as the modal axis approximately reverses its orientation. Hence the polarization point remains near the north pole, and the polarization remains approximately right circular. In the case of weak mode coupling, the rotation around the modal axis is fast compared with the rate of change of orientation of the modal axis. As the QT region is crossed, the modal axis 'drags' the polarization point with it, such that the polarization point rotates rapidly in a small circle about the modal pole that was originally near the north pole of the sphere. Far on the other side of the QT region, the polarization point has followed this modal pole, which is now located near the south pole of the sphere. Hence the polarization is nearly left hand circular. Thus, in weak mode coupling, the rotation of the representative point about the modal axis is so fast that the polarization point remains at essentially the same latitude relative to the modal axis as it changes due to changes in the plasma parameters, and in strong mode coupling the rotation of the representative point about the modal axis is so slow that the polarization point remains essentially stationary and ignores changes in the modal axis.

2.2. Formal Treatment of Mode Coupling

Following Melrose and Robinson (1994), mode coupling at a QT region may be treated using (1) with (2). It is convenient to write $\rho_V/\rho_Q = f$ and to regard $f = f(\tilde{z})$ as a function of the dimensionless variable $\tilde{z} = \rho_Q z$. Then, on denoting a derivative with respect to $\tilde{z}$ by a prime, (1) with (2) becomes

$$\begin{pmatrix} Q' \\ U' \\ V' \end{pmatrix} = \begin{pmatrix} 0 & -f & \sin(2\phi) \\ f & 0 & -\cos(2\phi) \\ -\sin(2\phi) & \cos(2\phi) & 0 \end{pmatrix} \begin{pmatrix} Q \\ U \\ V \end{pmatrix} . \tag{6}$$

The properties of the natural modes are related to to the properties of the eigensolutions of (6). The eigenvalues, $\pm i(1+f^2)^{1/2}$, determine the rate of Faraday rotation in $\tilde{z}$, and the eigenvectors

$$\begin{pmatrix} Q_0 \\ U_0 \\ V_0 \end{pmatrix} = \pm \frac{1}{(1+f^2)^{1/2}} \begin{pmatrix} \cos(2\phi) \\ \sin(2\phi) \\ f \end{pmatrix} \tag{7}$$

define the end points of the modal axis. The angle, ρ, between the modal axis and the polarization point for an arbitrary polarization is given by

$$\cos\rho = Q_0 Q + U_0 U + V_0 V . \tag{8}$$

2.3. A SIMPLE QT REGION

Let us define a *simple QT region* as a region in which the component of $\mathbf{B}$ along the ray direction passes through zero with the magnetic field and the ray in the same plane, which may be chosen as the plane $\phi = 0$. For a simple QT region (6) becomes

$$\begin{pmatrix} Q' \\ U' \\ V' \end{pmatrix} = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Q \\ U \\ V \end{pmatrix} . \tag{9}$$

The function f and our new independent variable $\tilde{z}$ both depend on the magnitude and orientation of the magnetic field: $f \sim \cos\theta/(B\sin^2\theta)$, $\tilde{z} \sim B^2 \sin^2\theta$. Within the QT region f may be regarded as a simple function of $\tilde{z}$.

Melrose and Robinson (1994) showed that for a standard assumed form for the spatial variation, specifically for

$$f(\tilde{z}) = f_\infty \tanh(\tilde{z}/\tilde{z}_0) , \tag{10}$$

where f_∞ and $\tilde{z}_0$ are constants, numerical solution of (9) reproduces the results of an analytic solution derived using mode-coupling theory (Zheleznyakov and Zlotnik, 1963). This establishes the equivalence of the two theories in this specific case.

For semi-quantitative purposes it is useful to define a mode coupling coefficient, C, which is such that $C \gg 1$ corresponds to strong mode coupling and $C \ll 1$ to weak mode coupling. From the viewpoint of the motion of points on the Poincaré sphere, C is of order the ratio of the speed (rate of change of position as a function of $\tilde{z}$) of the end points of the modal axis as they cross the equator due to the change in orientation of the magnetic field to the speed of the polarization point as it rotates around the modal axis due to Faraday rotation. From an algebraic viewpoint, a simple approximate equation for C in the present context is

$$C = |(\mathrm{d}f/\mathrm{d}\tilde{z})|_{\tilde{z}=0}| \,, \tag{11}$$

which reduces to $C = |f_\infty|/\tilde{z}_0$ for the particular case (10).

Note that (11) implies that the strength of the coupling at the QT region is determined by the ratio $|f_\infty|/\tilde{z}_0$, and not by f_∞ and $\tilde{z}_0$ separately. This may be understood by noting that in the idealized QT region, ρ_Q is assumed constant, and $\rho_V \sim \cos\theta$ is assumed to change sign linearly with z. If one introduces a length, L_θ, to characterize this change by $\cos\theta = z/L_\theta$ near $z = 0$, then (11) implies $C = 1/\rho_Q L_\theta$. In the following, $|f_\infty|$ and $\tilde{z}_0$ are defined only in terms of the ratio $|f_\infty|/\tilde{z}_0 = 1/\rho_Q L_\theta$.

2.4. Numerical Results ($\phi' = 0$)

The basis of our treatment of mode coupling is the numerical integration of (9) with (10) through a region with specified spatial dependence of θ and ϕ on $\tilde{z}$. All our results start with $V = 1$ at $\tilde{z} = -5\tilde{z}_0$. The function f, defined by (10), is plotted in Figures 1(a) and 1(b) for $f_\infty > 0$ and $f_\infty < 0$, respectively. In the absence of a twist (ϕ = constant) there is symmetry under simultaneous reversal of the initial signs of f_∞ and V, and we consider only the case $f_\infty > 0$. For each choice of initial variables we show plots of V, of the angle χ, cf. (4), which characterizes the ellipticity of the modes, and $\cos\rho$, cf. (8), which is the cosine of the angle between the polarization point and the pole of the modal axis. Plots of Q and U are shown only in the first example; in the other examples the oscillations in Q and U due to Faraday rotation are too rapid to be shown.

To explore the transition from strong to weak mode coupling we plot a sequence of three cases with $f_\infty = 10$, and (a) $\tilde{z}_0 = 1$, (b) $\tilde{z}_0 = 10$, and (c) $\tilde{z}_0 = 100$ in Figures 2–4. These correspond to (a) strong mode coupling ($C = 10$), (b) an intermediate case ($C = 1$), and (c) weak mode coupling ($C = 0.1$), cf. (11) with (10). The plots of V show that, as expected, V changes by a small fraction when the mode coupling is strong and approximately reverses in sign when the mode coupling is weak. The plot of $I - 1$, which should be equal to zero, is plotted as a numerical check. The main feature of the plots of Q and U is the oscillation due to Faraday rotation. Note that the apparent increase in the frequency of these oscillations from Figure 2 to Figure 4 is an artifact associated with the change in scale of the horizontal axis. The oscillations in Q and U do not affect χ, and the oscillations in this quantity are due to oscillations in V due to the ellipticity of the natural modes ($|f| < \infty$). The mode coupling is demonstrated most clearly by the

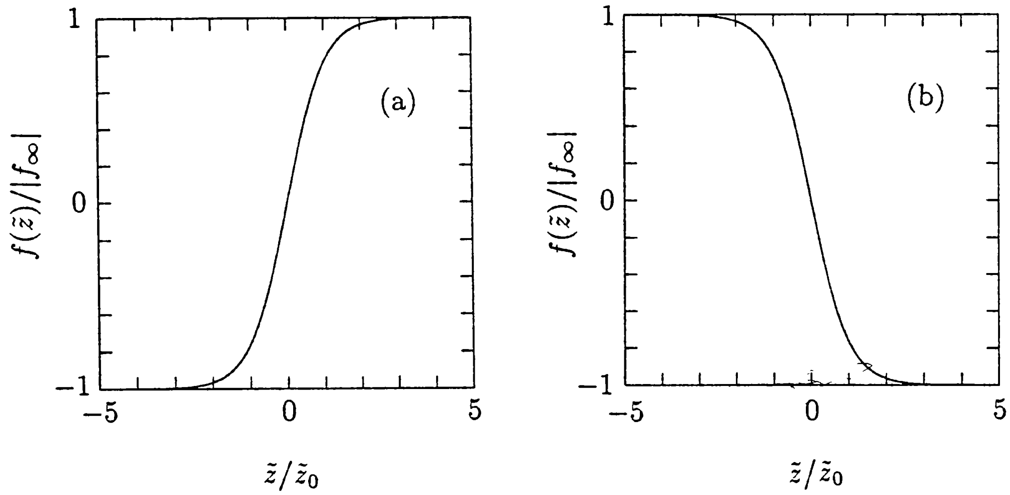

Fig. 1. The model (10) for the variation in θ across a QT region is illustrated for (a) $f_\infty > 0$, and (b) $f_\infty < 0$. Note that distance in all figures is expressed in terms of the dimensionless variable $\tilde{z} = \rho_Q z$, which is effectively in units of the Faraday rotation length. The dimensionless parameters f_∞ and $\tilde{z}_0$ appearing in (10) determine the strength of the coupling; only their ratio, $f_\infty/\tilde{z}_0$, is important and it is identified with the coupling constant, C, at the QT region, cf. (11).

parameter $\cos\rho$. Initially, we have $\cos\rho$ close to -1 due to the natural modes being nearly circularly polarized. (The initial sign of $\cos\rho$ for $V = 1$ is determined by the sign of f_∞, which is negative in our case.) As the QT region is crossed, the modal axis rotates across the equator. For (a) strong mode coupling $\cos\rho$ flips from $\cos\rho \approx -1$ to $\cos\rho \approx +1$ due predominantly to the change in the properties of the modes. For (c) weak mode coupling $\cos\rho$ remains close to -1, implying that the radiation remains in the same mode as the polarization of the mode changes.

As implied by (11), the strength of the mode coupling depends only on the combination $|f_\infty|/\tilde{z}_0$ and not on f_∞ and $\tilde{z}_0$ separately. In the sequence in Figures 2–4 we vary $\tilde{z}_0$ for fixed f_∞. In discussing the twist below we fix $\tilde{z}_0 = 10$ and vary f_∞. In Figure 5 we show the case $f_\infty = 100$, $\tilde{z}_0 = 10$, which has the same value of $C = |f_\infty|/\tilde{z}_0 = 10$ as for Figure 2. Apart from the scale of the horizontal axis affecting the apparent frequency of the oscillations, the two cases give similar results for the final average value of V, with the main difference being in the amplitude of the oscillations, whose magnitude scales as f_∞^{-1} for large f_∞.

3. Inclusion of a Twist

A twist corresponds to $\phi' = \mathrm{d}\phi/\mathrm{d}\tilde{z} \neq 0$. In treating the transfer of polarized radiation in a twisted magnetic field using (6), it is convenient to transform the problem into a form that is equivalent to (9).

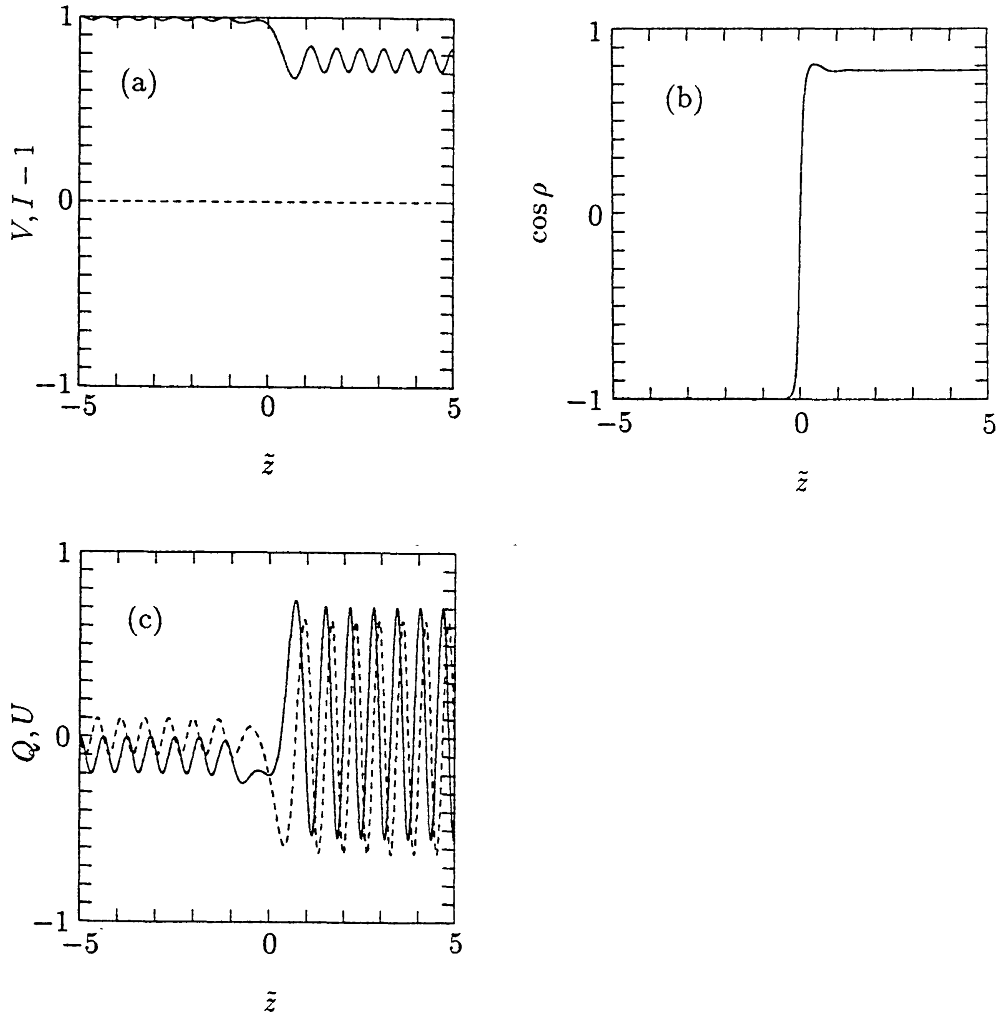

Fig. 2. Stokes parameters as a function of $\tilde{z}$ in the untwisted case ($\phi' = 0$) for $f_\infty = 10$ and $\tilde{z}_0 = 1$, giving $|f_\infty|/\tilde{z}_0$. In (a), the four plots show V (solid curve) and $I - 1$ (dashed curve), Q (solid curve), and U (dashed curve), and the cosine of the angle ρ between the polarization point and the modal axis, defined by (8). In (a) and (c), the oscillations in are due to Faraday rotation. In (b), the parameter $\cos\rho$ describes the relative motion on the Poincaré sphere of the polarization point of the radiation and the modal axis: the initial choices fix this at -1 at large negative $\tilde{z}_0$: in this case, the change in sign to $\cos\rho \approx 1$ implies that the polarization point does not move much as the orientation of the modal axis reverses across the QT region.

3.1. Reduction to an Equivalent Untwisted Case

On introducing new variables

$$\tilde{Q} = Q\cos(2\phi) + U\sin(2\phi)\,, \qquad \tilde{U} = -Q\sin(2\phi) + U\cos(2\phi)\,, \tag{12}$$

the general form (6) for a twisted field reduces to

$$\begin{pmatrix} \tilde{Q}' \\ \tilde{U}' \\ V' \end{pmatrix} = \begin{pmatrix} 0 & -g & 0 \\ g & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{Q} \\ \tilde{U} \\ V \end{pmatrix}\,, \tag{13}$$

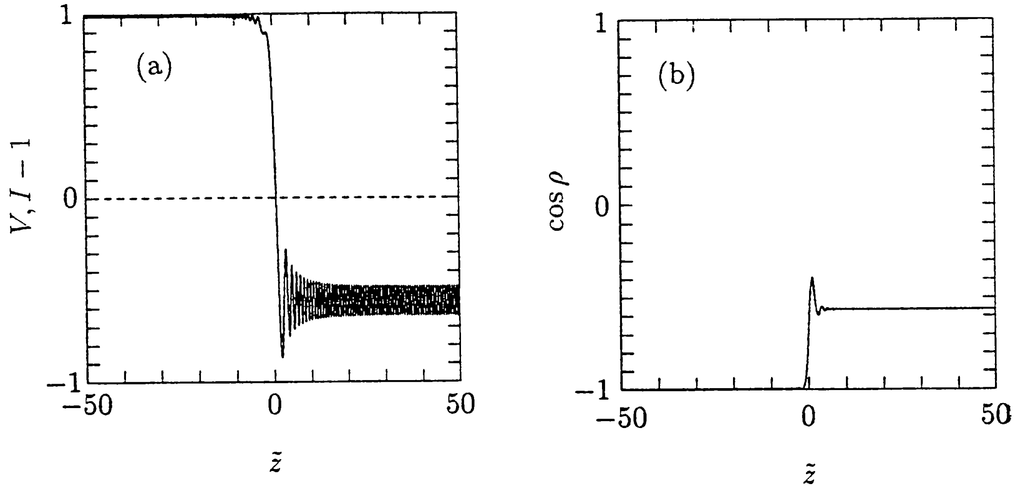

Fig. 3. As for Figure 2 but for $\tilde{z}_0 = 10$, giving $|f_\infty|/\tilde{z}_0 = 1$. In terms of the parameter $\tilde{z}$, the rate of Faraday rotation is fixed, and the apparent increase in this rate from Figure 2 to Figure 4 is an artifact of the change in scale on the horizontal axis. In this case the relatively small change in $\cos\rho$ implies that the polarization of the radiation remains approximately in the same natural wave mode as the polarization of the natural mode changes across the QT region.

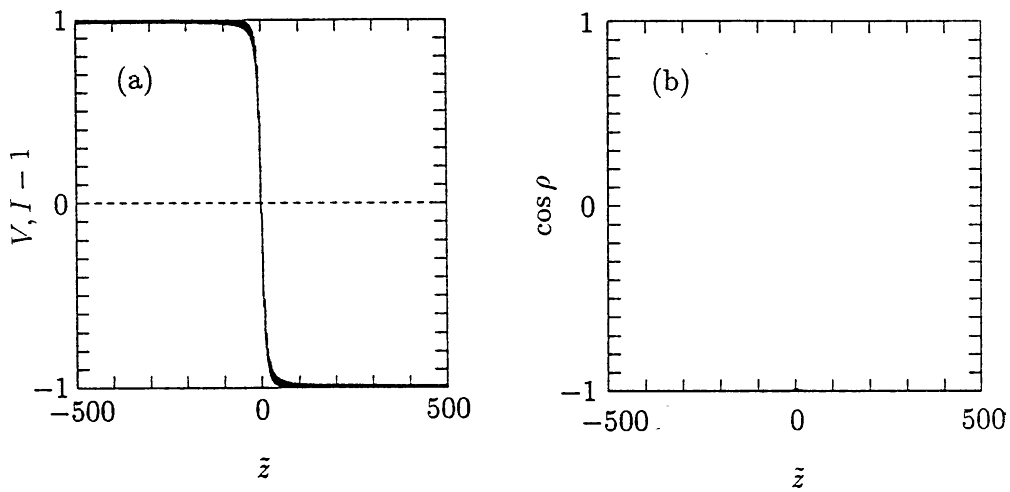

Fig. 4. As for Figure 2 but $\tilde{z}_0 = 100$, giving $|f_\infty|/\tilde{z}_0 = 0.1$.

with

$$g = f - 2\phi' \,. \tag{14}$$

In this way, the case of a twisted field is reduced to the same mathematical form as for an untwisted field; that is, (13) is formally the same as (9).

A subtle point relates to (13): the eigenvalues, $\pm i(1+g^2)^{1/2}$, and eigenvectors,

$$\begin{pmatrix} \tilde{Q}_0 \\ \tilde{U}_0 \\ \tilde{V}_0 \end{pmatrix} = \pm \frac{1}{(1+g^2)^{1/2}} \begin{pmatrix} 1 \\ 0 \\ g \end{pmatrix}, \tag{15}$$

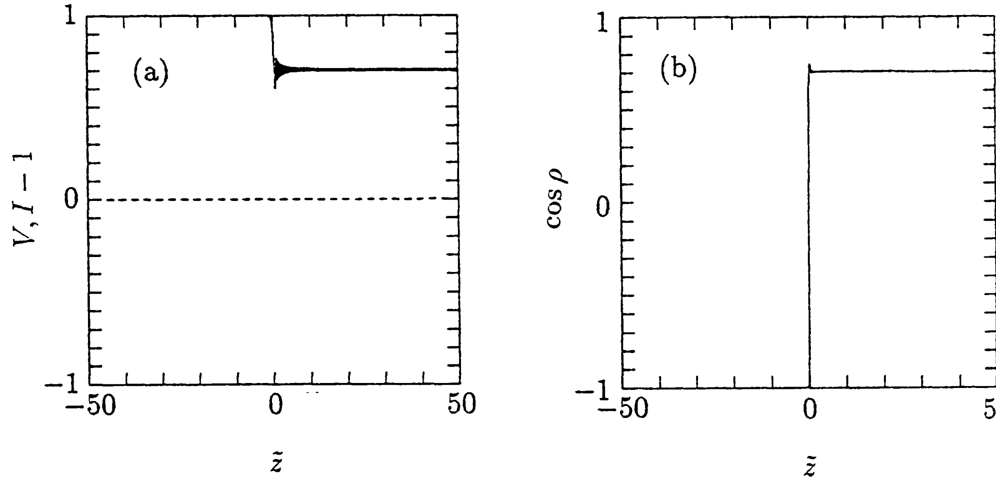

Fig. 5. As for Figure 2 except for $\phi' = 0$, $f_\infty = 100$, and $\tilde{z}_0 = 10$, giving $|f_\infty|/\tilde{z}_0 = 10$.

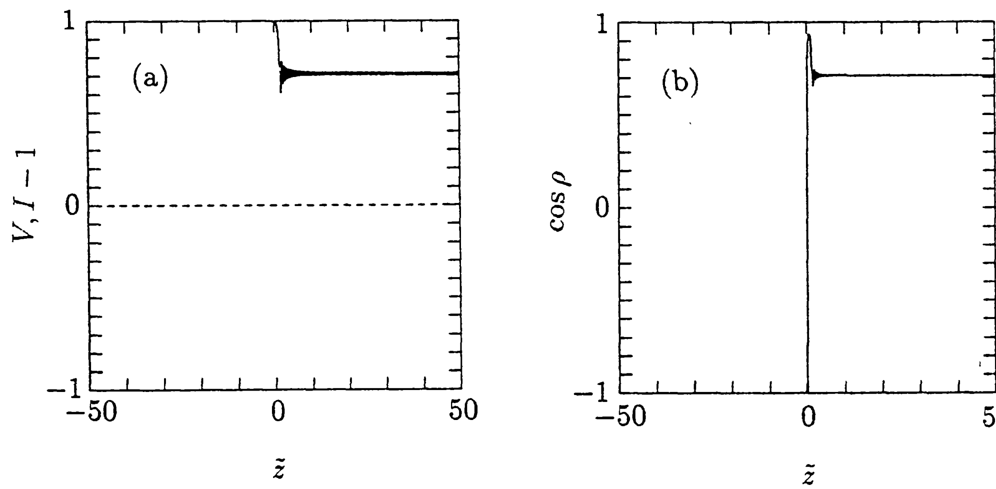

Fig. 6. As for Figure 2 except for $\phi' = 5$, $f_\infty = 100$, and $\tilde{z}_0 = 10$, giving $|f_\infty|/\tilde{z}_0 = 10$.

are different from those given by constructing $\tilde{Q}_0$ and $\tilde{U}_0$ from (7) using (12). The two are related by $f \leftrightarrow g$. This raises the question as to which are the appropriate natural modes to choose: (7) for a locally homogeneous system, or (15) for a system that incorporates the twist? The ambiguity implied by this question is related to the fact that it is not possible in general to define modes uniquely in a inhomogeneous system. The differences between these two choices are real. For example, the two definitions give different results for the angle between the modal axis and the polarization point: the angle $\tilde{\rho}$, defined by $\cos\tilde{\rho} = \tilde{Q}_0\tilde{Q} + \tilde{U}_0\tilde{U} + \tilde{V}_0 V$, is not equal to the angle ρ defined by (8), except for an untwisted field ($\phi' = 0$). In mode coupling theory one is concerned with the effect of inhomogeneities in causing a coupling between the modes in a locally homogeneous system. Hence, it is clear that the eigenvectors (8) are to be used to define the modes when discussing mode coupling, and the eigenvectors (15) are not to be used for this purpose. From this

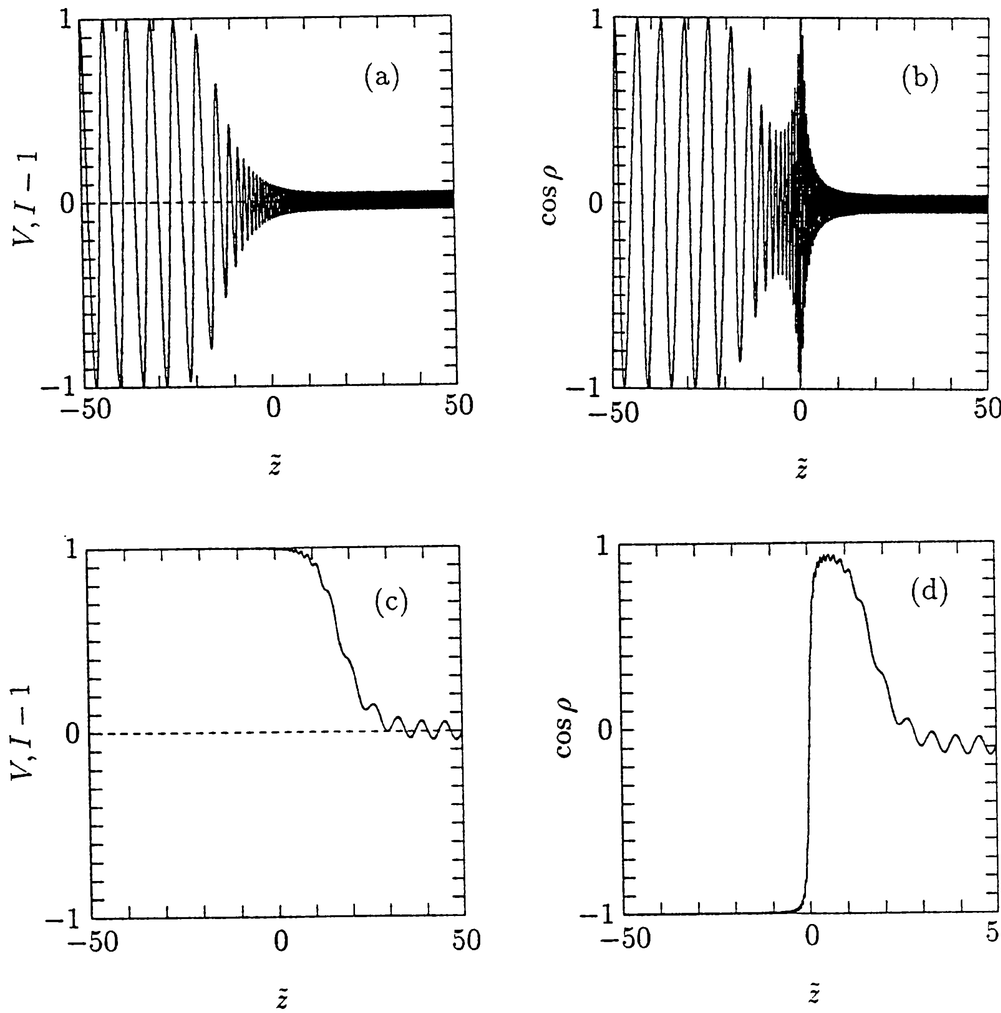

Fig. 7. As for Figure 4 except for $\phi' = 5$, $|f_\infty| = 10$, and $\tilde{z}_0 = 10$, giving $|f_\infty|/\tilde{z}_0 = 10$, with (a, b) $f_\infty = -10$, and (c, d) $f_\infty = +10$.

viewpoint, the eigenvectors (15) are simply for mathematical convenience and are not regarded as being of physical significance in themselves.

3.2. Resonant Twisting

For the particular case of a uniform twist ($\phi' =$ constant) in an otherwise uniform plasma ($f =$ constant), (13) implies a resonant behavior of the generalized Faraday rotation for $f = 2\phi'$. At this resonance, the rotation of the plane of linear polarization due to the Faraday effect occurs at the same rate as the change in the orientation of the magnetic field due to the twisting. This may be seen by noting that the transformation (12) corresponds to transforming to effective Stokes parameters as would be measured in a frame rotating with the twist in the magnetic field. At the resonance (13) one has $g = 0$, which implies $\tilde{Q} =$ constant. Then generalized Faraday reduces to a mathematical form equivalent to that which occurs in a quarter-wave plate. That is, $\tilde{U}$ and $\tilde{V}$ oscillate periodically with $\tilde{U}^2 + \tilde{V}^2 =$ constant.

3.3. Mode coupling in a twisted field

In comparing the twisted and untwisted cases, there are three important qualitative differences between (13) and (9). First, the function g, unlike the function f, need not pass through zero; specifically, for the form (10) and $\phi' =$ constant, there is no zero for $|f_\infty| < 2|\phi'|$. Second, if g does have a zero, then this zero is not at the center, $\tilde{z} = 0$, of the QT region. Third, the ellipticity of the natural modes still changes as in the untwisted case, so that the modal axis crosses the equator at $\tilde{z} = 0$, but this change is now partly decorrelated from the changes associated with g passing through zero.

The effect of $\phi' \neq 0$ on mode coupling is relatively unimportant provided that the ratio $|\phi'/f_\infty|$ is small. This may be seen by noting that $|\phi'| \ll |f_\infty|$ implies that the difference between f and g is $\ll |f_\infty|$. Hence, the point where g vanishes, $\tilde{z}_g$ say, is close to $\tilde{z} = 0$. As $|\phi'/f_\infty|$ is increased the resonance is approached and one expects large variations in polarization near the resonance (e.g., Ulrich and Simon, 1979).

In the opposite limit, $|\phi'| \gg |f_\infty|$, one has $g \approx \phi'$, implying that g has no zero and that it remains approximately constant across the QT region. Hence, essentially the only change in the polarization that occurs is due to generalized Faraday rotation, in the form implied by (13). That is, $\tilde{Q}$, $\tilde{U}$, and $\tilde{V}$ continue to vary approximately periodically in $\tilde{z}$ at the rate $(1 + g^2)^{1/2}$ as the QT region is crossed. Near $\tilde{z} = 0$ the modal axis rotates through $\approx \pi$. This suggests (and numerical calculations discussed below confirm) that the polarization of the radiation evolves essentially ignoring this reorientation of the modal axis.

The characteristic feature of strong mode coupling is that the reversal of the sense of circular polarization of the natural modes has little effect on the polarization of the radiation. Hence, the regime $|\phi'| \gg |f_\infty|$ corresponds to a new and distinct region of strong mode coupling, due specifically to the twisting of the magnetic field.

3.4. Numerical results ($\phi' \neq 0$))

In our numerical calculations for a twisted field we fix $\tilde{z}_0 = 10$ and $\phi' = 5$, and consider a range of values of f_∞. The resonance occurs for $|f_\infty| \geq 2\phi' = 10$.

Figure 6 shows results for $f_\infty = 100$, which corresponds to a weak twist and to strong mode coupling analogous to the untwisted case illustrated in Figure 3. Numerical results for $f_\infty = -100$ are almost indistinguishable from those for $f_\infty = 100$ (except for the sign of $\cos\rho$), and are not shown.

Figure 7 shows results for $|f_\infty| = 10$, when the resonance is encountered but only asymptotically, far from the center of the QT region. For $f_\infty = -10$ (Figure 7(b)) the resonance is encountered at the initial point, and the initial circular polarization oscillates between right and left circular; Q and U have similar oscillations with the ellipticity taking on all values periodically. The emerging radiation has essentially no circular component. The parameter $\cos\rho$ has a complicated behavior: it oscillates with unit amplitude for $\tilde{z} \ll -\tilde{z}_0$, then the amplitude of the oscillations decreases for $|\tilde{z}| \lesssim \tilde{z}_0$ and increases again as $\tilde{z} = 0$ is approached, and then decreases to a small amplitude on the other side of the QT region. Such complicated behavior is due in part to the intrinsically complicated nature of the resonant system, and in part due to the choice of initial conditions. For $f_\infty = 10$ (Figures 7(c) and 7(d)) the resonance is encountered as the radiation leaves the QT region, and the behavior of $\cos\rho$ is quite different: it is unaffected until $\tilde{z} = 0$ is approached where it nearly reverses in sign, and then changes to a value close to zero for $\tilde{z} \gtrsim \tilde{z}_0$ as the resonance is approached.

Figure 8 shows results for $f_\infty = \pm 1$ and $2\phi' > 1$ where no resonance occurs. For $|f_\infty| \ll 2\phi'$ there is negligible change in the polarization, corresponding to strong mode coupling. This strong mode coupling is not implied by (11) because we have $C = |f_\infty|/\tilde{z}_0 \ll 1$ in this case. As explained above, this case corresponds to a new regime of strong mode coupling associated with a twisted magnetic field.

Figure 9 illustrates how the additional region of strong mode coupling arises as ϕ' is increased. In the absence of any twist ($\phi' = 0$), the circular polarization changes by $\Delta V = 2$ for $|f_\infty|/z_0 = 0$, corresponding to weak mode coupling and a complete reversal of V across the QT region, and decreases with increasing $|f_\infty|/z_0$, so that for $|f_\infty|/z_0 \gg 1$ one has $\Delta V \ll 1$, corresponding to strong mode coupling. As shown by Melrose and Robinson (1994), the numerical results in this case closely follow the analytic form $\Delta V = 2(1 - e^{-x})$, $x = \pi|f_\infty|/2\tilde{z}_0$ (Zheleznyakov and Zlotnik, 1963). For $\phi' \neq 0$ an additional region of strong mode coupling ($\Delta V \ll 1$) occurs at $|f_\infty|/z_0 < 2\phi'$. As ϕ' is increased, the region of weak mode coupling shrinks, and disappears for $\phi' \gg 30$.

4. Application to Bipolar Radio Sources

Our results show that there is a previously unrecognized regime of strong mode coupling at a QT region associated with a twist in the magnetic field. In this section we consider whether this type of mode coupling might resolve the discrepancy discussed above between theory and observation of radio emission from bipolar regions in the solar corona. We start by comparing the conditions required for mode coupling to be strong due to a twist with the corresponding condition for an untwisted field.

4.1. Condition for Strong Mode Coupling due to a Twist

The newly recognized regime of strong mode coupling requires that the twist satisfy

$$|\mathrm{d}\phi/\mathrm{d}z| > \tfrac{1}{2}|\rho_V| \, . \tag{16}$$

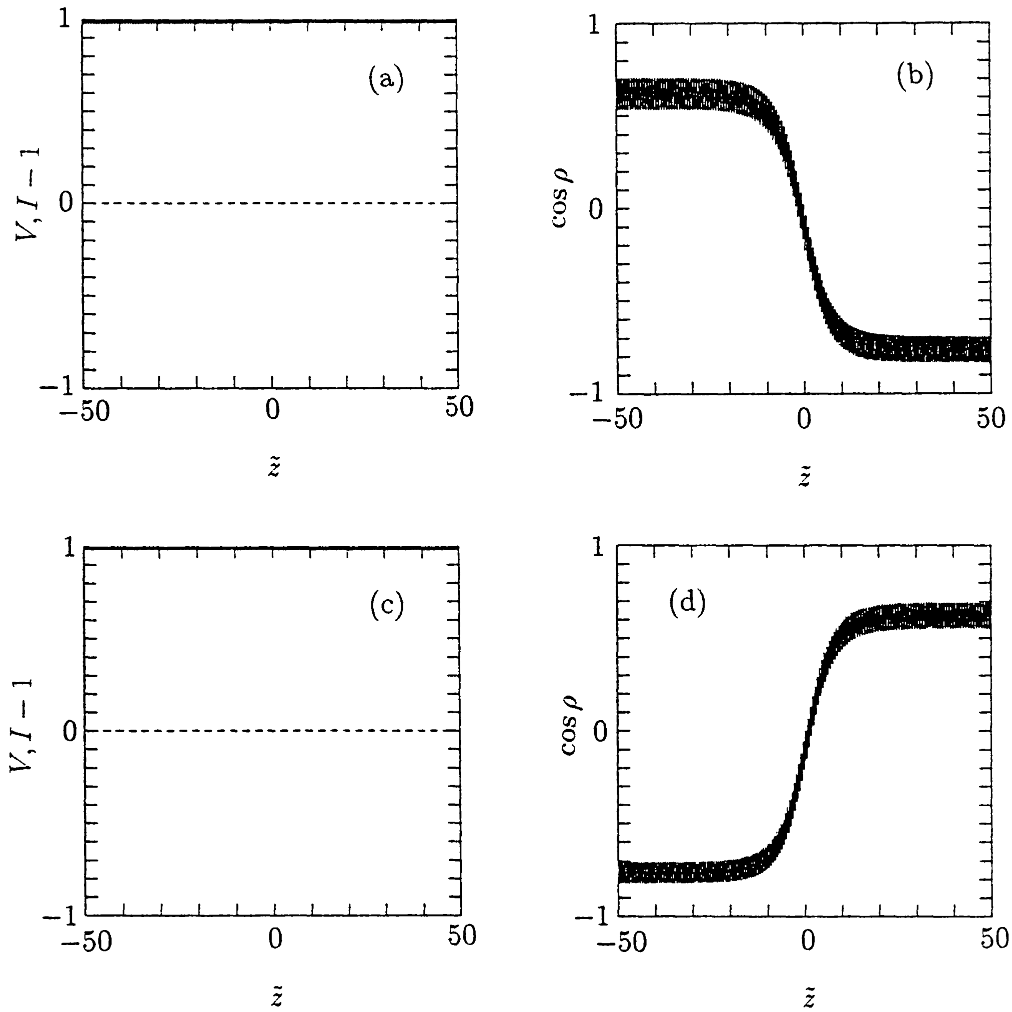

Fig. 8. As for Figure 5 except for $\phi' = 5$, $f_{\infty|} = 1$, and $\tilde{z}_0 = 10$, giving $|f_\infty|/\tilde{z}_0 = 10$, with (a, b) $f_\infty = -1$, and (c, d) $f_\infty = +1$.

That is, the rate of twisting, $d\phi/dz$, of the magnetic field with distance, z, along the ray path must exceed the rate $2\rho_V$, at which Faraday rotation of the plane of linear polarization occurs outside the QT region. Physically, this condition may be interpreted in terms of the induced circular birefringence due to the twist (e.g., Ulrich and Simon, 1979) causing the effective modes in the twisted field to remain approximately circularly polarized. From this viewpoint, the natural modes of a locally homogeneous medium are a poor approximation to the effective modes in the inhomogeneous medium, cf. (7) and (15). The QT region is then of little physical significance because the dominant induced circular birefringence is unaffected when the ray passes through $\theta = \pi/2$.

The condition (16) is to be applied at the boundary of the QT region. This boundary may be identified as the region where the angle θ, which passes through $\pi/2$ at the center of the QT region, has some characteristic value θ_0. We identify $\theta = \theta_0$ as the point at which the modes in a locally homogeneous medium satisfy

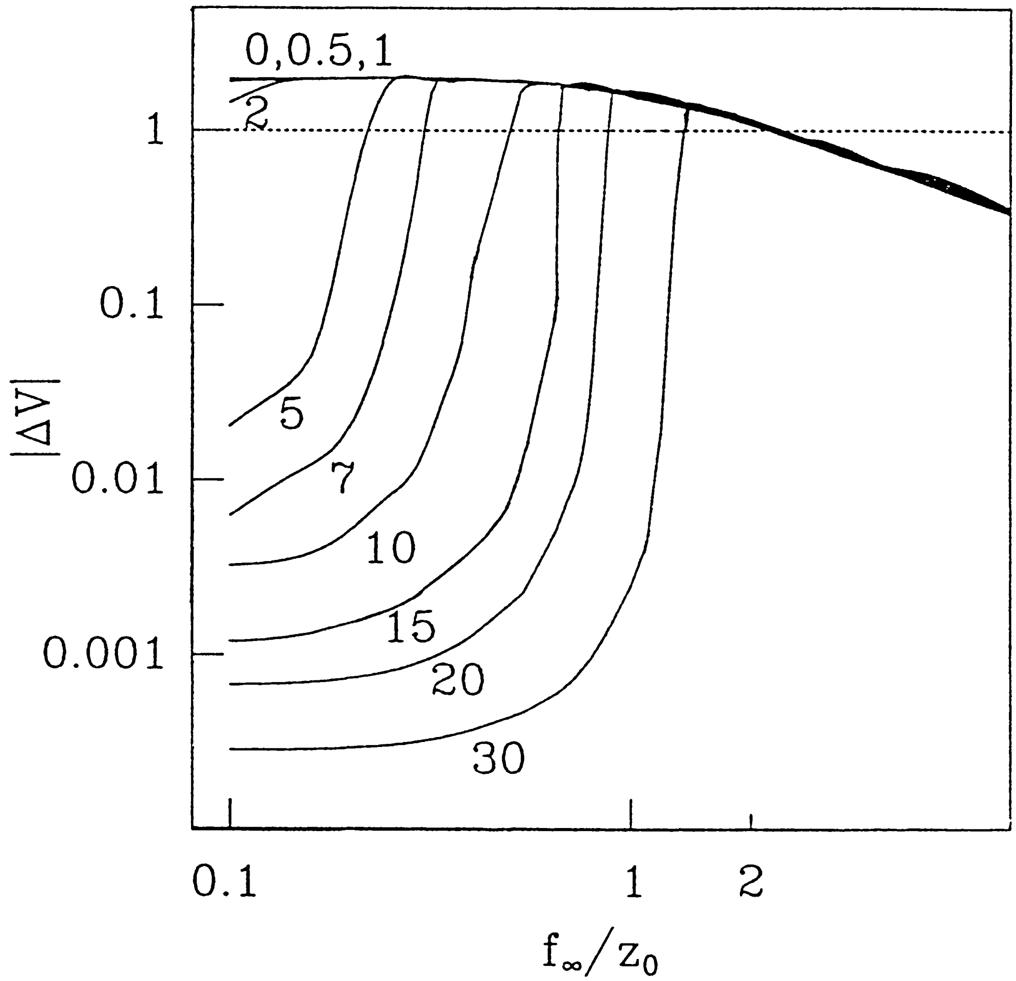

Fig. 9. The change ΔV in V as a function of $|f_\infty|/\tilde{z}_0$ for several values of ϕ', as labelled on the curves. The dashed line separates regions of weak mode coupling (above) from regions of strong mode coupling (below).

$\rho_V = (\rho_Q^2 + \rho_U^2)^{1/2}$. This corresponds to the point $2\chi_0 = 45°$ that separates the 'quasi-circular' approximation to the polarization of the modes ($2\chi_0 < 45°$ and $2\chi_0 > 135°$) from the 'quasi-linear' approximation ($45° < 2\chi_0 < 135°$). Then, outside the QT region the only important effect is Faraday rotation due to the circular birefringence, which does not affect the circular polarization, and inside the QT region the important effect is the linear birefringence, which converts circular into linear polarization and back in a periodic manner.

With this definition of θ_0, (3) implies $\cos\theta_0 \approx \Omega_e/2\omega$ for $\Omega_e \ll \omega$. Thus the condition (16) reduces to

$$\left|\frac{\mathrm{d}\phi}{\mathrm{d}z}\right| \gtrsim \frac{\omega_p^2\Omega_e^2}{4c\omega^3} . \tag{17}$$

One may invert (17) to define a characteristic frequency above which this type of mode coupling is strong:

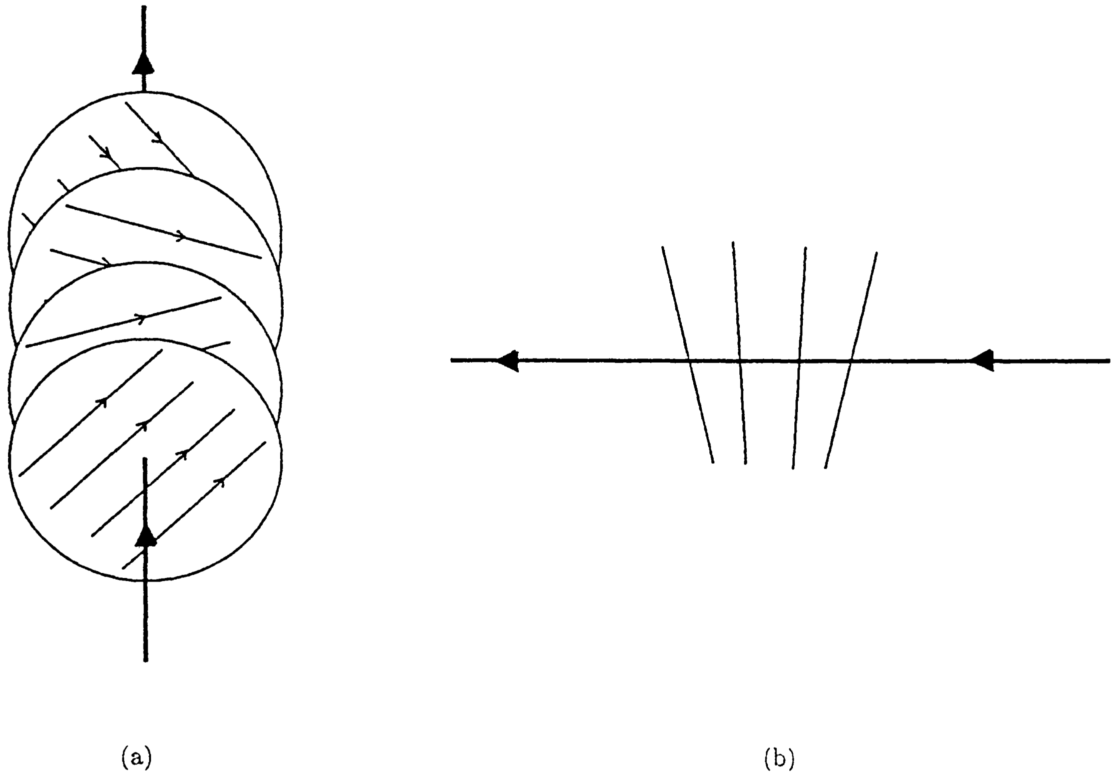

Fig. 10. (a) Four planes showing the orientation of the magnetic field lines along the ray path (solid line with arrow); (b) side view showing the fanning of the planes in (a).

$$\omega_{t2} = \left(\frac{\omega_p^2 \Omega_e^2}{4c\,|\mathrm{d}\phi/\mathrm{d}z|} \right)^{1/3} . \tag{18}$$

For comparison, the transition frequency implied by $C = 1$ in (11) is

$$\omega_{t1} = \left(\frac{\omega_p^2 \Omega_e^3}{c\,|\mathrm{d}\theta/\mathrm{d}z|} \right)^{1/4} . \tag{19}$$

Let us compare the ratios of these transition frequencies to Ω_e for comparable values of $|\mathrm{d}\phi/\mathrm{d}z| = 1/L_\phi$ and $|\mathrm{d}\theta/\mathrm{d}z| = 1/L_\theta$. Then (18) gives $\omega_{t2}/\Omega_e = (\omega_p^2 L_\phi/4c\Omega_e)^{1/3}$ and (19) gives $\omega_{t1}/\Omega_e = (\omega_p^2 L_\theta/4c\Omega_e)^{1/4}$. Thus ω_{t2}/Ω_e and ω_{t1}/Ω_e are determined by the one third and one fourth powers, respectively, of the large numbers $\omega_p^2 L_\phi/4c\Omega_e$ and $\omega_p^2 L_\theta/c\Omega_e$. It follows that for $L_\phi \sim L_\theta$ one has $\omega_{t2} > \omega_{t1}$, that is, the transition frequency is higher for the twisted case than for the untwisted case. Hence, mode coupling due to a twisted field can be the more effective only for $L_\phi \ll L_\theta$. Thus, to have $\omega_{t2} < \omega_{t1}$ requires $L_\phi/L_\theta < 4(\omega_p^2 L_\theta/c\Omega_e)^{-3/4}$.

The gradients $|\mathrm{d}\phi/\mathrm{d}z|$ and $|\mathrm{d}\theta/\mathrm{d}z|$ are determined by different properties of the magnetic field. The gradient in θ is limited by $\mathrm{div}\,\mathbf{B} = 0$, which implies $\mathrm{d}(B\cos\theta)/\mathrm{d}z = 0$, so that one has $|\mathrm{d}\theta/\mathrm{d}z| = |\mathrm{d}B/\mathrm{d}z|$ at $\theta = \pi/2$. Hence, the characteristic length, L_θ, is the same as the characteristic length L_B over which B changes. The gradient in ϕ is determined by $|\mathrm{curl}\,\mathbf{B}|/B$. Thus, in order to have $L_\phi \ll L_\theta$ one requires $|\mathrm{curl}\,\mathbf{B}| \gg |\mathrm{grad}\,B|$.

4.2. A Model for a Twisted Field

A simple model for the structure of the magnetic field that can produce a twist of the form required in the QT region is illustrated schematically in Figure 10. This model has field lines confined to planes with (a) a shear, in the sense that the orientation of the magnetic field rotates from one plane to the next, and (b) a fan in the planes, such that the planes intersect on a line a distance L away. The shear in the magnetic field corresponds to a twist for $\theta \approx \pi/2$, and the fanning of the planes corresponds to a variation of θ with distance along the ray path. Thus, for mode coupling due to a twist in the field to be effective for radio emission from a bipolar region, the ray from the more distant footpoint must encounter a QT region where the magnetic field is sheared in the sense illustrated in Figure 10.

A twist or shear in a magnetic field implies a nonzero local current density $\mathbf{J}$. For the model illustrated in Figure 10, at $\theta = \pi/2$ one finds

$$\left|\frac{\mathrm{d}\phi}{\mathrm{d}z}\right| = \frac{\mu_0\,|\mathbf{B}\cdot\mathbf{J}|}{B^2}\,. \tag{20}$$

Magnetic fields in the solar corona are approximately force free, implying that $\mathbf{J} = \mathrm{curl}\,\mathbf{B}/\mu_0$ is along $\mathbf{B}$. Then, using (3) and (20), the condition (17) reduces to

$$\frac{\mu_0 J}{B} \gtrsim \frac{\omega_p^2\Omega_e^2}{c\omega^3}\,. \tag{21}$$

The ratio J/B is approximately constant along a flux tube, due to J and B both varying inversely as the area of the flux tube.

4.3. Shearing Solar Flux Tubes

The current density in a possible QT region in a sheared coronal magnetic structure may be estimated from vector magnetogram data on the current density near the photosphere. Current densities as high as $\sim 2\times 10^{-2}$ A m^{-2} are found to flow through the photosphere (e.g., de la Beaujardière, Canfield, and Leka, 1993), where the magnetic field is $B \gtrsim 0.1$ T. As the current flows along magnetic field lines through the corona, both J and B should vary inversely as the cross-sectional area of the magnetic flux tube, implying $J \sim B$. On inserting $J = 2\times 10^{-2}$ A m^{-2} and $B = 0.1$ T in (21), one can estimate the frequency below which mode coupling is strong due to twisting. For example, with $\omega_p/2\pi = 10$ MHz and $\Omega_e/2\pi = 3$ MHz, one finds strong coupling only at $\omega/2\pi > 500$ MHz. Thus the twisting implied by the measured currents is too small to be important for mode coupling. Hence, if strong mode coupling is attributed to twisting or shearing of the magnetic field, this must be much stronger than implied by the average current densities estimated from vector magnetograms.

It may be that there are much smaller-scale twisting or shearing of the magnetic field, and that these might contribute to the mode coupling. Small-scale twists and shears tend to propagate away as Alfvén waves, and a discussion of mode coupling due to small-scale structures should include a discussion of the spectrum of such waves. We do not consider this here.

5. Conclusions

In this paper we use a new method to treat mode coupling in an inhomogeneous plasma. This method involves numerically integrating the equation for the transfer of the Stokes parameters through an inhomogeneous plasma. We use this method to treat mode coupling at a QT region for a twisted magnetic field. The main result is that the twist introduces a new regime where mode coupling is strong and explicitly dependent on the amount of twisting. This regime correspond to the rate of rotation of the plane of polarization due to Faraday rotation being less than the rate of rotation of the polarization vectors of the natural modes due to the twisting of the magnetic field lines. We define a transition frequency ω_{t2}, given by (18), above which this type of mode coupling is important, and compare it with the transition frequency ω_{t1}, given by (19), for the traditional form of mode coupling.

We discuss the application of this form of mode coupling in the solar corona. There is a long-standing problem in that mode coupling appears to be strong at frequency where simple theory suggests that it should be weak. Put another way, the estimate of ω_{t1} that one infers from observation is much lower than one estimates from theory. However, in order for the transition frequency ω_{t2} to be low enough to account for the strong mode coupling inferred for solar bipolar regions, one needs to appeal to conditions similar to those required for ω_{t1} to be correspondingly low. That is, one requires that the QT region be located where the magnetic field is weak with a large spatial gradient. A qualitative change is the recognition that mode coupling due to a twist allows the spatial variation in $\mathbf{B}$ to be due to curl $\mathbf{B}$, rather than to grad B as required by the traditional form of mode coupling. The new regime of mode coupling recognized here does not lead to an obvious solution of this long-standing problem.

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