Publ. Astron. Soc. Aust., 1996, 13, 144–9

Langmuir Wave Emission by Neutrinos in a Medium

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Received 1995 November 23, accepted 1996 February 13

Abstract: Expressions for the probability per unit time of the emission of a Langmuir wave by both massive and massless neutrinos moving in a medium are derived using finite-temperature field theory. A comparison between this process and the analogous emission from an electron is made, and possible implications of this process for the explosion mechanisms of type II supernovae are briefly discussed.

Keywords: neutrinos — plasmas — stars: supernovae : general

1 Introduction

The electromagnetic properties of neutrinos propagating through a medium have received wide attention in the recent literature (Oraevsky & Semikoz 1987; D'Olivo, Nieves & Pal 1989; Nieves & Pal 1994). It has been shown that neutrinos acquire an induced electric charge due to resonant interactions with the electrons in the medium allowing them to couple to the electromagnetic field, albeit weakly. This was noted before the introduction of the electroweak theory. Tsytovich (1964) determined the rate at which Langmuir waves were absorbed by neutrinos in a medium using a four-point effective Hamiltonian approach. These calculations were superseded by the advent of electroweak theory, which has been used to calculate the coupling of transverse waves in a medium to both massless (Kim & Pevsner 1993) and massive neutrinos (D'Olivo, Nieves & Pal 1990; Giunti, Kim & Lam 1991).

Recently, a scenario has been proposed in which these interactions hold astrophysical significance. Bingham et al. (1994; hereinafter BDSB) suggested that the production of Langmuir waves through a neutrino beam instability may be an effective method of depositing energy and momentum from the intense neutrino flux generated in a type II supernova (SN) into the plasma behind the SN shock wave, thereby allowing the explosion to proceed. However, BDSB's analysis of this effect relied on an effective potential method (Bethe 1986; D'Olivo, Nieves & Torres 1992) to model the neutrino–Langmuir wave coupling, a method which, in this circumstance, obscures the physical processes occurring. Furthermore, BDSB's calculation is performed using a monoenergetic neutrino distribution, which is overly restrictive in view of the approximately thermal spectrum expected to be emitted from the SN core.

In this paper we show that the probability for neutrino–Langmuir wave interaction may be calculated explicitly through finite-temperature field theory (FTFT) (Dolan & Jakiw 1974; Weldon 1982). We present here an explicit calculation of the probability of the emission of a longitudinal plasmon (in particular a Langmuir wave) by a neutrino in a medium to leading order in $1/M_{\rm W}^2$, where $M_{\rm W}$ is the mass of the W boson. The dispersive effects of matter on the neutrinos are included. We propose to use this result to perform a rigorous calculation of the growth rate of Langmuir waves due to a thermal distribution of neutrinos (Hardy & Melrose, in preparation). We begin with a brief discussion of the current theory of type II SNe (Bethe 1990).

A type II SN may occur at the end of the life of a massive star, when the star has completed the nuclear burning process, leaving a predominantly iron core. The core evolves until it is no longer able to thermally support its own mass against gravity and thus collapses. This collapse continues until the core densities reach approximately 10^{17} kg m^{-3} where the infall halts due to a sudden stiffening of the equation of state. During the collapse, electrons and protons coalesce to form neutrons and neutrinos. The neutrons eventually form a compact object. The neutrinos are initially free to escape the star, providing significant cooling. However, as the density climbs, the core material becomes opaque to neutrinos, leading to their trapping and thermalisation within the core.

The neutrinos thus trapped slowly diffuse outwards, to a region where the density is low enough to allow them to escape. This region is a sphere at a radius of ~70 km called the neutrinosphere. The neutrinos generated in the core appear to propagate from the neutrinosphere with a temperature of typically a few MeV and a flux of around 10^{38} W m^{-2}. Within a few seconds of the collapse, the neutrinos generated in this way absorb around 99% of the gravitational binding energy released in the collapse of the star.

Some of the remaining energy appears in the form of a bounce shock, which propagates away from the core when the stellar collapse halts. It is this shock which eventually blows away the outer layers of the star to form a visible SN remnant. Unfortunately, there is a problem with the energetics of the bounce shock. As the shock propagates through infalling material, it dissociates this material into its constituent nuclei, and this removes energy from the shock. In many simulations, for a large range of progenitor masses, this causes the shock to stall so that no visible explosion occurs and a black hole rather than a neutron star forms (Baron & Cooperstein 1990; Bethe 1990).

The problem of the stalling of the bounce shock was partially resolved when Wilson (1984) postulated that scattering between neutrinos and nuclei behind the shock deposits energy and momentum in this material, driving the shock outwards. However, this process has proved only marginally capable of providing the required heating for a class of observationally interesting cases of progenitor mass. It has been surmised (Bethe 1990) that around 1% of the neutrino energy needs to deposited behind the shock to ensure the success of the explosion.

More recently, two-dimensional models of the fluid transport in the SN envelope have suggested that convection plays an important role in the dynamics of shock propagation, perhaps even allowing the shock to escape (Miller, Wilson & Mayle 1993; Burrows, Hayes & Fryxell 1995). However, the heat deposition mechanism which allows the convective overturn to take place is again the scattering of neutrinos off nuclei in the post-shock region. Thus it remains important to examine other sources of energy deposition that might disrupt or supplant this process.

In Section 2 we calculate the probability per unit time of the emission of a Langmuir wave by a massless neutrino travelling through an isotropic medium. The case of massive neutrinos is discussed in Section 3. Emission by neutrinos is compared with emission by electrons and photons in Section 4. Natural units ($c = \hbar = 1$) are used throughout.

2 Emission by Massless Neutrinos

The emission of a plasmon by a massless neutrino may be represented diagrammatically as in Figure 1, where $p = (\varepsilon, \mathbf{p})$ and p' are the momenta of the incoming and outgoing neutrinos respectively, and $k = (\omega, \mathbf{k})$ is the plasmon momentum. *In vacuo* both these processes are forbidden kinematically. However, in a medium the internal electron lines (propagators) may be replaced by averages over the electron distribution in the medium, and the process is then allowed. This average is inherent in FTFT, which leads to a scattering matrix for plasmon emission which may be written (D'Olivo et al. 1989)

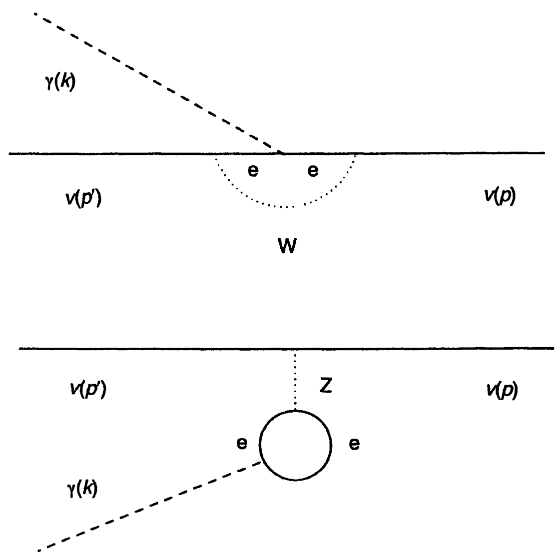

Figure 1—Emission of a plasmon from a neutrino through one-loop diagrams in the standard model.

$$M_{\rm fi} = \frac{G_{\rm F}}{\sqrt{s}}\frac{ic_V}{\sqrt{4\pi\alpha}}\,\bar{\nu}\gamma_\mu(1-\gamma_5)\nu\alpha^{\mu\nu}(k)\,\varepsilon_\nu(k)\,, \quad (1)$$

where $\alpha^{\mu\nu}(k)$ is the polarisation tensor for the electron gas, $\varepsilon_\nu(k)$ represents the polarisation 4-vector of the longitudinal Langmuir wave (plasmon), $G_{\rm F}$ is the Fermi constant, α is the fine-structure constant, and

$$c_V = \begin{cases} 2\sin^2\theta_{\rm W} + \frac{1}{2} & \text{for } \nu_{\rm e}, \\ 2\sin^2\theta_{\rm W} - \frac{1}{2} & \text{for } \nu_\mu, \nu_\tau\,. \end{cases} \quad (2)$$

2·1 Kinematics

The presence of the medium also modifies the energy–momentum relations for the plasmons and the neutrinos, as well as the normalisation of the plane wave states. For a Langmuir wave in a thermal nonrelativistic medium, the dispersion relation is

$$\omega_{\rm L}(\mathbf{k}) = (\omega_{\rm p}^2 + 3|\mathbf{k}|^2 V_{\rm e}^2)^{1/2}\,, \quad (3)$$

and the normalisation factor is

$$|a_M(\mathbf{k})| = \left(\frac{\mu_0\,\omega_{\rm L}(\mathbf{k})}{2V\omega_{\rm p}^2}\right)^{1/2}, \tag{4}$$

where $\omega_{\rm p}$ is the plasma frequency, $V_{\rm e}$ is the thermal speed of the electrons in the medium (Melrose 1986), and V is a volume normalisation factor. For a relativistic medium, equation (3) must be replaced by the relativistic dispersion relation (Silin 1960). In the ultrarelativistic limit, this is given by the solutions to the transcendental equations (Braaten 1991)

$$\omega_{\rm L}(\mathbf{k})^2 = \omega_{\rm p}^2\frac{3\omega_{\rm L}(\mathbf{k})^2}{|\mathbf{k}|^2} \times \left(\frac{\omega_{\rm L}(\mathbf{k})}{2|\mathbf{k}|}\ln\frac{\omega_{\rm L}(\mathbf{k})+|\mathbf{k}|}{\omega_{\rm L}(\mathbf{k})-|\mathbf{k}|} - 1\right). \tag{5}$$

The plasma frequency, $\omega_{\rm p}$, which appears in equation (5) is modified from the nonrelativistic expression which appears in equation (3) to take into account effects of order $T_{\rm e}/m_{\rm e}$, where $T_{\rm e}$ is the temperature of the relativistic plasma.

For the neutrinos, the dispersion relation and normalisation factor have been evaluated by Nieves (1989), who found that to lowest order in $1/M_{\rm W}^2$ the normalisation factor is unity and the dispersion relation is given by

$$\varepsilon(\mathbf{p}) = |\mathbf{p}| + \sqrt{2}G_{\rm F}\,n_-\,, \tag{6}$$

where n_- is the number density of electrons. Assuming an electron density of $n_- = 10^{36}\ {\rm m}^{-3}$ (as used by BDSB), the correction due to dispersion to a neutrino with energy around 1 eV is less than 10^{-7} eV and is therefore negligible. Thus we may use both the vacuum normalisation and the vacuum dispersion relation for the neutrino plane wave states. Hence, for massless neutrinos, $p^2 = p'^2 = 0$ and conservation of momentum, $p' = p - k$, leads to

$$pk = \tfrac{1}{2}k^2\,. \tag{7}$$

2.2 Calculation of Probability

The electromagnetic contribution to the scattering matrix element, equation (1), may be simplified by considering only the longitudinal part of the linear response tensor, $\alpha^{\mu\nu}(k)$, as it is this component which couples to the longitudinal plasmons (Braaten & Segel 1993). We choose the temporal gauge and write the polarisation 4-vector for the plasmons as

$$\varepsilon^\nu(k) = \frac{k^\nu - (ku)u^\nu}{[(ku)^2 - k^2]^{\frac{1}{2}}} = \left(0, \frac{\mathbf{k}}{|\mathbf{k}|}\right)^\nu, \tag{8}$$

where the second form of the expression is evaluated in the rest frame of the plasma, that is, the frame in which the 4-velocity of the plasma is $u^\mu = (1, \mathbf{0})^\mu$.

The longitudinal part of the linear response tensor is given by (Melrose 1982; Nieves & Pal 1989)

$$\alpha^{\rm L}(k) = \frac{(ku)^4}{k^4}\,L_{\mu\nu}(k,u)\alpha^{\mu\nu}(k)\,, \tag{9}$$

where

$$L^{\mu\nu} = -\frac{k^2}{k^2-(ku)^2} \times \left\{\frac{k^\nu u^\mu}{ku} + \frac{k^\mu u^\nu}{ku} - \frac{k^2 u^\mu u^\nu}{(ku)^2} - \frac{k^\mu k^\nu}{k^2}\right\}. \tag{10}$$

For an isotropic plasma, the longitudinal part of the linear response tensor may be reconstructed through

$$\alpha^{\mu\nu}(k) \approx \alpha^{\rm L}(k)L^{\mu\nu}(k,u)\,. \tag{11}$$

Thus we may write

$$\begin{aligned}\alpha^{\mu\nu}(k)\varepsilon_\nu(k) &= \alpha^{\rm L}(k)\,L^{\mu\nu}(k,u)\varepsilon_\nu(k,u)\\ &= -\frac{1}{\mu_0}\frac{(ku)^2k^\mu - (ku)k^2u^\mu}{[(ku)^2-k^2]^{\frac{1}{2}}}\\ &= -\frac{\omega^2}{\mu_0}\left(\frac{|\mathbf{k}|}{\omega}, \frac{\mathbf{k}}{|\mathbf{k}|}\right)^\mu,\end{aligned} \tag{12}$$

where the final expression applies in the rest frame, and where we use the dispersion relation for the plasmons in the form (Melrose 1986)

$$\alpha^{\rm L}(k) = -\frac{(ku)^2}{\mu_0} = -\frac{\omega^2}{\mu_0}. \tag{13}$$

The probability per unit time of the emission of a Langmuir wave may be written as

$$\omega_{\rm L}(\mathbf{p},\mathbf{k}) = |M_{\rm fi}|^2\frac{V|a_M(\mathbf{k})|^2}{2\varepsilon(\mathbf{p})\,2\varepsilon'(\mathbf{p})} \times 2\pi\delta[\varepsilon'(\mathbf{p}) + \omega_{\rm L}(\mathbf{k}) - \varepsilon(\mathbf{p})]\,, \tag{14}$$

where $\varepsilon(\mathbf{p})$ and $\omega_{\rm L}(\mathbf{k})$ denote the neutrino and Langmuir wave dispersion formulae, equations (6) and (3) respectively.

Explicit evaluation of $|M_{\rm fi}|^2$ through equation (12) gives

$$|M_{\rm fi}|^2 = \frac{G_{\rm F}^2}{2}\frac{c_{\rm V}^2}{4\pi\alpha}|\mathbf{k}|^2\omega^2\,M_{\mu\nu}\,N^{\mu\nu}\,, \tag{15}$$

with

$$M_{\mu\nu} = 8[2p_\mu p_\nu - k_\nu p_\mu - p_\nu k_\mu + (pk)g_{\mu\nu} + i\varepsilon_{\alpha\nu\beta\mu}k^\alpha p^\beta]\,, \qquad (16)$$

and

$$N^{\mu\nu} = \left(1, \frac{\omega\mathbf{k}}{|\mathbf{k}|^2}\right)^\mu \left(1, \frac{\omega\mathbf{k}}{|\mathbf{k}|^2}\right)^\nu, \qquad (17)$$

where $p^\mu = (\varepsilon(\mathbf{p}), \mathbf{p})^\mu$ and $k^\mu = (\omega_\mathrm{L}(\mathbf{k}), \mathbf{k})^\mu$, and $\omega = \omega_\mathrm{L}(\mathbf{k})$ is understood from now on. Gathering terms and substituting equation (7) yields

$$\omega_\mathrm{L}(\mathbf{p}, \mathbf{k}) = \frac{G_\mathrm{F}^2 c_V^2}{16\pi\alpha} \frac{|\mathbf{k}|^2}{\omega_\mathrm{p}^2} \frac{\omega^3}{\varepsilon\varepsilon'} 2\pi\delta(\varepsilon' + \omega - \varepsilon) \times [(2\varepsilon - \omega)^2 - |\mathbf{k}|^2]\left(1 - \frac{\omega^2}{|\mathbf{k}|^2}\right)^2, \qquad (18)$$

which is the probability per unit time of the emission of a Langmuir wave by a neutrino travelling in a medium.

From equations (1) and (12) it is clear that $\omega_\mathrm{L}(\mathbf{p}, \mathbf{k}) \propto \alpha^\mathrm{L}(k)^2 = \mu_0^2(\varepsilon^\mathrm{L} - 1)^2$, where ϵ_L is the longitudinal dielectric function of the medium. Thus, as the electron density vanishes, the dielectric function approaches unity, and hence the probability of emission vanishes with vanishing electron density.

3 Emission by Massive Neutrinos

We consider now the decay of a massive neutrino from one mass eigenstate (ν_2) with mass m to another, less massive, eigenstate (ν_1). These mass eigenstates are related to the electroweak eigenstates through the lepton mixing matrix

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \qquad (19)$$

where θ is the mixing angle (Kim & Pevsner 1993). We follow D'Olivo et al. (1990) in assuming that the mass of the final state is negligible in comparison to the mass of the initial state.

The calculation of the emission probability for a massive neutrino follows that of the massless neutrino with the replacements

$$pk = \tfrac{1}{2}(m^2 + k^2)$$

for equation (7), and

$$c_V = \tfrac{1}{2}$$

for equation (2), as only the first diagram of Figure 1 contributes to this process. Finally, we include a factor $\sin^2\theta\cos^2\theta$ in the decay probability to account for the projection onto the electroweak eigenstates given in equation (19).

Hence, the probability is given by

$$\omega_\mathrm{L}(\mathbf{p}, \mathbf{k}) = \frac{G_\mathrm{F}^2 c_V^2}{16\pi\alpha} \sin^2\theta\cos^2\theta \times \frac{|\mathbf{k}|^2}{\omega_\mathrm{p}^2} \frac{\omega^3}{\varepsilon\varepsilon'} 2\pi\delta(\varepsilon' + \omega - \varepsilon) \times \left[\left((2\varepsilon - \omega)\left(1 - \frac{\omega^2}{|\mathbf{k}|^2}\right) + \frac{\omega m^2}{|\mathbf{k}|^2}\right)^2 - (m^2 - \omega^2 + |\mathbf{k}|^2)\left(1 - \frac{\omega^2}{|\mathbf{k}|^2}\right)\right]. \qquad (22)$$

Clearly, this expression agrees with equation (18) in the limit $m \to 0$ and where the appropriate numerical factors are replaced.

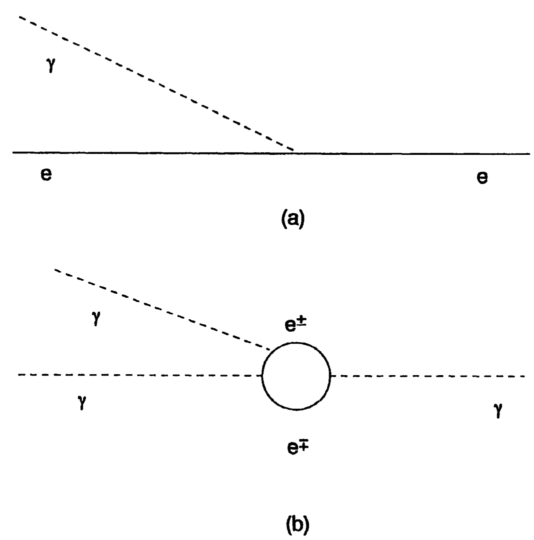

Figure 2—Emission of a plasmon from (a) an electron and (b) a photon.

4 Comparison with Cerenkov Emission

It is instructive to compare the emission from a massless neutrino with the analogous electron and photon beam instabilities (Melrose & Stenhouse 1977; Gedalin & Eichler 1993). The diagrams for these processes are shown in Figure 2. To allow more direct comparison we assume that the recoil of the neutrino is negligible compared to the initial momentum. To zeroth order in the recoil, the

δ-function in equation (18) may be replaced with $\delta(\omega - \mathbf{k}\cdot\mathbf{v})$, leading to

$$w_{\rm L}(\mathbf{k},\mathbf{p}) = \frac{G_{\rm F}^2\, c_V^2}{2\alpha}\,\frac{|\mathbf{k}|^2\,\omega^3}{\omega_{\rm p}^2}\left(1-\frac{\omega^2}{|\mathbf{k}|^2}\right)^2 \times \delta(\omega-\mathbf{k}\cdot\mathbf{v})\,, \qquad (23)$$

where $\mathbf{v} = \mathbf{p}/\epsilon$. For a massless neutrino we have $|\mathbf{v}| = 1$.

Equation (23) is similar in form to the probability of emission of a Langmuir wave from an electron and from a photon. The probability of emission by an electron is given by (Melrose 1986)

$$w_{\rm L}(\mathbf{k},\mathbf{p}) = 4\pi^2\alpha\,\frac{\omega_{\rm p}^2}{|\mathbf{k}|^2\omega}\,\delta(\omega-\mathbf{k}\cdot\mathbf{v}_{\rm e})\,, \qquad (24)$$

where $\mathbf{v}_{\rm e}$ is the velocity of the electron. The probability of emission by a photon is given by (Melrose 1994)

$$w_{\rm L}(\mathbf{k},\mathbf{p}) = \pi^2\alpha\,\frac{\omega_{\rm p}^2\,|\mathbf{k}|^2}{m_{\rm e}^2\,\epsilon(\mathbf{p})^2\,\omega_{\rm L}(\mathbf{k})} \times \delta(\omega-\mathbf{k}\cdot\mathbf{v}_{\rm g})\,, \qquad (25)$$

where $\mathbf{p}$ is identified as the photon momentum, $\epsilon(\mathbf{p})$ is the photon dispersion relation, and $\mathbf{v}_{\rm g} = \partial\epsilon(\mathbf{p})/\partial\mathbf{p}$ is the photon group velocity. All forms of emission are allowed only at angles where $|\mathbf{k}|$ times the component of the particle velocity along the wave vector is equal to the wave frequency. This similarity may be explained through the presence of an induced charge on a neutrino travelling through an electron medium (Nieves & Pal 1994). Thus the three systems are kinematically similar, though the specific details of the interactions are different.

Given the similarities between equations (23)–(25), the generation of Langmuir turbulence by a monoenergetic beam of neutrinos in a medium, as shown by BDSB, is unsurprising, as an equivalent instability exists for electrons. However, it is well known that for a monoenergetic beam of electrons the only instabilities are reactive (Melrose 1986). These forms of instabilities are problematical, as the growth rate is phase-coherent and saturates due to loss of this coherence. Hence quasilinear theory cannot be used to estimate the level of turbulence at which the instability saturates. In contrast, conventional treatments of electron–Langmuir wave (and photon–Langmuir wave) instabilities assume kinetic growth, which requires $\mathbf{k}\cdot(\partial f(\mathbf{p})/\partial\mathbf{p})$ to be positive [$f(\mathbf{p})$ is the distribution function of the electrons]. A kinetic instability may be analysed using quasilinear theory, e.g. to discuss saturation and thereby to determine the rate of energy transfer to the plasma. A kinetic version of the instability proposed by BDSB will be presented elsewhere (Hardy & Melrose, in preparation).

5 Conclusion

In equations (18) and (22) we have expressions for the probability per unit time of the emission of a Langmuir wave by massless and massive neutrinos travelling through an electron medium, including the dispersive properties of both the plasmon and the neutrinos. The method used to calculate the couplings between neutrinos and any collective mode in a medium represents a rigorous rederivation, using the methods of FTFT, of the Langmuir wave–neutrino coupling suggested by BDSB.

To zeroth order in the neutrino recoil, the probability of emission from a massless neutrino is given by equation (23). Comparison with the equivalent expression for Cerenkov emission by electrons confirms that a neutrino beam instability is possible, in principle. The existence of such an instability provides a mechanism for the reheating of plasma behind the bounce shock of a type II SN, as proposed by BDSB. However, their use of a monoenergetic distribution of electrons to treat the wave growth through this instability needs to be generalised to obtain a kinetic version of the instability. Further work is required to determine the actual heating rate due to this instability, and whether its effect is sufficient to ensure the success of the bounce shock in blowing off the outer layers of the star.

More generally, the calculation presented here illustrates a general method for synthesising the electroweak interactions of neutrinos with the theory of quantum plasmadynamics, and this may be generalised to calculate neutrino–plasmon coupling to any plasma mode.

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