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The response tensor for a highly relativistic magnetized thermal plasma

D. B. MELROSE

Research Centre for Theoretical Astrophysics, School of Physics, University of Sydney, NSW 2006, Australia

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The transverse part of the linear response tensor is evaluated for a highly relativistic thermal electron gas, starting from Trubnikov's response tensor for an arbitrary temperature. Three contributions to the response tensor are important. The diagonal components are dominated by an unmagnetized term, which gives the familiar dispersion $k^2 = \omega_{p0}^2$, where ω_{p0} is the proper plasma frequency. The difference between the diagonal components is larger in magnitude than the off-diagonal components, implying that the natural modes are nearly linearly polarized. This leads to a generalized form of Faraday rotation in which linear polarization is partially converted into circular polarization at a rate per unit path length $\propto \lambda^3$ (λ = wavelength).

1. Introduction

A characteristic feature of a highly relativistic magnetized electron plasma is the emission and absorption of synchrotron radiation. Synchrotron absorption may be treated in terms of the anti-Hermitian part of the response tensor for the highly relativistic electron gas. The Hermitian part of the corresponding response tensor may be used to describe dispersion and birefringence in the electron gas. In this paper an explicit expression for the response tensor for a highly relativistic magnetized thermal plasma is derived, starting from Trubnikov's response tensor for arbitrary temperature and making the high-temperature approximation. Relevant literature on dispersion in a synchrotron-emitting gas is relatively sparse (Sazonov 1969; Skilling 1971), and appears to contain errors, as discussed below. There is a much more extensive literature on dispersion in a highly relativistic strongly magnetized electron gas, which is of relevance to pulsar magnetospheres (see e.g. Melrose 1979; Volokitin *et al.* 1985; Arons and Barnard 1986; Lominadze *et al.* 1986; Beskin *et al.* 1987; Kazbegi *et al.* 1991). However, the strongly magnetized limit is not relevant here, since it presupposes that the frequencies of interest are smaller than or comparable to the cyclotron frequency Ω_0. The frequencies of interest in a synchrotron-emitting gas are comparable to the frequency of the synchrotron emission itself, $\omega \approx \Omega_0\gamma^2$, where γ is the Lorentz factor of the radiating particles.

Explicit evaluation of the response tensor for a synchrotron-emitting electron gas has presented technical problems. The detailed derivation by Sazonov (1969) started from the expression for the response tensor derived using the Vlasov equation and integrated over gyrophase after expanding in Bessel functions. As Sazonov remarked, the analytic connection between the anti-Hermitian and Hermitian parts is lost in the subsequent calculation. This precludes an important check on the correctness of the result: the anti-Hermitian part of the response tensor must reproduces the expression for the synchrotron absorption coefficient derived from synchrotron emission formulae by appealing to detailed balance. Although Sazonov (1969) stated that his method reproduced this result, because the analytic connection between the anti-Hermitian and Hermitian parts is lost in Sazonov's calculation, this provides no actual check on the validity of the expressions for the Hermitian part of the response tensor. Another possible check on the resulting expression for the response tensor is to apply it to a highly relativistic thermal distribution and to compare the results with Trubnikov's response tensor in the appropriate limit. This is the motivation for the derivation presented here of the relevant limiting case of Trubnikov's response tensor. Skilling's (1971) discussion of this problem was predicated on an argument that the natural modes of a highly relativistic plasma are circularly polarized, which is inconsistent with Sazonov's results and with the results obtained here. Skilling's (1971) expression for the off-diagonal components of the response tensor for a relativistic thermal distribution has a logarithmic dependence on temperature that is reproduced by the result found here, but his treatment of the diagonal components seems to be incorrect.

In order to avoid the technical difficulties encountered in Sazonov's (1969) treatment, an alternative approach has been developed and is to be presented elsewhere (Melrose 1997*b*). The alternative approach involves evaluating the integrals over gyrophase and pitch angle using the method of stationary phase. This approach preserves the analytic properties of the tensor, so that the check that the anti-Hermitian part reproduce the known synchrotron absorption formulae is also a check on the Hermitian part in this case. The additional check, that the expression reproduce the relevant limit of Trubnikov's response tensor, is confirmed here.

The physical motivation for the investigation of dispersion and birefringence in a synchrotron-emitting gas is connected with possible interpretations of the circular polarization of synchrotron radiation from astrophysical radio sources (Sazonov 1969; Skilling 1971). An earlier investigation of the circularly polarized component of astrophysical sources of synchrotron radiation (see e.g. Roberts *et al*. 1975) led to results just above the then available observational threshold of detection. The Australia Telescope Compact Array now allows measurement of a degree of circular polarization as small as 0.1%, and there is renewed interest in measuring circular polarization. The simplest interpretation of the circular polarization is that intrinsic to synchrotron emission, which should should vary $\propto \omega^{-1/2}$ (see e.g. Melrose 1980, p. 122). The preliminary data do not appear to support this prediction (R. Norris, private communication, 1997), and alternative causes for the circular polarization need to be explored. The only known alternative is propagation through a medium with elliptically (or linearly) polarized natural modes. This allows a cyclic conversion of linear into circular polarization, as in a quarter-wave plate. An investigation of the dispersion and birefringence of a synchrotron-emitting gas is needed to model this process.

2. The limit $\rho \ll 1$ of Trubnikov's response tensor

The highly relativistic limit of Trubnikov's (1958) response tensor is obtained here by making three related approximations. First, the limit $\rho \ll 1$ is taken (ρ is the inverse temperature in units of the rest mass). Secondly, only the transverse components of the response tensor are retained, on the assumption that only relatively high frequencies are of interest, in which case the waves are approximately transverse. Thirdly, Trubnikov's's response tensor involves an integration over proper time ξ, and the only important contribution to the integral over proper time is from the range $\Omega_0\xi \ll 1$; hence one expands the integrand in powers of $\Omega_0\xi$.

2.1. *The transverse components*

It is convenient to define two transverse polarization vectors, and to project the polarization tensor onto the transverse plane spanned by these two vectors. In a frame in which the magnetostatic field is along the 3-axis and the wave vector is in the 1–3 plane (at an angle θ to the 3-axis), the chosen polarization vectors are

$$\mathbf{e}^1 = (\cos\theta, 0, -\sin\theta), \qquad \mathbf{e}^2 = (0, 1, 0). \tag{2.1}$$

The approximation $|\mathbf{k}| = \omega$ is also to be made, corresponding to the dispersion of transverse waves in vacuo. However, this approximation is not made until after the unmagnetized limit of the response tensor has been subtracted.

2.2. *Trubnikov's response tensor*

The starting point for the present calculation is Trubnikov's response tensor in the form written down by Melrose (1997*a*). After projecting onto the transverse plane in the rest frame of the plasma, this becomes

$$\alpha^{\mu\nu}(k) = \frac{iq^2 n\omega\rho^2}{mK_2(\rho)} \int_0^\infty d\xi \left[t^{\mu\nu}(\xi)\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} - R^\mu(\xi)\tilde{R}^\nu(\xi)\,\frac{K_3\big(r(\xi)\big)}{r^3(\xi)} \right], \tag{2.2}$$

$$t^{\mu\nu}(\xi) = \begin{pmatrix} -\cos^2\theta\,\cos\Omega_0\xi - \sin^2\theta\,\Omega_0\xi & -\eta\cos\theta\sin\Omega_0\xi \\ \eta\cos\theta\sin\Omega_0\xi & -\cos\Omega_0\xi \end{pmatrix}, \tag{2.3}$$

$$R^\mu(\xi) = \frac{|\mathbf{k}|\sin\theta}{\Omega_0}\big(\cos\theta\,(\sin\Omega_0\xi - \Omega_0\xi), -\eta(1-\cos\Omega_0\xi)\big), \tag{2.4a}$$

$$\tilde{R}^\nu(\xi) = \frac{|\mathbf{k}|\sin\theta}{\Omega_0}\big(\cos\theta\,(\sin\Omega_0\xi - \Omega_0\xi), \eta(1-\cos\Omega_0\xi)\big), \tag{2.4b}$$

$$r(\xi) = \left[(\rho - i\omega\xi)^2 + |\mathbf{k}|^2\cos^2\theta\,\xi^2 + \frac{2|\mathbf{k}|^2\sin^2\theta}{\Omega_0^2}\,(1-\cos\Omega_0\xi)\right]^{1/2}, \tag{2.5}$$

where η is the sign of the charge. In the notation used here, the units are chosen to give $c = 1$, and the components $\alpha^{\mu\nu}(k)$ for μ, $\nu = 1, 2$ are numerically equal to minus the components $\alpha^\mu{}_\nu(k)$ in the relation $J^\mu(k) = \alpha^\mu{}_\nu(k)A^\nu(k)$ between the transverse components of the induced current and the vector potential.

2.3. *The highly relativistic approximation*

The highly relativistic limit involves assuming $\rho \ll 1$, with $K_2(\rho) = 2/\rho^2$ in (2.2), and $\Omega_0\xi \ll 1$, so that only the leading terms in the expansion (in $\Omega_0\xi$) of the trignometric functions in the integrand in (2.2) are retained. An important approximation is that made to the argument (2.5) of the Macdonald functions. Expanding $r(\xi)$ in $\Omega_0\xi$ gives

$$r(\xi) = \left[\rho^2 - 2i\rho\omega\xi - (\omega^2 - |\mathbf{k}|^2)\xi^2 - \frac{\omega^2\sin^2\theta}{\Omega_0^2}\,\frac{(\Omega_0\xi)^4}{12}\right]^{1/2}. \tag{2.6}$$

The terms ρ^2 and $-(\omega^2 - |\mathbf{k}|^2)\xi^2$ are to be neglected below, but are retained for the present in order to discuss the limit $\xi = 0$. For $(\Omega_0\xi)^3 \ll 1/(\rho\omega\Omega_0) \ll 1$ the final term in (2.6) is a small correction, and the basic approximation made below is $r(\xi) = (-2i\rho\omega\xi)^{1/2}$.

The ξ integration in (2.2) may be performed by deforming the contour of integration so that $r(\xi)$ is real. This involves rotating the contour so that it is along the imaginary ξ axis, and changing the variable of integration from ξ to $r(\xi)$, which is now real.

2.4. Subtraction of the unmagnetized response tensor

The foregoing procedure leads to a singular term. Specifically, with $t^{\mu\nu}(\xi) = -\delta^{\mu\nu}$ to lowest order in $\Omega_0\xi$, and $d\xi \to i(\omega/\rho)r\,dr$, the integral $\int dr\, r\, K_2(r)/r^2$ diverges as $2\int dr/r^3$ at the lower limit $r = 0$. The term ρ^2 retained in (2.5) avoids the divergence, and ensures that the integral is finite. However, this term is then of a different character from the other (non-singular) terms in (2.2). Let this term be denoted by $\alpha_0^{\mu\nu}(k)$. The integral gives

$$\alpha_0^{\mu\nu}(k) = -\frac{\omega_{p0}^2}{\mu_0}\, g^{\mu\nu}, \tag{2.7}$$

with the proper plasma frequency given by $\omega_{p0}^2 = q^2 n\rho/m$ in the limit $\rho \ll 1$. The contribution (2.7) is independent of the magnetostatic field. Thus the quasi-singular term is associated with the response of a highly relativistic unmagnetized plasma. Subtracting the unmagnetized part of the response, by writing

$$\alpha_{\rm mag}^{\mu\nu}(k) = \alpha^{\mu\nu}(k) - \alpha_0^{\mu\nu}(k), \tag{2.8}$$

isolates the part, $\alpha_{\rm mag}^{\mu\nu}(k)$, that depends explicitly on the magnetostatic field.

2.5. The diagonal components

There are two contributions to the diagonal components of the response tensor. One is from the term proportional to $t^{\mu\nu}(\xi)$ in (2.2), and this term contributes equally to the 11- and 22-components. The subtraction of the unmagnetized response involves only this term. Writing

$$r^2 = r_0^2 + \delta r^2, \qquad r_0^2 = -2i\omega\rho\xi, \qquad \delta r^2 = -\frac{\omega^2 \sin^2\theta}{\Omega_0^2}\,\frac{(\Omega_0\xi)^4}{12}, \tag{2.9}$$

one subtracts $K_2(r_0)/r_0^2$ from $K_2(r)/r^2$, and expands in $\delta r^2/r_0^2$ to find

$$\frac{K_2(r)}{r^2} - \frac{K_2(r_0)}{r_0^2} = -\frac{\delta r^2}{2}\,\frac{K_3(r_0)}{r_0^3}. \tag{2.10}$$

The integral over $r \to r_0$ of the resulting function may be evaluated exactly using the standard integral

$$\int_0^\infty dx\, x^\mu K_\nu(ax) = 2^{\mu-1} a^{-\mu-1}\,\Gamma\!\left(\frac{1+\mu+\nu}{2}\right)\Gamma\!\left(\frac{1+\mu-\nu}{2}\right). \tag{2.11}$$

The other term in (2.2) involves the product of the vectors $R^\mu(k)$ and $\tilde{R}^\nu(k)$, and, after expanding (2.4) in powers of $\Omega_0\xi$, one finds that the 1-component of each vector is of order $\Omega_0\xi$ smaller than the 2-component. Hence, to lowest order, this term contributes only to the 22-component. Explicit evaluation shows that this term is of the same form the other term, and is a factor 6 larger than it.

Thus, after evaluating the integrals, the diagonal terms reduce to

$$\alpha^{11}_{\rm mag}(k) = \frac{2q^2 n}{m\rho}\,\frac{\Omega_0^2}{\omega^2}\sin^2\theta, \qquad \alpha^{22}_{\rm mag}(k) = 7\alpha^{11}(k), \tag{2.12}$$

where only the the leading correction due to the presence of the magnetostatic field is retained.

2.6. *The off-diagonal components*

The leading contribution to the off-diagonal components, $\alpha^{12}(k) = -\alpha^{21}(k)$, is from the off-diagonal term in (2.3). This leading term gives an integral

$$\int d\xi\,\xi\,\frac{K_2(r)}{r^2} \propto i\int dr\,r^3\,\frac{K_2(r)}{r^2},$$

and with $K_2(r) = 2/r^2$, this term is logarithmically divergent. Cutting the integral off, for $r < r_0$ and $r > r_1$ say, one has

$$\int dr\,r^3\,\frac{K_2(r)}{r^2} \approx 2\ln\left(\frac{r_1}{r_0}\right);$$

alternatively, one may partially integrate twice to find

$$\int_{r_0}^{r_1} dr\,r^3\,\frac{K_2(r)}{r^2} = -2[K_0(\tfrac{1}{2}r_1) - K_0(\tfrac{1}{2}r_0)] = 2\ln\left(\frac{r_1}{r_0}\right),$$

when only the logarithmic terms are retained. In either case, separate arguments are required to determine the cutoffs r_1 and r_0. The choices $r_1 = 1$ and $r_0 = \rho$ are suggested by the limit on the range of validity of the approximation $K_2(r) = 2/r^2$ and the value $r(0) = \rho$ respectively.

Thus the off-diagonal terms reduce to

$$\alpha^{12}_{\rm mag}(k) = -\alpha^{21}_{\rm mag}(k) = -i\tfrac{1}{2}\eta\cos\theta\,\frac{2q^2 n}{m}\,\frac{\Omega_0}{\omega}\,\rho^2\,\ln\left(\frac{r_1}{r_0}\right), \qquad \ln\left(\frac{r_1}{r_0}\right) \approx \ln\left(\frac{1}{\rho}\right), \tag{2.13}$$

where only the simplest approximation to the logarithmic factor is considered.

2.7. *The final form for the response tensor*

The final form for the transverse part of the response tensor for a highly relativistic thermal magnetized plasma is

$$\begin{bmatrix}\alpha^{11}(k)\\ \alpha^{22}(k)\end{bmatrix} = \frac{\omega_{p0}^2}{\mu_0}\left(1 + \frac{2\Omega_0^2\sin^2\theta}{\rho^2\omega^2}\begin{bmatrix}1\\ 7\end{bmatrix}\right), \tag{2.14a}$$

$$\alpha^{12}(k) = -\frac{i}{2}\eta\cos\theta\,\frac{\omega_{p0}^2\Omega_0}{\mu_0\omega}\,\rho\,\ln\left(\frac{1}{\rho}\right). \tag{2.14b}$$

The off-diagonal term is a factor of order $(\omega/\Omega_0)\rho^3\ln(1/\rho)$ smaller than the diagonal terms; for the typical frequencies of relevance, $\omega \approx \Omega_0/\rho^2$, this factor is of order $\rho\ln(1/\rho)$. The logarithmic dependence on temperature, $\ln(1/\rho)$, reproduces a result derived by Skilling (1971) in a different manner.

3. Alternative derivation of the response tensor

The transverse components of the response tensor for an arbitrary distribution of relativistic particles is derived elsewhere (Melrose 1997*b*) using the method of stationary phase. A motivation for the present investigation is to provide a check on the validity of the result obtained in Melrose (1997*b*): the result should reproduce (2.14) for a relativistic thermal distribution.

3.1. *Dispersion functions*

The result obtained in Melrose (1997*b*) can be summarized as follows. After evaluating the integrals over gyrophase and pitch angle by stationary phase, the integral over proper time reduces to integrals of the form

$$I^{(n)}(a,b) = \int_0^\infty dy\, y^n \, \exp[iay + i\tfrac{1}{3}by^3], \tag{3.1}$$

with $n = 0$, 1 and -1 (for $n = -1$, $\exp[iay+i\frac{1}{3}by^3]$ is replaced by $\exp[iay+i\frac{1}{3}by^3]-1$ to remove the divergence), and with

$$y = \mathbf{\Omega}_0\xi, \qquad a = \gamma(\omega - |\mathbf{k}|v), \qquad b = \tfrac{1}{8}\gamma|\mathbf{k}|v\sin^2\theta. \tag{3.2}$$

The functions (3.1) play the role of plasma dispersion functions in this theory. These functions may be expressed in terms of the Airy functions (see e.g. Abramowitz and Stegun 1965),

$$\mathrm{Ai}(z) = \frac{1}{\pi}\int_0^\infty dt\, \cos(zt + \tfrac{1}{3}t^3), \qquad \mathrm{Gi}(z) = \frac{1}{\pi}\int_0^\infty dt\, \sin(zt + \tfrac{1}{3}t^3), \tag{3.3}$$

with $z = a/b^{1/3}$. Thus the Hermitian part of the resulting response tensor may be obtained from the anti-Hermitian part (which is known from formulae for synchrotron absorption) by the following simple prescription: write the anti-Hermitian part in terms of the Airy function $\mathrm{Ai}(z)$, and then replace $\mathrm{Ai}(z)$ by $\mathrm{Ai}(z) + i\,\mathrm{Gi}(z)$. The specific integrals involved are

$$I^{(0)}(a,b) = \pi b^{-1/3}\big[\mathrm{Ai}(z) + i\,\mathrm{Gi}(z)\big], \tag{3.4a}$$

$$I^{(1)}(a,b) = -i\pi b^{-1/3}\big[\mathrm{Ai}'(z) + i\,\mathrm{Gi}'(z)\big], \tag{3.4b}$$

$$I^{(-1)}(a,b) = i\pi\int_0^z dz'\,\big[\mathrm{Ai}(z') + i\,\mathrm{Gi}(z')\big]. \tag{3.4c}$$

The parts involving $\mathrm{Ai}(z)$ contribute to the anti-Hermitian part of the response tensor, and parts involving $\mathrm{Gi}(z)$ contribute to the Hermitian part of the response tensor. An important point is that the functions (3.1) are causal, so these two parts must be related to each other by the Kramers–Kronig relations; this approach does not suffer the weakness of that used in Sazonov (1969).

The known formulae for synchrotron absorption arise from exact expressions for $\mathrm{Ai}(z)$ in terms of Macdonald functions:

$$\mathrm{Re}\,I^{(0)}(a,b) = \frac{1}{\sqrt{3}}\left(\frac{a}{b}\right)^{1/2} K_{1/3}(R), \qquad \mathrm{Im}\,I^{(1)}(a,b) = \frac{1}{\sqrt{3}}\,\frac{a}{b}\,K_{2/3}(R), \tag{3.5a, b}$$

$$\mathrm{Im}\,I^{(-1)}(a,b) = -\frac{1}{\sqrt{3}}\int_R^\infty dt\,K_{1/3}(t), \qquad R = \frac{2a^{3/2}}{3b^{1/2}}. \tag{3.5c, d}$$

There is no known analogous form for the functions $\mathrm{Gi}(z)$. In the application of interest here we are concerned with the high-frequency response, when the asymptotic expansion for $z \gg 1$ applies. The relevant approximations are

$$I^{(0)}(a,b) = \frac{i}{a}\left(1 + \frac{2b}{a^3}\right), \qquad I^{(1)}(a,b) = -\frac{1}{a^2}, \qquad I^{(-1)}(a,b) = -\ln a, \tag{3.6}$$

with $a = \omega/2\Omega_0\gamma$ for $\gamma \gg 1$ and $|\mathbf{k}| = \omega$.

3.2. *The alternative form of the response tensor*

The response tensor is written in terms of the particle spectrum $N(\gamma)$:

$$4\pi d|\mathbf{p}|\,|\mathbf{p}|^2 f(|\mathbf{p}|) = d\gamma\, N(\gamma), \tag{3.7}$$

where the distribution is assumed to be isotropic. The resulting expression for the response tensor is

$$\alpha^{\mu\nu}_{\rm mag}(k) = -i\frac{q^2\Omega_0}{m\omega}\int d\gamma\,\frac{N(\gamma)}{\gamma^2 v}\,J^{\mu\nu}(a,b), \tag{3.8}$$

$$J^{11}(a,b) = -\tfrac{4}{3}ibI^{(1)}(a,b), \tag{3.9a}$$

$$J^{22}(a,b) = -4ibI^{(1)}(a,b) + \tfrac{8}{3}a^2[I^{(0)}(a,b) - I^{(0)}(a,0)], \tag{3.9b}$$

$$J^{12}(a,b) = -\tfrac{1}{2}\eta\cos\theta\,[\tfrac{16}{9}a^2 I^{(1)}(a,b) + \tfrac{20}{9}iaI^{(0)}(a,b) - 2I^{(-1)}(a,b) + 2], \tag{3.9c}$$

where the term $I^{(0)}(a,0)$ is subtracted in accord with (2.8), where the unmagnetized limit corresponds to $b = 0$. The result (3.8) is similar in form to the result obtained in Sazonov (1969), but differs from Sazonov's result in specific details most of which may be ascribed to Sazonov's expression not satisfying the causal condition.

On making the approximations (3.6), (3.9) simplify to

$$J^{11}(a,b) = i\frac{2}{3}\frac{\Omega_0}{\omega}\gamma^3\sin^2\theta, \qquad J^{22}(a,b) = 7J^{11}(a,b), \tag{3.10a, b}$$

$$J^{12}(a,b) = \tfrac{1}{2}\eta\cos\theta\,\ln\!\left(\frac{\omega}{2\Omega_0\gamma}\right), \tag{3.10c}$$

where the logarithmic term is assumed to dominate in $J^{12}(a,b)$.

3.3. *Highly relativistic thermal distribution*

A highly relativistic thermal distribution corresponds to

$$N(\gamma) = \tfrac{1}{2}n\rho^3\,\gamma^2 e^{-\gamma\rho}. \tag{3.11}$$

On evaluating the response tensor (3.8), making the asymptotic approximation (3.10), the result (2.14) is reproduced except for the argument of the logarithm in the off-diagonal term, which is replaced according to

$$\ln\!\left(\frac{1}{\rho}\right) \to \ln\!\left(\frac{\omega\rho}{2\Omega_0}\right). \tag{3.12}$$

In one sense, this difference is a significant disagreement between the two approaches, and needs to be explained. In another sense, the difference is quite minor. This is because the logarithmic terms in(3.12) are comparable for typical frequencies, $\omega \approx \Omega_0\rho^{-2}\sin\theta$, of relevance here, and in either approach the logarithmic factor is determined only to within a factor of order unity inside the logarithm.

4. Waves in a highly relativistic plasma

High-frequency waves in a highly relativistic plasma have some similarities and some important difference from waves in a cold plasma.

4.1. Response of a cold plasma

The response tensor (2.14) may be compared with the corresponding response tensor for a non-relativistic thermal plasma. In the high-frequency limit, the response of such a plasma is independent of the temperature, and is the same as for a cold plasma. In the present notation, the transverse part of the response of a cold plasma reduces to

$$\begin{bmatrix}\alpha^{11}(k)\\ \alpha^{22}(k)\end{bmatrix} = \frac{\omega_p^2}{\mu_0}\left(1+\frac{\Omega_0^2}{\omega^2}+\frac{\Omega_0^2\sin^2\theta}{\omega^2}\begin{bmatrix}-1\\ 0\end{bmatrix}\right),$$

$$\alpha^{12}(k) = -i\eta\cos\theta\,\frac{\omega_p^2\Omega_0}{\mu_0\omega}, \tag{4.1}$$

which applies for $\omega \gg \Omega_0$, ω_p with the plasma frequency ω_p equal to the proper plasma frequency for a cold plasma.

There are obvious similarities in form between (2.14) and (4.1). The most important difference between them is in the ratio $|\alpha^{11}(k)-\alpha^{22}(k)| : |\alpha^{12}(k)|$. For (2.14) this ratio is large for typical frequencies of interest, $\omega \approx \Omega_0\rho^{-2}\sin\theta$, except for very small $\sin\theta$, and for (4.1) this ratio is small, except for very small $|\cos\theta|$.

4.2. The wave properties

The high-frequency waves may be regarded as two transverse wave modes with slightly different dispersion relations,

$$k^2 = k_\pm^2 = \frac{\mu_0}{2}\{\alpha^{11}+\alpha^{22}\pm[(\alpha^{11}-\alpha^{22})^2+4\alpha^{12}\alpha^{21}]^{1/2}\}, \tag{4.2}$$

where $\pm$ labels the two modes, and with orthogonal elliptical polarization vectors

$$e_\pm^\mu = \frac{T_\pm e^1 + ie^2}{(1+T_\pm^2)^{1/2}}, \qquad T_\pm = \frac{\alpha^{11}-\alpha^{22}\mp[(\alpha^{11}-\alpha^{22})^2+4\alpha^{12}\alpha^{21}]^{1/2}}{2i\alpha^{12}}, \tag{4.3}$$

where $T_\pm$, with $T_+T_- = -1$, denote the axial ratios of the polarization ellipses. With the response tensor (2.14), (4.2) becomes

$$k^2 = k_\pm^2 = \omega_{p0}^2\left\{1+\frac{2\Omega_0^2\sin^2\theta}{\rho^2\omega^2}[4\pm(9+\Delta^2)^{1/2}]\right\}, \tag{4.4}$$

$$\Delta = \frac{\eta\cos\theta}{2\sin^2\theta}\,\frac{\omega\rho^2}{\Omega_0}\,\rho\ln\!\left(\frac{1}{\rho}\right), \tag{4.5}$$

and (4.3) gives

$$e_\pm^\mu = \frac{T_\pm e^1 + ie^2}{(1+T_\pm^2)^{1/2}}, \qquad T_\pm = \frac{3\pm(9+\Delta^2)^{1/2}}{2\Delta}. \tag{4.6}$$

One has $\Delta^2 \ll 1$, except for very small $\sin\theta$, so that the natural modes are nearly linearly polarized.

4.3. Effect of an admixture of cold plasma

Let the number densities of the cold and relativistic plasmas be n_{cold} and n_{rel} respectively. Then for $n_{\text{cold}} \ll n_{\text{rel}}(\rho/2)\ln(1/\rho)$ the cold plasma is negligible, and for $n_{\rm cold} \gg 12n_{\rm rel}/\rho$ the relativistic plasma is negligible. There is an intermediate regime, for

$$n_{\rm rel}\frac{\rho}{2}\ln\left(\frac{1}{\rho}\right) \ll n_{\rm cold} \ll \frac{12n_{\rm rel}}{\rho}, \tag{4.7}$$

in which the cold plasma makes the dominant contribution to α^{12} and the relativistic plasma makes the dominant contribution to $\alpha^{11}-\alpha^{22}$. In this intermediate regime, as $n_{\rm cold}/n_{\rm rel}$ increases, the natural modes of the medium change over from being nearly linear, which is characteristic of the highly relativistic plasma, to nearly circular, which is characteristic of the cold plasma.

4.4. Generalized Faraday rotation

In a cold plasma, the difference $\Delta|\mathbf{k}|$ in wavenumber between the two circularly polarized natural modes leads to Faraday rotation, in which the plane of linear polarization rotates at a rate $\frac{1}{2}\Delta|\mathbf{k}|$ per unit distance along the ray path. In astrophysical applications the net angle through which the rotation occurs is used to define a rotation measure RM by writing that the net rotation in the form $\int ds\, \frac{1}{2}\Delta|\mathbf{k}| = \mathrm{RM}\,\lambda^2$, where ds is an element of distance along the ray path and where the dependence on wavelength λ follows from $\alpha^{12} \propto 1/\omega$ implying $\Delta|\mathbf{k}| \propto 1/\omega^2$. In a highly relativistic plasma, in which the modes are linearly polarized, the counterpart of Faraday rotation is the cyclic conversion of linear into circular polarization for the component of incident radiation linearly polarized at an angle $\frac{1}{4}\pi$ to either of the natural modes. One may define a 'relativistic rotation measure' RRM to describe this effect, by writing $\int ds\, \frac{1}{2}\Delta|\mathbf{k}| = \mathrm{RRM}\,\lambda^3$, where the dependence on λ^3 follows from $\alpha^{11}-\alpha^{22} \propto 1/\omega^2$ implying $\Delta|\mathbf{k}| \propto 1/\omega^3$. The explicit expression for RRM is (in SI units)

$$\mathrm{RRM} = \frac{6e^4}{(2\pi)^3\varepsilon_0 m^3 c^4}\int ds\, \frac{n_{\rm rel}B^2\sin^2\theta}{\rho}. \tag{4.8}$$

The result (4.8) is not sensitive to the form of the distribution function. For nonthermal distributions $1/\rho$ may be interpreted as the mean Lorentz factor of the particles that contribute to $\alpha^{11}-\alpha^{22}$. In principle, RRM is measurable in any synchrotron source in which the circular polarization is due to this effect, which would be characterized by a degree of circular polarization varying $\propto \lambda^3$.

5. Discussion and conclusions

The main result of the present paper is the expression (2.14) for the high-frequency response of a highly relativistic thermal plasma. The response (2.14) may be compared with the analogous result (4.1) for a cold plasma. The response (2.14) is characterized by

(*a*) dispersion similar to that in a cold plasma, with the relevant plasma frequency being the proper plasma frequency $\omega_{p0} = (q^2 n\rho/\varepsilon_0 m)^{1/2}$; and

(*b*) natural wave modes that are approximately linearly polarized, for $\omega \lesssim (\Omega_0\sin\theta)/\rho^2\ln(1/\rho)$, with

(*c*) a difference in wavenumber $\Delta k \propto 1/\omega^3$ between the two modes, whereas in a cold plasma the modes are circular, with $\Delta k \propto 1/\omega^2$.

These properties are potentially observable for any synchrotron source in which the birefringence is dominated by the relativistic gas. The observable effect may be described by the 'relativistic rotation measure' defined by (4.8). The possible significance in the interpretation of the circular polarization of astrophysical sources of synchrotron radiation will be discussed elsewhere.

An important motivation for the present investigation is to provide a check on an alternative calculation (Melrose 1997*b*) of the response tensor for an arbitrary highly relativistic gas. The basis of the check is that the response of a highly relativistic thermal gas may be calculated in two independent ways: from Trubnikov's exact result for a thermal gas of any temperature, and by applying the alternative procedure to a highly relativistic thermal gas. The two results agree except in the argument of the logarithm in the off-diagonal term: the logarithmic factor in (2.14) from Trubnikov's tensor is replaced according to (3.12) in the alternative method. As already remarked, the two logarithmic factors are approximately equal for the frequencies of relevance, and exact agreement in the argument of the logarithm is not necessarily expected, because the logarithmic factor arises from cutting off integrals, and the choice of cutoffs has some arbitrariness. The difference is probably due to an actual difference in the two theories. Trubnikov's theory includes the effect of mildly relativistic particles and it treats nonrelativistic particles correctly. Nonrelativistic particles are present even in a highly relativistic thermal distribution. On the other hand, only the effect of the relativistic particles is treated correctly in the alternative theory. Although nonrelativistic particles have no significant effect on the (synchrotron) emission and absorption at high frequencies, they can affect the Hermitian part of the response at high frequencies. Specifically, the Kramers–Kronig relations imply that the Hermitian part of the response is related to the absorption through an integral that extends over all frequencies. It may be that minor difference in the form of the logarithmic terms can be attributed to the different treatment of the effect of the nonrelativistic particles in the two theories.

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