J. Plasma Physics (1997), *vol.* 57, *part* 2, *pp.* 479–488. Printed in the United Kingdom

Covariant form of Trubnikov's response tensor for a relativistic magnetized thermal plasma

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(Received 6 March 1996 and in revised form 22 April 1996)

A covariant version of Vlasov theory is used to derive the linear response 4-tensor for a relativistic magnetized thermal plasma, generalizing Trubnikov's 3-tensor form. The charge-continuity and gauge-invariance conditions imply a set of integral identities, and use of these identities allows one to derived a manifestly gauge invariant form for the response 4-tensor. This form implies an alternative version of Trubnikov's 3-tensor. The different versions lead to different expressions in the weakly relativistic small-gyroradius limit. Arguments are given as to how this inconsistency should be resolved.

1. Introduction

The linear response tensor for a relativistic magnetized thermal plasma was calculated by Trubnikov (1958). Trubnikov's tensor involves an integral (over a variable $0 < \xi < \infty$ here) over Macdonald functions (modified Bessel functions) $K_n\big(r(\xi)\big)/r^n(\xi)$ with $n = 2, 3$, where $r(\xi)$ is a complex function of ξ. This expression is too cumbersome for most practical applications. An approximate form for this tensor was calculated in the weakly relativistic small-gyroradius limit by Shkarofsky (1966), and Shkarofsky's tensor has formed the basis for much of the subsequent literature on cyclotron waves and cyclotron absorption in relativistic plasmas (see e.g. Robinson 1988; Bornatici and Engelmann 1994).

In this paper a covariant and gauge-invariant form of Trubnikov's tensor is derived. A covariant and gauge-invariant description of the linear response of an arbitrary medium is provided by the polarization 4-tensor $\alpha^{\mu\nu}(k)$ that relates the Fourier transforms of the 4-current and the 4-potential in the form $J^\mu = \alpha^\mu{}_\nu(k)A^\nu(k)$, where $k^\mu = [\omega, \mathbf{k}]$ is the wave 4-vector. (The 4-tensor notation here has Greek indices running over 0 to 3, Latin indices over 1 to 3, the metric tensor $g^{\mu\nu}$ is diagonal $(1, -1, -1, -1)$, and units with $c = 1$ are used.) A prescription for constructing $\alpha^{\mu\nu}(k)$ from a non-covariant description is (see e.g. Melrose 1973): write the response as a relation between the 3-current $\mathbf{J}(k)$ and the 3-potential $\mathbf{A}(k)$ in the temporal gauge to define a polarization 3-tensor, which is identified with the space component $\alpha^i{}_j(k)$, and then impose the charge-continuity and gauge-invariance conditions

$$k_\mu\alpha^\mu{}_\nu(k) = 0, \qquad \alpha^\mu{}_\nu(k)k^\nu = 0 \tag{1}$$

to construct the remaining components of the tensor. However, the advantages of a covariant description of the plasma response become evident only when a covariant version of the kinetic theory of plasmas is used to calculate the response tensors. A covariant version of orbit theory may be used to derive the linear and nonlinear response tensors by the forward-scattering method for both unmagnetized (see e.g. Melrose 1983) and magnetized (see e.g. Melrose 1987) plasmas. Similarly, a covariant version of Vlasov theory (see e.g. Dewar 1977) may be used to calculate covariant expressions for the response tensors. A covariant version of Vlasov theory for a magnetized plasma is used here to calculate the covariant form of Trubnikov's tensor.

In an earlier paper (Melrose 1982) a covariant generalization of Trubnikov's tensor for an unmagnetized plasma was compared with other covariant forms for the response tensor. An aspect of the calculation that was not emphasized in that earlier paper is that the use of covariant Vlasov theory to calculate Trubnikov's tensor leads to an expression that does not manifestly satisfy the conditions (1). As a consequence, one infers from (1) that certain integral identities must be satisfied. The corresponding identities are derived in this paper for the magnetized case. Using these identities, the response tensor may be written in a manifestly gauge-invariant form, that is, a form that obviously satisfies (1). This form involves an integral only over $K_n\big(r(\xi)\big)/r^n(\xi)$ with $n = 3$, and it implies an alternative expression for Trubnikov's 3-tensor. The use of this alternative form in the derivation of the weakly relativistic small-gyroradius limit (Shkarofsky 1966) leads to some apparent inconsistencies, in that the different versions of the exact expression lead to different results when Shkarofsky's procedure is applied to them. These inconsistencies are discussed here, and it is argued that Shkarofsky's procedure is valid only when applied to the original version of Trubnikov's tensor.

Covariant Vlasov theory is introduced in Sec. 2 and used to calculate the covariant generalization of Trubnikov's tensor for a magnetized plasma in Sec. 3. The integral identities implied by (1) are written down, an independent proof of them is given, and they are used to derive the manifestly gauge-invariant form of the response tensor in Sec. 4. The choice of starting point for the derivation of the weakly relativistic small-gyroradius limit is discussed in Sec. 5, and resolved in favour of the choice made implicitly by Shkarofsky (1966).

2. Covariant Vlasov theory

A covariant version of Vlasov theory may be obtained by introducing the distribution $F(x,p)$ of world lines in an 8-dimensional (x,p) phase space (see e.g. Dewar 1977, and references therein). The relation with the more familiar distribution $f(\mathbf{p})$ in the 6-dimensional $(\mathbf{x},\mathbf{p})$ phase space is

$$F(x,p) = 2\varepsilon\,\delta(p^2 - m^2)\,f(\mathbf{p}), \tag{2}$$

with $p^\mu = [\varepsilon, \mathbf{p}]$ and $p^2 = p^\mu p_\mu = \varepsilon^2 - |\mathbf{p}|^2$. Including both a magnetostatic field $F_0^{\mu\nu} = Bf^{\mu\nu}$, $B = (\frac{1}{2}F_0^{\mu\nu}F_{0\,\mu\nu})^{1/2}$, and a fluctuating field $F^{\mu\nu}(x)$, the covariant Vlasov equation is

$$\left[u^\mu\frac{\partial}{\partial x^\mu} + q\,F_0^{\mu\nu}u_\nu\frac{\partial}{\partial p^\mu} + q\,F^{\mu\nu}(x)u_\nu\frac{\partial}{\partial p^\mu}\right]F(x,p) = 0, \tag{3}$$

with $p^\mu = mu^\mu$. After Fourier transforming, expressing $F^{\mu\nu}(k)$ in terms of the

4-potential $A^\mu(k)$, and linearizing in $A(k)$, (3) gives

$$\left(ku + iq\,F_0^{\mu\nu}u_\nu\frac{\partial}{\partial p^\mu}\right)F(k,p) = q\,ku\,G^{\mu\nu}(k,u)A_\nu(k)\frac{\partial F(p)}{\partial p^\mu}, \tag{4}$$

$$G^{\mu\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu}{ku}. \tag{5}$$

The unperturbed distribution function $F(p)$ can depend only on constants of the motion. In the case of a thermal plasma it depends only on the energy ε in the rest frame of the plasma. Let the 4-velocity of the rest frame of the plasma be $\tilde{u}^\mu$ in an arbitrary inertial frame ($\tilde{u}^\mu = [1, \mathbf{0}]$ in the rest frame of the plasma). Then a covariant expression for the thermal distribution follows from (2) by replacing ε by the invariant $p\tilde{u} = mu\tilde{u}$.

It is convenient to integrate (4) along the unperturbed orbit. The orbit of a particle in a magnetostatic field is found by solving the equation of motion

$$\frac{du^\mu(\tau)}{d\tau} = \frac{q}{m}F_0^{\mu\nu}u_\nu(\tau), \tag{6}$$

where τ is the proper time for the particle. Let u_0^μ be the 4-velocity of the particle at an initial time $\tau = 0$. Then integrating (6) once, the 4-velocity at an arbitrary proper time τ may be written in the form

$$u^\mu(\tau) = \dot{t}^{\mu\nu}(\tau)\,u_{0\nu}. \tag{7}$$

A further integration gives the orbit $x^\mu = X^\mu(\tau)$, with

$$X^\mu(\tau) = x_0^\mu + t^{\mu\nu}(\tau)u_{0\nu}, \tag{8}$$

where x_0 is a constant 4-vector. Explicit evaluation gives

$$t^{\mu\nu}(\tau) = g_\parallel^{\mu\nu}\tau + g_\perp^{\mu\nu}\frac{\sin\Omega_0\tau}{\Omega_0} - \eta f^{\mu\nu}\frac{\cos\Omega_0\tau}{\Omega_0}, \tag{9}$$

$$\dot{t}^{\mu\nu}(\tau) = g_\parallel^{\mu\nu} + g_\perp^{\mu\nu}\,\cos\Omega_0\tau + \eta f^{\mu\nu}\,\sin\Omega_0\tau, \tag{10}$$

with $\Omega_0 = |q|B/m$, $\eta = q/|q|$, $g_\perp^{\mu\nu} = -f^{\mu\alpha}f_\alpha{}^\nu$ and $g_\parallel^{\mu\nu} = g^{\mu\nu} - g_\perp^{\mu\nu}$.

Integrating (4) once gives

$$F\big(k,p(\tau)\big) = -iq\,e^{ikX(\tau)}\int^\tau d\tau'\,e^{-ikX(\tau')}\,ku(\tau')G^{\mu\nu}\big(k,u(\tau')\big)A_\nu(k)\frac{\partial F(p)}{\partial p^\mu(\tau')}. \tag{11}$$

One may regard $F(p)$ as a function of any two of the constants of the motion $p^0 = \varepsilon$, $p_\perp$ and p_z, with $p_\perp = (p_x^2 + p_y^2)^{1/2}$, in the rest frame of the plasma. Note that in (11), and wherever p^μ-derivatives appear below, there are derivatives with respect to three variables, p^0, $p_\perp$ and p_z; because of the delta function in (2), one needs to choose to integrate over one of these variables, and then the derivative with respect to this variable is omitted (or set identically zero). The first-order term in an expansion of (11) in powers of A is

$$F^{(1)}\big(k,p(\tau)\big) = q\,A_\nu(k)\left(\frac{p_\parallel^\nu}{p_\perp}\frac{\partial}{\partial p_\perp} + \frac{\partial}{\partial p_{\parallel\nu}}\right)F(p)$$
$$+i\frac{q}{m}\,A_\nu(k)\int_0^\infty d\xi\,u^\nu(\tau-\xi)\,e^{ik[X(\tau)-X(\tau-\xi)]}\left[\frac{(kp)_\parallel}{p_\perp}\frac{\partial}{\partial p_\perp} + k_\parallel^\alpha\frac{\partial}{\partial p_\parallel^\alpha}\right]F(p), \tag{12}$$

with $p_\parallel^\mu = p_\nu g_\parallel^{\mu\nu}$ and $(kp)_\parallel = k_\mu p_\nu g_\parallel^{\mu\nu}$.

The linear term in the 4-current $J^\mu(k) = q\int d^4p\, u^\mu\, F^{(1)}(k,p)$ is written in the form $J^\mu(k) = \alpha^\mu{}_\nu(k)A^\nu(k)$ to identify the linear response tensor. This gives

$$\alpha^{\mu\nu}(k) = q^2\int d^4p\,\left\{p_\parallel^\mu\left(\frac{\partial}{\partial p_{\parallel\nu}} + \frac{p_\parallel^\nu}{p_\perp}\frac{\partial}{\partial p_\perp}\right)\right.$$
$$\left. + i\int_0^\infty d\xi\, u^\mu(0)u^\nu(-\xi)\, e^{ik[X(0)-X(-\xi)]}\left[\frac{(kp)_\parallel}{p_\perp}\frac{\partial}{\partial p_\perp} + k_\parallel^\alpha\frac{\partial}{\partial p_\parallel^\alpha}\right]\right\} F(p), \quad (13)$$

with $\xi = \tau - \tau'$, and where the independence of the result on time is used to choose $\tau = 0$ without loss of generality. The result (13) is a covariant generalization of a well-known result for the linear response tensor derived using a relativistic but non-covariant version of Vlasov theory (see e.g. Bornatici *et al.* 1983, Sec. 2.3).

3. Covariant form of Trubnikov's calculation

Trubnikov's tensor is evaluated for the relativistic counterpart of a Maxwellian distribution of particles, which is the Jüttner–Synge distribution

$$f(\gamma) = \frac{n\rho\, e^{-\rho\gamma}}{4\pi m^3 K_2(\rho)}, \quad (14)$$

where $\rho = m/T$ is the inverse temperature in units of the rest energy of the particle and n is to the number density in the rest frame. (The proper number density is $n_0 = n\,K_1(\rho)/K_2(\rho)$.) The corresponding 8-dimensional distribution function is

$$F(p) = \frac{n\rho}{2\pi m^2 K_2(\rho)}\,\delta(p^2 - m^2)\,\exp\left[-\rho(u\tilde{u})\right]. \quad (15)$$

On inserting (15) into (13), the derivatives of $\delta(p^2 - m^2)$ give zero identically, as is apparent from (11) with $(\partial/\partial p^\mu)\delta(p^2 - m^2) \propto u_\mu$ and $G^{\mu\nu}(k,u)u_\mu = 0$ from (5). The differentiations of the exponential function in (15) are trivial, and (13) gives

$$\alpha^{\mu\nu}(k) = -\frac{q^2\rho}{m}\int d^4p\,F(p)\left[\tilde{u}_\parallel^\mu\tilde{u}_\parallel^\nu + i(k\tilde{u})_\parallel\int_0^\infty d\xi\, u^\mu\dot{t}^{\nu\rho}(-\xi)u_\rho\, e^{iR(\xi)u}\right], \quad (16)$$

with $R^\mu(\xi) = k_\alpha[t^{\alpha\mu}(-\xi) - t^{\alpha\mu}(0)]$. For later purposes, it is convenient to write $R^\mu(\xi)$ in the form

$$R^\mu(\xi) = k_\alpha T^{\mu\alpha}(\xi),$$
$$T^{\mu\nu}(\xi) = -[t^{\mu\alpha}(-\xi) - t^{\mu\alpha}(0)]\dot{t}_\alpha{}^\nu(\xi) = t^{\mu\nu}(\xi) - t^{\mu\nu}(0), \quad (17)$$

with $t^{\mu\nu}(\xi)$ and $\dot{t}^{\mu\nu}(\xi)$ given by (9) and (10) respectively.

Trubnikov's method for evaluating the integral in (16) involves introducing a function

$$I(\xi) = \int d^4p\,F(p)\,e^{iR(\xi)u}, \quad (18)$$

so that (17) becomes

$$\alpha^{\mu\nu}(k) = -\frac{\rho q^2}{m}\left[n\tilde{u}_\parallel^\mu \tilde{u}_\parallel^\nu - i(k\tilde{u})_\parallel \int_0^\infty d\xi\, t^\nu{}_\rho(\xi)\frac{\partial^2 I(\xi)}{\partial R_\mu(\xi)\partial R_\rho(\xi)}\right]. \tag{19}$$

The integral in (17) may be evaluated by writing $\gamma = \cosh\chi$, $|\mathbf{u}| = \sinh\chi$ and $v = \tanh\chi$, and using the integral representation

$$K_\nu(x) = \int_0^\infty d\chi \,\cosh(\nu\chi)\, e^{-x\cosh\chi}. \tag{20}$$

This gives

$$I(\xi) = \frac{n\rho}{K_2(\rho)}\,\frac{K_1\big(r(\xi)\big)}{r(\xi)}, \tag{21}$$

with $r^2(\xi) = [\rho\tilde{u}^\mu - iR^\mu(\xi)][\rho\tilde{u}_\mu - iR_\mu(\xi)]$, implying

$$r(\xi) = \left[(\rho - i\omega\xi)^2 + k_\parallel^2\xi^2 + \frac{2k_\perp^2}{\mathbf{\Omega}_0^2}\,(1-\cos\mathbf{\Omega}_0\xi)\right]^{1/2} \tag{22}$$

in the rest frame of the plasma.

The resulting expression for the response tensor is

$$\alpha^{\mu\nu}(k) = -\frac{q^2 n\rho}{m}\left\{\tilde{u}_\parallel^\mu\tilde{u}_\parallel^\nu - i(k\tilde{u})_\parallel\frac{\rho}{K_2(\rho)}\int_0^\infty d\xi\, t^\nu{}_\rho(-\xi)\frac{\partial^2}{\partial R_\mu(\xi)\partial R_\rho(\xi)}\left[\frac{K_1\big(r(\xi)\big)}{r(\xi)}\right]\right\}. \tag{23}$$

Carrying out the differentiations using

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\frac{K_n(\rho)}{\rho^n}\right] = -\frac{K_{n+1}(\rho)}{\rho^{n+1}} \tag{24}$$

then leads to

$$\alpha^{\mu\nu}(k) = -\frac{q^2 n\rho}{m}\left\{\tilde{u}_\parallel^\mu\tilde{u}_\parallel^\nu - \frac{i\rho(k\tilde{u})_\parallel}{K_2(\rho)}\int_0^\infty d\xi\,\left[t^{\mu\nu}(\xi)\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} - S^{\mu\nu}(\xi)\,\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\right]\right\}, \tag{25}$$

with $t^{\nu\mu}(-\xi) = t^{\mu\nu}(\xi)$ from (10), and with

$$S^{\mu\nu}(\xi) = [\rho\tilde{u}^\mu - ik_\alpha T^{\mu\alpha}(\xi)][\rho\tilde{u}^\nu - ik_\beta T^{\beta\nu}(\xi)]. \tag{26}$$

The result (25) is the covariant generalization of Trubnikov's response 3-tensor.

For some purposes it is convenient to introduce a matrix representation of the two tensorial quantities inside the integrand in (26). The following matrix representations apply in the rest frame of the plasma, $\tilde{u}^\mu = (1, \mathbf{0})$, with the axes oriented to give $\mathbf{k} = (k_\perp, 0, k_\parallel)$. Then one has

$$t^{\mu\nu}(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos\Omega_0\xi & -\eta\sin\Omega_0\xi & 0 \\ 0 & \eta\sin\Omega_0\xi & -\cos\Omega_0\xi & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{27}$$

$$T^{\mu\nu}(\xi) = \begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & -\dfrac{\sin\Omega_0\xi}{\Omega_0} & -\eta\dfrac{1-\cos\Omega_0\xi}{\Omega_0} & 0 \\ 0 & \eta\dfrac{1-\cos\Omega_0\xi}{\Omega_0} & -\dfrac{\sin\Omega_0\xi}{\Omega_0} & 0 \\ 0 & 0 & 0 & -\xi \end{pmatrix}, \tag{28}$$

$$\left.\begin{aligned} k_\alpha T^{\mu\alpha}(\xi) &= \left(\omega\xi, k_\perp\,\frac{\sin\Omega_0\xi}{\Omega_0}, -\eta k_\perp\,\frac{1-\cos\Omega_0\xi}{\Omega_0}, k_\parallel\xi\right), \\ k_\beta T^{\beta\nu}(\xi) &= \left(\omega\xi, k_\perp\,\frac{\sin\Omega_0\xi}{\Omega_0}, \eta k_\perp\,\frac{1-\cos\Omega_0\xi}{\Omega_0}, k_\parallel\xi\right). \end{aligned}\right\} \tag{29}$$

4. Manifestly gauge-invariant form

Superficially, the response tensor (25) appears not to satisfy the charge-continuity and gauge-invariance relations (1). However, these relations are built into the theory, and hence they must be satisfied. On imposing the relations (1) on (25), one then infers a set of identities:

$$\frac{K_2(\rho)}{\rho^2} = -i\int_0^\infty d\xi\,[\rho\omega - i\beta(\xi)]\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}, \tag{30}$$

$$0 = \int_0^\infty d\xi\left\{\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} + i[\rho\omega - i\beta(\xi)]\xi\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\right\}, \tag{31}$$

$$0 = \int_0^\infty d\xi\left\{\cos(\Omega_0\xi)\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} + i[\rho\omega - i\beta(\xi)]\,\frac{\sin\,\Omega_0\xi}{\Omega_0}\,\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\right\}, \tag{32}$$

$$0 = \int_0^\infty d\xi\left\{\sin(\Omega_0)\xi\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} + i[\rho\omega - i\beta(\xi)]\,\frac{1-\cos\,\Omega_0\xi}{\Omega_0}\,\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\right\}, \tag{33}$$

$$\beta(\xi) = k_\alpha R^\alpha(\xi) = (\omega^2 - k_\parallel^2)\xi - k_\perp^2\,\frac{\sin\Omega_0\xi}{\Omega_0}. \tag{34}$$

Having identified these integral identities, one may prove them directly by partially integrating using

$$\frac{dr(\xi)}{d\xi} = -i\frac{\rho\,\omega - i\beta(\xi)}{r(\xi)} \tag{35}$$

and the identity (24). This proof is straightforward for $k_\parallel^2 > \omega^2$, when $\xi \to \infty$ implies $r(\xi) \to \infty$ with $r(\xi)$ approaching the real axis. The asymptotic form

$$K_\nu(z) \sim (\pi/2z)^{1/2}e^{-z} \tag{36}$$

then implies that the Macdonald functions are well behaved at the upper limit of integration. As no singularities are encountered, analytic continuation allows one to extend this proof to $k_\parallel^2 \leqslant \omega^2$.

Using the integral identities (30)–(33), one may rewrite (25) in the form

$$\alpha^{\mu\nu}(k) = i\frac{q^2 n\rho}{m}\,\frac{\rho\, k\tilde{u}}{K_2(\rho)}\int_0^\infty d\xi\,\big[k_\alpha T^{\mu\alpha}(\xi)\,k_\beta T^{\beta\nu}(\xi) - k_\alpha k_\beta T^{\alpha\beta}(\xi)\,T^{\mu\nu}(\xi)$$
$$-i\rho\, k\tilde{u}\,\tilde{T}_{\mu\nu}(\xi)\big]\,\frac{K_3(r(\xi))}{r^3(\xi)}, \tag{37}$$

$$\tilde{T}^{\mu\nu}(\xi) = T_{\alpha\beta}(\xi)\,G^{\alpha\mu}(k,\tilde{u})G^{\beta\nu}(k,\tilde{u}), \tag{38}$$

with $k_\alpha k_\beta T^{\alpha\beta}(\xi) = \beta(\xi)$ given by (34). Contracting this form with either k_μ or k_ν gives zero, as required by (1). Thus (37) is a manifestly covariant and gauge-invariant form of Trubnikov's response 3-tensor.

5. Weakly relativistic small-gyroradius limit

The general form of Trubnikov's response tensor is too cumbersome to be of direct use in most applications, and approximations need to be made. An important set of approximations was introduced by Shkarofsky (1966), who expanded Trubnikov's general expression for the linear response tensor in the weakly relativistic ($\rho \gg 1$) and small-gyroradius ($k_\perp^2/\Omega_0^2\rho \ll 1$) limits. Shkarofsky's procedure was applied to the 3-tensor components of (25). As shown below, applying Shkarofsky's procedure to the alternative form (37) gives a different result. This raises two questions. First, from which form is it appropriate to start when introducing Shkarofsky's approximations? Secondly, is there a prescription that allows one to apply Shkarofsky's procedure, starting from any of the different forms, without obtaining inconsistent results?

Shkarofsky's procedure involves the following four steps. First, assume $|r(\xi)| \approx \rho \gg 1$ and approximate the Macdonald functions by their asymptotic limit (36). Secondly, expand $r(\xi)$ for $k_\perp^2/\Omega_0^2\rho \ll 1$, corresponding to the small gyroradius limit. This approximation applied to (22) gives

$$r(\xi) \approx r_0(\xi) + \Lambda(1 - \cos\,\Omega_0\xi), \qquad \Lambda = \frac{k_\perp^2}{\Omega_0^2 r_0(\xi)},$$
$$r_0(\xi) = [(\rho\tilde{u} - ik_\parallel\xi)^\mu(\rho\tilde{u} - ik_\parallel\xi)_\mu]^{1/2} = [(\rho - i\omega\xi)^2 + k_\parallel^2\xi^2]^{1/2}, \tag{39}$$

where the final expression applies in the rest frame of the plasma. When inserted in the asymptotic form for the Macdonald functions, a function $\exp(\Lambda\,\cos\,\Omega_0\xi)$ appears. Thirdly, use the generating function and recursion relations for the modified Bessel functions $I_n(z)$ to write

$$\begin{bmatrix} 1 \\ \cos\,\Omega\xi \\ \sin\,\Omega\xi \\ \cos^2\,\Omega\xi \\ \sin^2\,\Omega\xi \\ \sin\,\Omega\xi\,\cos\,\Omega\xi \end{bmatrix} e^{\Lambda\,\cos\,\Omega\xi} = \sum_{n=-\infty}^{\infty} \begin{bmatrix} I_n(\Lambda) \\ I_n'(\Lambda) \\ -i(n/\Lambda)I_n(\Lambda) \\ I_n''(\Lambda) \\ (1/\Lambda)I_n'(\Lambda) - (n^2/\Lambda^2)I_n(\Lambda) \\ i(n/\Lambda^2)I_n(\Lambda) - i(n/\Lambda)I_n'(\Lambda) \end{bmatrix} e^{in\Omega\xi}. \tag{40}$$

The final step in Shkarofsky's procedure concerns the terms in the integrand in (25) with (26)–(29) that are linear and quadratic in ξ. These are evaluated by considering them as operating on exponential functions $\exp(in\Omega_0\xi)$ and $\exp[-r_0(\xi)]$ respectively. Hence one may make the replacements $\xi \;\rightarrow\; -(i/\Omega_0)(\partial/\partial n)$ and

$\xi^2 \rightarrow -[r_0(\xi)/k_\parallel](\partial/\partial k_\parallel)$, where it is understood that these derivatives operate only on the exponential functions.

To see how and why inconsistencies arise with Shkarofsky's procedure, it is helpful to consider a specific example. The example chosen is the longitudinal part of the response tensor. In the rest frame of the plasma this may be calculated either from the longitudinal part of the 3-tensor, $\alpha^{\rm L}(k) = -k_i k_j \alpha^{ij}(k)/|\mathbf{k}|^2$, or by using the Coulomb gauge, $\alpha^{\rm L}(k) = -\omega^2\alpha^{00}(k)/|\mathbf{k}|^2$. These two forms are equivalent in view of (1). Three superficially different forms are obtained:

$$\alpha^{\rm L}(k) = i\frac{q^2 n\rho^2\omega}{mK_2(\rho)|\mathbf{k}|^2}\int_0^\infty d\xi\left[(k_\perp^2\cos\boldsymbol{\Omega}_0\xi + k_\parallel^2)\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} - \left(k_\perp^2\frac{\sin\boldsymbol{\Omega}_0\xi}{\boldsymbol{\Omega}_0} + k_\parallel^2\xi\right)^2\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\right], \tag{41}$$

$$\alpha^{\rm L}(k) = \frac{q^2 n\rho\omega^2}{m|\mathbf{k}|^2}\left\{1 - \frac{i\rho\omega}{K_2(\rho)}\int_0^\infty d\xi\left[\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} - (\rho - i\omega\xi)^2\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\right]\right\}, \tag{42}$$

$$\alpha^{\rm L}(k) = \frac{q^2 n\rho^2\omega^2}{mK_2(\rho)|\mathbf{k}|^2}\int_0^\infty d\xi\,(\rho - i\omega\xi)\left(k_\perp^2\frac{\sin\boldsymbol{\Omega}_0\xi}{\boldsymbol{\Omega}_0} + k_\parallel^2\xi\right)\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}. \tag{43}$$

The form (41) is from the longitudinal part of (25), that is, from Trubnikov's 3-tensor, (42) is from the 00-component of (25), and (43) is from the manifestly gauge-invariant form (37).

Applying Shkarofsky's procedure to (41) gives

$$\alpha^{\rm L}(k) = i\frac{q^2 n\omega}{m|\mathbf{k}|^2}\int_0^\infty d\xi\,\frac{e^{\rho-\Lambda}}{[r_0(\xi)/\rho]^{5/2}}\sum_{n=-\infty}^{\infty}\left[k_\perp^2\frac{n^2 I_n(\Lambda)}{\Lambda} + k_\parallel^2 I_n(\Lambda)\left(1 + k_\parallel\frac{\partial}{\partial k_\parallel}\right) + 2k_\parallel^2 I_n(\Lambda)\,n\frac{\partial}{\partial n}\right]e^{-r_0(\xi)+in\Omega_0\xi}, \tag{44}$$

which, apart from notation, is the standard form in the literature. The integral in (44) may be written in terms of relevant plasma dispersion functions whose properties were reviewed by Robinson (1986).

Applying Shkarofsky's procedure to (42) and (43) leads to quite different results, which do not reduce to (44) even when the properties of these plasma dispersion functions are used. To see why this is the case, consider the order in ρ of the various terms. In (42) the leading terms inside the curly brackets are the unit term and the term in the integrand $\propto \rho^2$. Individually these two terms are of order ρ^2 larger than the leading terms in (41). Hence equality of (42) and (41) requires that in (42) not only the terms of leading order in ρ cancel, but also the terms of next order. The first of these cancellations follows from the approximate form of (30): applying Shkarofsky's procedure to (30) gives

$$\frac{e^{-\rho}}{\rho^{5/2}} \approx \int_0^\infty d\xi\,\frac{dr(\xi)}{d\xi}\,\frac{e^{-r(\xi)}}{r^{5/2}(\xi)} = \frac{e^{-\rho}}{\rho^{5/2}}\left(1 - \frac{5}{2\rho} + \dots\right), \tag{45}$$

which is clearly an identity only if terms of first and higher order in the expansion in $1/\rho$ are ignored. When one uses (45) in (42), the leading terms cancel trivially. However, because of spurious correction terms introduced by the approximations, the next order terms in (42) do not cancel. These terms are of order ρ larger than the leading terms in (41), and so (42) is clearly incompatible with (41) when only the leading term in (45) and in the corresponding approximate forms of (31)–(33) are used. A similar inconsistency arises when one applies Shkarofsky's procedure to (43) and uses the approximate forms of (30)–(33).

The foregoing discussion may be summarized as follows. The forms (41)–(43) are equivalent when one uses the exact forms of the identities (30)–(33). However, when Shkarofsky's procedure is applied to (41)–(43) and to (30)–(33), inconsistent results are obtained. These differences may be attributed to the fact that the leading terms in (41)–(43) in an expansion in $1/\rho$ are of different order, and the approximate forms of the identities (30)–(33) are themselves identities only to lowest order in this expansion. Hence spurious terms that do not cancel are introduced by the approximation procedure.

Now consider the two questions posed at the start of this section. First, of the three forms (41)–(43) for the longitudinal part of the response tensor, the foregoing arguments imply that (41) (which requires no cancellation of higher terms in ρ) is the only acceptable starting point for Shkarofsky's procedure. Similar arguments apply to the full tensor, rather than just its longitudinal part. Moreover, Shkarofsky's procedure should be applied only to the 3-tensor components of Trubnikov's tensor (25), because the other (00, $i0$, $0j$) terms introduced in the covariant formulation are of higher order in ρ (the 00-term is order ρ^2, and the $i0$-terms and the $0j$-terms are of order ρ compared with the ij-terms). However, the discussion is further complicated by the fact that the terms involving K_2 and K_3 in (41) are of superficially different order in ρ. The two terms are of the same order if one writes the variable of integration as $\zeta = \Omega_0 \xi$ and assumes that $k^2/\Omega_0^2 \rho \approx \Lambda$ is an independent parameter. Secondly, a prescription that allows one to introduce Shkarofsky's procedure in a way that does not lead to inconsistencies is: use the exact forms of the identities (30)–(33) to rewrite all components of $\alpha^{\mu\nu}(k)$ such that the leading terms in an expansion in $1/\rho$ are all of the same order. (For this purpose, the parameter Λ is assumed to be of zeroth order in ρ.) This turns out to be equivalent to calculating the approximate forms for the ij-terms by applying Shkarofsky's procedure to (25) and then calculating the 00-, $i0$- and $0j$-terms from them by imposing the conditions (1).

6. Conclusions

The use of a covariant formalism to derive plasma response tensors not only leads to neat and concise forms, but also can lead to alternative forms that may be useful in practice. Here these points are illustrated here by the derivation of the manifestly covariant generalization (25) of Trubnikov's response tensor. The form does not manifestly satisfy the charge-continuity and gauge-invariance relations (1), implying a set of identities (30)–(33). Use of these identities allows one to rewrite (25) in the manifestly gauge-invariant form (37), and the 3-tensor part of (37) is an alternative version of Trubnikov's 3-tensor.

In the one application discussed here this alternative form turns out not to be useful. The two forms lead to expressions that are not equivalent when one takes the weakly relativistic small-gyroradius limit. It is argued that Trubnikov's original form is the more appropriate starting point for these approximations, as assumed implicitly in the literature (Shkarofsky 1966). However, the range of validity of these approximations, especially the small-gyroradius limit, needs further investigation. In contrast, in the ultrarelativistic limit ($\rho \to 0$), (37) provides a somewhat simpler starting point than (25) for an expansion in powers of ρ.

Acknowledgements

I thank Peter Robinson and Stephen Hardy for helpful comments on the manuscript.

References

Bornatici, M., Cano, R., De Barbieri & O., Engelmann, F. 1983 Electron cyclotron emission and absorption in fusion plasmas. *Nucl. Fusion* **23**, 1153–1257.

Bornatici, M. & Engelmann, F. 1994 Electron-cyclotron absorption and emission: 'vexatae quaestiones'. *Phys. Plasmas* **1**, 189–198.

Dewar, R.L. 1977 Energy-momentum tensors for dispersive electromagnetic waves. *Aust. J. Phys.* **30**, 533–575.

Melrose, D.B. 1973 A covariant formulation of wave dispersion. *Plasma Phys.* **15**, 99–106.

Melrose, D.B. 1982 Covariant description of dispersion in a relativistic thermal electron gas. *Aust. J. Phys.* **35**, 41–52.

Melrose, D.B. 1983 Quantum electrodynamics in strong magnetic fields. II. Photon fields and interactions. *Aust. J. Phys.* **36**, 775–798.

Melrose, D.B. 1987 The cubic response tensor for a magnetized plasma: a covariant forward-scattering approach. *Aust. J. Phys.* **40**, 139–156.

Robinson, P.A. 1986 Relativistic plasma dispersion functions. *J. Math. Phys.* **27**, 1206–1214.

Robinson, P.A. 1988 Weakly relativistic dispersion of Bernstein waves. *Phys. Fluids* **31**, 107–114.

Shkarofsky, I.P. 1966 Dielectric tensor in Vlasov plasmas near cyclotron harmonics. *Phys. Fluids* **9**, 561–580.

Trubnikov, B.A. 1958 Magnetic emission of high temperature plasma. Thesis, Moscow Institute of Engineering and Physics [English translation in AEC-tr-4073, US Atomic Energy Commission, Oak Ridge, Tennessee, 1960].