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Response of a relativistic anisotropic thermal plasma. Part 1. Unmagnetized particles

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Expressions for the linear response tensor are derived using a covariant forward-scattering method for unmagnetized relativistic thermal distributions restricted to one-dimensional (strictly parallel) and two-dimensional (strictly perpendicular) motions. The freedom to make Lorentz transformations is used to generalize the strictly perpendicular distribution to a counterpart of the nonrelativistic Dory–Guest–Harris (DGH) distribution.

1. Introduction

In this paper and an accompanying paper, Melrose (1997*a* hereinafter referred to as Part 2) a covariant forward-scattering method is used to evaluate the response tensor for one-dimensional (strictly parallel), two-dimensional (strictly perpendicular) and three-dimensional (isotropic) relativistic thermal distributions. A covariant method of calculation of the response tensor is used in this paper, and this method differs from that used by Melrose (1997*a*) in that the starting point for the calculation is an expression for the response tensor for an arbitrary distribution function derived using a covariant forward-scattering method. This method is applied in this paper to an unmagnetized plasma in such a way that the generalization to the magnetized case in Part 2 is straightforward. The result for the isotropic case is shown to be equivalent to that derived by Melrose (1997*a*), and the results for the strictly parallel and strictly perpendicular cases are derived here for the first time.

A covariant expression for the response tensor for a 'one-dimensional' relativistic thermal distribution was derived by Godfrey *et al.* (1975*a*), and a possible application to relativistic beam instabilities was suggested by Godfrey *et al.* (1975*b*). The case treated by Godfrey *et al.* (1975*a*) is 'one-dimensional' in the sense that it is the relativistic thermal distribution function in an artificial 2-dimensional space–time. In the present paper, 'one-dimensional' refers to particles whose motion is restricted to a 2-dimensional subspace of 4-dimensional space–time. The alternative description 'strictly parallel' is used here to avoid possible confusion with the case considered by Godfrey *et al.* (1975*a*). For a similar reason, the two-dimensional case is referred to here as the 'strictly perpendicular' distribution.

One motivation for this investigation is the possibility of identifying a relativistic counterpart for the nonrelativistic bi-Maxwellian streaming distribution used extensively to discuss the response of hot anisotropic plasmas (see e.g. Stix 1962).

A more general nonrelativistic anisotropic thermal-like distribution is that introduced by Dory *et al.* (1965), and referred to here as a DGH distribution. A specific nonrelativistic ($\mathbf{p} = m\mathbf{v}$) DGH distribution is

$$f(\mathbf{p}) \propto G(v_\parallel)\, v_\perp^{2j} \exp\left(-\frac{v_\perp^2}{2V_\perp^2}\right), \qquad G(v_\parallel) = \exp\left[-\frac{(v_\parallel - v_0)^2}{2V_\parallel^2}\right], \tag{1.1}$$

where v_0, $V_\parallel$ and $V_\perp$ are constants. In the evaluation of the response tensor for the distribution (1.1) the integrals over the parallel and perpendicular components in momentum space factorize, and standard integrals for Bessel functions enable one to evaluate those over the perpendicular components. For the particular $G(v_\parallel)$ chosen in (1.1), the integral over the parallel component is evaluated in terms of the familiar plasma dispersion function. For $j = 0$, (1.1) reduces to a bi-Maxwellian streaming distribution (see e.g. Stix 1962). A specific motivation for including relativistic effects in such an anisotropic distribution is discussed in Part 2.

In one sense there can be no relativistic counterpart of the distribution (1.1). The important feature of a DGH distribution is that it satisfies the following two conditions:

(*a*) the evaluation of the response tensor factorizes into parallel and perpendicular components;

(*b*) the distribution reduces to a thermal distribution in the appropriate special case.

The incompatibility of conditions (a) and (b) is seen by noting that the relativistic thermal distribution $f(\mathbf{p}) \propto \exp(-\rho\gamma)$, with ρ the dimensionless inverse temperature, does not factorize in either velocity space, $\gamma = (1 - v_\parallel^2 - v_\perp^2)^{-1/2}$, or momentum space, $\gamma = (1 + p_\parallel^2/m^2 + p_\perp^2/m^2)^{1/2}$, where units with $c = 1$ are used. In another sense, the following argument suggests that there must be a relativistic generalization of the DGH distribution. First note that Trubnikov and Yakubov (1963) found an exact expression for the response tensor for a relativistic two-dimensional (or strictly perpendicular) thermal-like distribution (cf. also Bornatici *et al.* 1983). This distribution is such that the particles have no motion along the magnetic field lines, and this is too artificial for practical purposes. However, given a covariant version of the response tensor found by Trubnikov and Yakubov (1963), one may apply a Lorentz transformation along the magnetic field lines to find the response tensor for particles with any specific parallel motion. Using this freedom, one can include an arbitrary distribution of parallel motions superimposed on the thermal motion in the perpendicular directions. This procedure, which is developed here, allows one to identify a specific class of relativistic distributions that may be regarded a counterparts of the DGH distribution functions (1.1).

In Sec. 2 the method of calculation of the linear response tensor is introduced and illustrated by rederiving the covariant form of Trubnikov's response tensor for a relativistic unmagnetized thermal distribution (cf. Melrose 1997*a*). In Sec. 3 this method is used to treat the strictly parallel distribution, and in Sec. 4 it is used to treat the strictly perpendicular distribution. The proposed relativistic counterpart of the DGH distribution (1.1) is discussed in Sec. 5. The reduction of the relativistic plasma dispersion functions to the nonrelativistic plasma dispersion function is discussed in Sec. 6.

2. Derivation of Trubnikov's response tensor

A covariant form of Trubnikov's response tensor for an unmagnetized thermal plasma was derived by Melrose (1982). In this section the isotropic thermal distribution is reconsidered in order to develop the method used to treat this and related distributions.

2.1. Response tensor for an arbitrary distribution

The starting point for the calculation is a general expression for the response tensor for an unmagnetized distribution of particles. The particles are described by their distribution $F(p)$ in 8-dimensional phase space (cf. Dewar 1997). The response tensor is

$$\alpha^{\mu\nu}(k) = -\frac{q^2}{m}\int d^4p\, F(p)\, a^{\mu\nu}(k,u), \tag{2.1}$$

$$a^{\mu\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + \frac{k^2\, u^\mu u^\nu}{(ku)^2}. \tag{2.2}$$

The number density n of particles in the rest frame of the plasma and the proper number n_{pr} are given by

$$n = \int d^4p\, u^0\, F(x,p) = \int d^3\mathbf{p}\, f(\mathbf{x},\mathbf{p},t), \tag{2.3}$$

$$n_{\mathrm{pr}} = \int d^4p\, F(x,p) = \int d^3\mathbf{p}\, \frac{f(\mathbf{x},\mathbf{p},t)}{\gamma} \tag{2.4}$$

respectively.

In evaluating the response tensor below, the method used requires that the denominators in (2.2) be rewritten as exponentials using

$$\frac{1}{ku} = -i\int_0^\infty d\xi\, e^{i\,ku\xi}, \qquad \frac{1}{(ku)^2} = -\int_0^\infty d\xi\,\xi\, e^{i\,ku\xi}, \tag{2.5}$$

where the causal condition is taken into account in choosing the sign in the exponent. The response tensor (2.1) with (2.2) becomes

$$\alpha^{\mu\nu}(k) = -\frac{q^2}{m}\left[n_{\mathrm{pr}}g^{\mu\nu} + i\int d^4p\, F(p)\int_0^\infty d\xi\, e^{iku\,\xi}\,(k^\mu u^\nu + k^\nu u^\mu + ik^2\xi u^\mu u^\nu)\right], \tag{2.6}$$

where (2.4) is used.

For the purpose of comparison with the magnetized case, it is useful to rewrite (2.6) further in the form

$$\alpha^{\mu\nu}(k) = \frac{q^2}{m}\int d^4p\, F(p)\int_0^\infty d\xi\,\xi\, e^{iku\,\xi}\,(ku)^2\, a^{\mu\nu}(k,u). \tag{2.7}$$

The starting point for the calculations below is the form (2.7). Results obtained by starting from the form (2.6) are quoted in Appendix A.

2.2. The Jüttner–Synge distribution

The relativistic counterpart of a nonrelativistic Maxwellian distribution of particles is a distribution $f(\mathbf{p}) \propto \exp(-\varepsilon/T)$, where $\varepsilon = \gamma m$ is the energy of a particle and T is the temperature in energy units, that is, with Boltzmann's constant set equal to unity. After normalization, this becomes the Jüttner–Synge distribution (see e.g. Synge 1957)

$$f(\gamma) = \frac{n\rho\, e^{-\rho\gamma}}{4\pi m^3 K_2(\rho)}, \tag{2.8}$$

where $K_\nu(x)$ is a Macdonald function of order ν, with $\rho = m/T$ the inverse temperature in units of the rest energy of the particle. The normalization in (2.8) follows by writing the integral over momentum space in terms of the variable χ:

$$\gamma = \cosh\chi, \qquad |\mathbf{p}| = m\sinh\chi, \qquad v = \tanh\chi. \tag{2.9}$$

One uses the integral representation of Macdonald functions

$$K_\nu(x) = \frac{(x/2)^\nu\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty d\chi\, \sinh^{2\nu}\chi\, \exp(-x\cosh\chi). \tag{2.10}$$

Also note the identity

$$\frac{1}{x}\frac{d}{dx}\left[\frac{K_\nu(x)}{x^\nu}\right] = -\frac{K_{\nu+1}(x)}{x^{\nu+1}}. \tag{2.11}$$

The ratio of the proper number density to the rest-frame density for this distribution is $n_{\rm pr}/n = K_1(\rho)/K_2(\rho)$.

In an arbitrary frame, in which the 4-velocity of the rest frame of the plasma is $\tilde{u}$, the 8-dimensional counterpart of (2.8) is

$$F(p) = \frac{n\rho}{2\pi m^2 K_2(\rho)}\,\delta(p^2 - m^2)\,\exp(-\rho\, u\tilde{u}). \tag{2.12}$$

A streaming motion may be included simply by choosing a frame in which the plasma is not at rest, specifically choose $\tilde{u} = u_0$ with $u_0 = [\gamma_0, \gamma_0\mathbf{v}_0]$, $\gamma_0 = (1-v_0^2)^{-1/2}$ to introduce streaming with velocity $\mathbf{v}_0$.

2.3. *Trubnikov's method of evaluation of the integral*

Trubnikov's procedure for evaluating the integral that appears when (2.12) is inserted into (2.1) involves expressing all dependence on u in exponential form, and evaluating the resulting integral in terms of the integral representation (2.10) of a Macdonald function. The resonant denominators are rewritten in the appropriate form in (2.6) and (2.7). The powers of the 4-velocity that appear in (2.6) and (2.7) are written as

$$u^\mu = \left.\frac{\partial e^{su}}{\partial s_\mu}\right|_{s=0}, \qquad u^\mu u^\nu = \left.\frac{\partial^2 e^{(s+s')u}}{\partial s_\mu \partial s'_\nu}\right|_{s=0,s'=0}. \tag{2.13}$$

(It might be remarked that the manner in which the limits as $s \to 0$, and $s' \to 0$ are taken is not important in the following, and all the results may be derived by alternative methods that do not involve introducing s and s'.) The basic integral that needs to be evaluated is then

$$\begin{aligned} I(\rho, \xi, s+s') &= \int d^4p\, F(p)\, e^{i\,ku\,\xi + (s+s')u} \\ &= \frac{n\rho}{2\pi m^2 K_2(\rho)}\int d^4p\,\delta(p^2 - m^2)\, e^{-[\rho\tilde{u} - ik\xi - (s+s')]u}. \end{aligned} \tag{2.14}$$

The integral in (2.14) is performed by first integrating over p^0 using the delta function, carrying out the angular integrals, rearranging the exponent to

$\cosh(\chi + i\chi_0)$, shifting the contour of integration by $-i\chi_0$, and performing the remaining χ integral using (2.10). This gives

$$I(\rho, \xi, s + s') = \frac{n\rho}{K_2(\rho)} \frac{K_1\big(r(\xi)\big)}{r(\xi)}, \tag{2.15}$$

$$r(\xi) = \{[a(\xi) - (s + s')]^2\}^{1/2}, \qquad a^\mu(\xi) = \rho\tilde{u}^\mu - ik^\mu\xi. \tag{2.16}$$

Let $\hat{u}^\mu$ denote the operation of differentiating with respect to s_μ and setting $s = 0$. In the integrand in (2.6), u^μ is then to be replaced by $\hat{u}^\mu$. The derivatives

$$\hat{u}^\mu \frac{K_1\big(r(\xi)\big)}{r(\xi)} = a^\mu(\xi) \frac{K_2\big(r(\xi)\big)}{r^2(\xi)}, \tag{2.17}$$

$$\hat{u}^\mu\hat{u}^\mu \frac{K_1\big(r(\xi)\big)}{r(\xi)} = -g^{\mu\nu} \frac{K_2\big(r(\xi)\big)}{r^2(\xi)} + a^\mu(\xi)a^\nu(\xi) \frac{K_3\big(r(\xi)\big)}{r^3(\xi)}, \tag{2.18}$$

are then required in the evaluation of (2.6) with (2.12). Although both (2.17) and (2.18) are required when the form (2.6) is used, in view of

$$(ku)^2 a^{\mu\nu}(k, u) = H^{\mu\nu}{}_{\sigma\tau} u^\sigma u^\tau, \tag{2.19}$$

$$H^{\mu\nu}{}_{\sigma\tau} u^\sigma u^\tau = (k^\sigma g^{\alpha\mu} - k^\alpha g^{\sigma\mu})\,(k^\tau g^{\beta\nu} - k^\beta g^{\tau\nu}), \tag{2.20}$$

only (2.18) is required when (2.7) is used.

2.4. Response tensor for a thermal plasma

The resulting expression for the linear response tensor for the isotropic thermal distribution (2.12) is

$$\alpha^{\mu\nu}(k) = \frac{q^2 n\rho}{mK_2(\rho)} \int_0^\infty d\xi\, \xi \left\{ -2(k^2 g^{\mu\nu} - k^\mu k^\nu) \frac{K_2\big(r(\xi)\big)}{r^2(\xi)} + [ka(\xi)]^2 a^{\mu\nu}\big(k, a(\xi)\big) \frac{K_3\big(r(\xi)\big)}{r^3(\xi)} \right\}, \tag{2.21}$$

where (2.20), (2.18) and (2.2) are used.

The form (2.21) for the response tensor is superficially different from the form obtained by starting from the response tensor derived using Vlasov theory. This alternative form is the unmagnetized version of the result obtained by Melrose (1997*a*), and is the covariant version of the result derived by Trubnikov (1958). It is

$$\alpha^{\mu\nu}(k) = \frac{q^2 n\rho}{m} \left\{ -\tilde{u}^\mu\tilde{u}^\nu - i\frac{k\tilde{u}\,\rho}{K_2(\rho)} \int_0^\infty d\xi \left[-g^{\mu\nu} \frac{K_2\big(r(\xi)\big)}{r^2(\xi)} + a^\mu(\xi)a^\nu(\xi) \frac{K_3\big(r(\xi)\big)}{r^3(\xi)} \right] \right\}. \tag{2.22}$$

The equivalence of the two forms may be established using the identity

$$f(0)\frac{K_\nu(\rho)}{\rho^\nu} + \int_0^\infty d\xi \left[\frac{df(\xi)}{d\xi} \frac{K_\nu\big(r(\xi)\big)}{r^\nu(\xi)} + if(\xi)\, ka(\xi) \frac{K_{\nu+1}\big(r(\xi)\big)}{r^{\nu+1}(\xi)} \right] = 0, \tag{2.23}$$

which applies for arbitrary $f(\xi)$ and ν. The identity may be confirmed by a partial integration using (2.11) and $\partial r(\xi)/\partial\xi = -ika(\xi)/r(\xi)$. The equivalence of (2.21) and

(2.22) is most easily established by using (2.23) to rewrite all terms in (2.21) and (2.22) in the form

$$\alpha^{\mu\nu}(k) = \frac{q^2 n\rho}{mK_2(\rho)} \int_0^\infty d\xi \, T^{(3)\mu\nu}(\xi) \, \frac{K_3\big(r(\xi)\big)}{r^3(\xi)}, \tag{2.24}$$

and showing that the resulting expressions for $T^{(3)\mu\nu}(\xi)$ are the same. One finds that (2.21) and (2.22) reduce to (2.24) with

$$T^{(3)\mu\nu}(\xi) = -i\rho\, k\tilde{u}\, \xi\, [\xi(k^2 g^{\mu\nu} - k^\mu k^\nu) + i\rho\, k\tilde{u}\, a^{\mu\nu}(k, \tilde{u})]. \tag{2.25}$$

The form (2.22) is the relevant one to start from when making the weakly relativistic approximation. As discussed by Melrose (1997*a*), if one starts from other forms then the expansion in $1/\rho$ in the weakly relativistic limit leads to inconsistencies when only the leading term in the asymptotic expansion of the Macdonald functions is retained. Hence, in treating other distributions below, it is important to identify forms that are the counterpart of (2.22).

3. Strictly parallel thermal distribution

The foregoing method is used in this section to derive the response tensor for a one-dimensional, or strictly parallel, relativistic thermal distribution. This corresponds to a distribution in which the motion of the particles is confined to one direction, with a thermal spread in velocities.

3.1. Strictly parallel distribution

The strictly parallel counterpart of the isotropic relativistic thermal distribution (2.8) is $f(\mathbf{p}) \propto \delta^2(\mathbf{p}_\perp)\exp(-\rho\gamma)$, where $\mathbf{p}_\perp$ is the projection of $\mathbf{p}$ onto the plane orthogonal to the axis of symmetry, which is specified by a spacelike unit 4-vector b^μ ($b^2 = -1$). The strictly parallel counterpart of (2.12) is

$$F(p) = \frac{2n}{K_1(\rho_\parallel)}\, \delta(p^2 - m^2)\, \delta^2(\mathbf{p}_\perp)\, \exp(-\rho_\parallel\, u\tilde{u}). \tag{3.1}$$

The separation of the 4-dimensional space into two 2-dimensional parallel and perpendicular subspaces follows by writing

$$g_\parallel^{\mu\nu} = \tilde{u}^\mu \tilde{u}^\nu - b^\mu b^\nu, \qquad g_\perp^{\mu\nu} = g^{\mu\nu} - g_\parallel^{\mu\nu}, \tag{3.2}$$

and then $(ab)_\parallel = a_\mu b_\nu g_\parallel^{\mu\nu}$, and so on. In (3.1) $\delta^2(\mathbf{p}_\perp)$ is to be interpreted as implying the projection of the integral d^4p onto the 2-dimensional parallel subspace. A streaming velocity v_0 is then included in (3.1) simply by interpreting the 4-velocity as $\tilde{u} = u_0$, $u_0 = [\gamma_0, \gamma_0\mathbf{v}_0]$.

3.2. The response tensor for the strictly parallel distribution

The derivation of the response tensor leading to (2.21) is readily modified to treat the strictly parallel case. The changes are that the orders of all the functions $K_\nu(z)/z^\nu$ are reduced by unity, and s^μ and s'^μ have no space components orthogonal to $\mathbf{b}$ in the rest frame. With these changes, the counterparts of (2.15), (2.16) and (2.17) are

$$I(\rho_\parallel,\xi) = \frac{n}{K_1(\rho_\parallel)}\,K_0\big(r_\parallel(\xi)\big), \tag{3.3}$$

$$r_\parallel(\xi) = \{[a_\parallel(\xi)]^2\}^{1/2}, \qquad a_\parallel^\mu(\xi) = \rho_\parallel\tilde{u}_\parallel^\mu - ik_\parallel^\mu\xi, \tag{3.4}$$

$$\hat{u}^\mu K_0\big(r_\parallel(\xi)\big) = a_\parallel^\mu(\xi)\,\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel^2(\xi)}, \tag{3.5}$$

$$\hat{u}^\mu\hat{u}^\nu K_0\big(r_\parallel(\xi)\big) = -g_\parallel^{\mu\nu}\,\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)} + a_\parallel^\mu(\xi)a_\parallel^\nu(\xi)\,\frac{K_2\big(r_\parallel(\xi)\big)}{r_\parallel^2(\xi)} \tag{3.6}$$

respectively, with $k_\parallel^\mu = g_\parallel^{\mu\nu}k_\nu$, and so on. (To distinguish the square of $k_\parallel^\mu$ from the square of the $\parallel$ component of $\mathbf{k}$, the notation $(k^2)_\parallel = \omega^2 - k_\parallel^2$ is used, and similarly $(k^2)_\perp = -k_\perp^2$.) The counterpart of (2.21) is

$$\begin{aligned}\alpha^{\mu\nu}(k) = \frac{q^2 n}{mK_1(\rho_\parallel)}\int_0^\infty d\xi\,\xi\,\bigg\{&[ka_\parallel(\xi)]^2\,a^{\mu\nu}\big(k,a_\parallel(\xi)\big)\,\frac{K_2\big(r_\parallel(\xi)\big)}{r_\parallel^2(\xi)}\\ &-[(k^2)_\parallel g^{\mu\nu} - k_\parallel^\mu k^\nu - k^\mu k_\parallel^\nu + k^2 g_\parallel^{\mu\nu}]\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)}\bigg\}.\end{aligned} \tag{3.7}$$

The result (3.7) may be written in a variety of different ways using the counterpart of (2.23), which is

$$f(0)\frac{K_\nu(\rho_\parallel)}{\rho_\parallel^\nu} + \int_0^\infty d\xi\left[\frac{df(\xi)}{d\xi}\,\frac{K_\nu\big(r_\parallel(\xi)\big)}{r_\parallel^\nu(\xi)} + if(\xi)\,ka_\parallel(\xi)\,\frac{K_{\nu+1}\big(r_\parallel(\xi)\big)}{r_\parallel^{\nu+1}(\xi)}\right] = 0. \tag{3.8}$$

One alternative form is given in Appendix A. The alternative form that is the nearest counterpart for the strictly parallel distribution to the form (2.22) for the isotropic distribution is

$$\begin{aligned}\alpha^{\mu\nu}(k) = \frac{q^2 n}{m}\bigg\{&-\frac{n_{\rm pr}}{n}\,g_\perp^{\mu\nu} - \rho_\parallel\tilde{u}_\parallel^\mu\tilde{u}_\parallel^\nu\\ &-\frac{i}{K_1(\rho_\parallel)}\int_0^\infty d\xi\,\bigg[(\rho_\parallel\,k\tilde{u}_\parallel - ik_\perp^2\xi)\bigg(-g_\parallel^{\mu\nu}\xi\,\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)}\\ &+a_\parallel^\mu(\xi)a_\parallel^\nu(\xi)\,\frac{K_2\big(r_\parallel(\xi)\big)}{r_\parallel^2(\xi)}\bigg) + [a_\parallel^\mu(\xi)k_\perp^\nu + k_\perp^\mu a_\parallel^\nu(\xi)]\,\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)}\bigg]\bigg\}.\end{aligned} \tag{3.9}$$

As in the isotropic case, the equivalence of the various forms for the response tensor is most easily established by using (3.8) to express them in a form analogous to (2.24), that is, a form that involves an integral over $K_2\big(r_\parallel(\xi)\big)/r_\parallel^2(\xi)$.

The form (3.9) exhibits a property that may be deduced directly from (2.1) with (2.2) and (2.4): if the particles have no motion along a given direction then the response tensor has diagonal component equal to $-q^2n_{\rm pr}/m$ along that direction. Here this applies to the two directions that span the perpendicular subspace. The form (3.9) reduces to a strictly one-dimensional counterpart of (2.22) for parallel propagation ($k_\perp = 0$).

4. Strictly perpendicular thermal distribution

In this section the method used above is applied to derive the response tensor for a two-dimensional, or strictly perpendicular, relativistic thermal distribution. This corresponds to a distribution in which the motion of the particles is confined to two spatial dimensions, with a thermal spread in velocities.

4.1. Strictly perpendicular distribution

The strictly perpendicular counterpart of the thermal distribution of (2.8) is $f(\mathbf{p}) \propto \delta(p_\parallel)\,\exp(-\rho_\perp\gamma_\perp)$, with $\gamma_\perp = (1 + p_\perp^2/m^2)^{1/2}$. The counterpart of (2.12) is then

$$F(p) = \frac{n\rho_\perp^{1/2}}{(2\pi)^{1/2} m K_{3/2}(\rho_\perp)}\,\delta(p^2 - m^2)\,\delta(pb)\,\exp(-\rho_\perp\, u\tilde{u}), \tag{4.1}$$

where b^μ is a unit vector ($b^2 = -1$) along the axis of symmetry in the rest frame. It is convenient to introduce the notation

$$a_{3\perp}^\mu = g_{3\perp}^{\mu\nu} a_\nu, \qquad g_{3\perp}^{\mu\nu} = g^{\mu\nu} + b^\mu b^\nu. \tag{4.2}$$

The metric tensor $g_{3\perp}^{\mu\nu}$ spans the 3-dimensional subspace orthogonal to the axis of symmetry, and allows one to make a separation into components in this subspace and components along the axis of symmetry. Note that $\tilde{u}^\mu$ is in the 3-dimensional subspace ($\tilde{u}_{3\perp}^\mu = \tilde{u}^\mu$) and that b^μ is orthogonal to it ($b_{3\perp}^\mu = 0$).

4.2. The response tensor for the strictly perpendicular distribution

The evaluation of the response tensor for the distribution (4.1) closely parallels the derivation of (2.21) for the isotropic case and (3.7) for the strictly parallel case. Compared with the isotropic case, the changes are that (i) the orders of all the functions $K_\nu(z)/z^\nu$ are reduced by one half, and (ii) the parallel components of s^μ and s'^μ are identically zero. With these changes, in place of (3.3), (3.4) and (3.6) one has

$$I(\rho_\perp, \xi) = \frac{n\rho_\perp^{1/2}}{K_{3/2}(\rho_\perp)}\,\frac{K_{1/2}\big(r_\perp(\xi)\big)}{r_\perp^{1/2}(\xi)}, \tag{4.3}$$

$$r_\perp(\xi) = \{[a_\perp(\xi)]^2\}^{1/2}, \qquad a_\perp^\mu(\xi) = \rho_\perp\tilde{u}^\mu - ik_{3\perp}^\mu\xi, \tag{4.4}$$

$$\hat{u}^\mu\,\frac{K_{1/2}\big(r_\perp(\xi)\big)}{r_\perp^{1/2}(\xi)} = a_\perp^\mu(\xi)\frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)}, \tag{4.5}$$

$$\hat{u}^\mu\hat{u}^\nu\,\frac{K_{1/2}\big(r_\perp(\xi)\big)}{r_\perp^{1/2}(\xi)} = -g_{3\perp}^{\mu\nu}\,\frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} + a_\perp^\mu(\xi)a_\perp^\nu(\xi)\,\frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \tag{4.6}$$

respectively.

The counterpart of (3.7) is

$$\alpha^{\mu\nu}(k) = \frac{q^2 n\rho_\perp^{1/2}}{mK_{3/2}(\rho_\perp)}\int_0^\infty d\xi\,\xi\,\bigg\{\big[ka_\perp(\xi)\big]^2\,a^{\mu\nu}\big(k, a_\perp(\xi)\big)\,\frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \\ -[(k^2)_{3\perp}g^{\mu\nu} - k_{3\perp}^\mu k^\nu - k^\mu k_{3\perp}^\nu + k^2 g_{3\perp}^{\mu\nu}]\frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)}\bigg\}. \tag{4.7}$$

As with the isotropic and strictly parallel cases, the result may be written in a variety of forms, related by an identity analogous to (2.23) or (3.8). One alternative form is given in Appendix A. Another, which is analogous to (3.9), is

$$\alpha^{\mu\nu}(k) = \frac{q^2 n}{m}\left\{\frac{n_{\rm pr}}{n}\, b^\mu b^\nu - \rho_\perp \tilde{u}^\mu \tilde{u}^\nu \right.$$
$$-i\frac{\rho_\perp^{1/2}}{K_{3/2}(\rho_\perp)}\int_0^\infty d\xi\, \left[(\rho_\perp\, k\tilde{u} - ik_\parallel^2\xi)\left(- g_{3\perp}^{\mu\nu}\,\frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)}\right.\right.$$
$$\left. + a_\perp^\mu(\xi) a_\perp^\nu(\xi)\,\frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)}\right)$$
$$\left.\left. + \big[a_\perp^\mu(\xi) k_\parallel b^\nu + k_\parallel b^\mu a_\perp^\nu(\xi)\big]\,\frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)}\right]\right\}, \tag{4.8}$$

with $n_{\rm pr} = nK_{1/2}(\rho_\perp)/K_{3/2}(\rho_\perp)$. The space components of (4.8) are equivalent to the unmagnetized (zero-B) limit of the components of the response tensor derived by Trubnikov and Yakubov (1963). The form (4.8) follows by using the identity (3.8), with subscripts $\|$ replaced by $\perp$, first with $\nu = \frac{3}{2}$ and

$$f(\xi) = -i\xi\, ka_\perp(\xi) g^{\mu\nu} + i\xi[k^\mu a_\perp^\nu(\xi) + k^\nu a_\perp^\mu(\xi)] + a^\mu(\xi) a^\nu(\xi)$$

and then with $\nu = \frac{1}{2}$ and

$$f(\xi) = -ka_\perp(\xi) b^\mu b^\nu.$$

The Macdonald functions that appear in the strictly perpendicular case reduce to simpler functions:

$$K_{1/2}(z) = \left(\frac{\pi}{2}\right)^{1/2}\frac{e^{-z}}{z^{1/2}}, \qquad K_{3/2}(z) = \left(\frac{\pi}{2}\right)^{1/2}\frac{e^{-z}}{z^{3/2}}(1+z), \tag{4.9a, b}$$

$$K_{5/2}(z) = \left(\frac{\pi}{2}\right)^{1/2}\frac{e^{-z}}{z^{5/2}}(3 + 3z + z^2). \tag{4.9c}$$

Thus, for example, the proper density is related to the density in the rest frame by $n_{\rm pr} = n\rho_\perp/(1+\rho_\perp)$.

5. A relativistic counterpart of a DGH distribution

As noted above there must exist a class of relativistic distributions that are related to DGH distributions and that are constructed by operating on the strictly perpendicular distribution. Here this class of distributions is identified and the response tensor is written down for them. There are two steps involved in starting from the strictly perpendicular distribution (4.1) and constructing a relativistic generalization of the DGH distribution (1.1). First, the counterpart of the factor $v_\perp^{2j}$ needs to be included. A factor $(p_\perp/m)^{2j}$ may be included in (4.1), but it is simpler to include a power of $\gamma_\perp = (1 + p_\perp^2/m^2)^{1/2}$. Here a factor $(\gamma_\perp)^r$ is included; the distribution with a factor $(p_\perp/m)^{2j}$ follows from the appropriate sum of distributions with factors $(\gamma_\perp)^r$, $r = 0, 2, \ldots, 2j$. Secondly, the ability to make Lorentz transformations is used to include an arbitrary distribution in parallel velocities $v_\|$.

5.1. *Inclusion of the factor* $(\gamma_\perp)^r$

The generalization of (4.1) to include a factor $(\gamma_\perp)^r$ follows by differentiating with respect to $\rho_\perp$. This leads to a class of distributions of the form

$$F(p) = \frac{n\,\delta(p^2 - m^2)\,\delta(pb)\,(\hat{d}_\perp)^r \exp(-\rho_\perp\, u\tilde{u})}{(2\pi)^{1/2} m(\hat{d}_\perp)^r [K_{3/2}(\rho_\perp)/\rho_\perp^{1/2}]}, \tag{5.1}$$

with $\hat{d}_\perp = -\partial/\partial\rho_\perp$ and where the resulting factor $(u\tilde{u})^r$ in the numerator reduces to $(\gamma_\perp)^r$ in the rest frame.

The generalization of (4.7) is

$$\begin{aligned} \alpha^{\mu\nu}(k) = &\frac{q^2 n}{m(\hat{d}_\perp)^r [K_{3/2}(\rho_\perp)/\rho_\perp^{1/2}]} \\ &\times \int_0^\infty d\xi\,\xi\,(\hat{d}_\perp)^r \left\{ [ka_\perp(\xi)]^2\, a^{\mu\nu}\big(k, a_\perp(\xi)\big) \frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \right. \\ &\left. -[(k^2)_{3\perp} g^{\mu\nu} - k^\mu_{3\perp} k^\nu - k^\mu k^\nu_{3\perp} + k^2 g^{\mu\nu}_{3\perp}] \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} \right\}. \end{aligned} \tag{5.2}$$

Explicit evaluation of the derivatives involves

$$\hat{d}_\perp r_\perp(\xi) = \frac{ka_\perp(\xi)}{r_\perp(\xi)}, \qquad \hat{d}_\perp a^\mu_\perp(\xi) = \tilde{u}^\mu. \tag{5.3}$$

However, the resulting explicit form of (5.2) becomes rapidly more cumbersome with increasing r. The generalization of the form (4.8) follows by analogy with (5.2) except for the two terms that precede the ξ integral: the term involving $b^\mu b^\nu$ is unchanged and in the remaining term $(\hat{d}_\perp)^r[K_{3/2}(\rho_\perp)\rho_\perp^{1/2}]/(\hat{d}_\perp)^r[K_{3/2}(\rho_\perp)/\rho_\perp^{1/2}]$ replaces $\rho_\perp$.

5.2. *Inclusion of a parallel distribution*

An arbitrary distribution of parallel velocities $v_\parallel$ may be included in (5.1) by using the freedom to make Lorentz transformations. Suppose that in (5.1), $\tilde{u}$ is replaced by u_0 corresponding to a speed v_0 along the axis, relative to some specific rest frame. In the rest frame, this corresponds to the particles all having velocity $v_\parallel = v_0$ along the axis. By considering a sum of a finite number of distributions of the form (5.1) each with a different u_0 and then taking the continuum limit with an appropriate weighting, an arbitrary distribution of parallel velocities may be simulated. The value of the perpendicular momentum and hence of $\gamma_\perp$ is invariant under Lorentz transformations along the axis. It follows that this procedure allows one to identify a class of distributions that are separable in the perpendicular and parallel motions, and for which the response tensor can be evaluated in closed form using the foregoing results for a strictly perpendicular distribution.

To identify this class of distributions, first note that in the rest frame one has

$$u_0^\mu = [\gamma_0, \gamma_0 v_0 \mathbf{b}], \qquad b^\mu = [\gamma_0 v_0, \gamma_0 \mathbf{b}], \tag{5.4}$$

and hence

$$pb = m\gamma_0\gamma(v_0 - v_\parallel), \qquad uu_0 = \gamma\gamma_0(1 - v_\parallel v_0) = \gamma/\gamma_0, \tag{5.5}$$

where in the final equality $v_\parallel = v_0$ is used, and this is implied by $\delta(pb)$ in (5.1).

One has

$$\gamma_0 = (1 - v_\parallel^2)^{-1/2} = \gamma/\gamma_\perp, \tag{5.6}$$

which gives $uu_0 = \gamma_\perp$, in accord with the fact that $p_\perp$ is invariant under Lorentz transformations parallel to $\mathbf{b}$.

Let the weighted distribution be such that the contribution from a particular range dv_0 to the proper number density is $dn_{\rm pr} = dv_0\, g(v_0)$. The expression (5.5) implies that $\delta(pb) = (m\gamma_0\gamma)^{-1}\delta(v_\parallel - v_0)$. On integrating (5.1) with $\tilde{u} = u_0$ over $dv_0\, g(v_0)$, one identifies the desired distribution for which the response tensor can be evaluated in the manner outlined above. The resulting relativistic counterpart of a DGH distribution follows from (5.1) simply by setting $\tilde{u} = u_0$ and including the weighting defined above:

$$F(p) = \int dv_0\, g(v_0) \frac{n\,\delta(p^2 - m^2)\,\delta(pb)\,(\hat{d}_\perp)^r \exp(-\rho_\perp\, uu_0)}{(2\pi)^{1/2} m (\hat{d}_\perp)^r [K_{3/2}(\rho_\perp)/\rho_\perp^{1/2}]}, \tag{5.7}$$

with u_0 and b given by (5.4) in the rest frame. On carrying out the v_0 integral over $\delta(pb)$, the resulting distribution in 6-dimensional phase space in the rest frame is

$$f(p_\parallel, p_\perp) = \frac{\partial v_\parallel}{\partial p_\parallel}\, g(v_\parallel)\, \frac{\gamma}{\gamma_\perp} \frac{n\,(\hat{d}_\perp)^r \exp(-\rho_\perp\gamma_\perp)}{(2\pi)^{1/2} 2m^2 (\hat{d}_\perp)^r [K_{3/2}(\rho_\perp)/\rho_\perp^{1/2}]}, \tag{5.8}$$

with $\partial v_\parallel/\partial p_\parallel = \gamma_\perp^2/m\gamma^3$. The normalization in (5.7) corresponds to

$$\int dv_\parallel\, g(v_\parallel) = 1, \qquad \frac{n_{\rm pr}}{(\hat{d}_\perp)^r [K_{1/2}(\rho_\perp)/\rho_\perp^{1/2}]} = \frac{n}{(\hat{d}_\perp)^r [K_{3/2}(\rho_\perp)/\rho_\perp^{1/2}]}. \tag{5.9}$$

For $r = 0$ the distribution (5.8) reduces to (1.1) with $j = 0$ in the nonrelativistic limit, and, as noted above, the factor $v_\perp^{2j}$ may be reproduced by an appropriate sum of distributions with different values of r.

5.3. Response tensor for the relativistic DGH distribution

The generalization of (5.2) for the relativistic DGH distribution function (5.8) now follows by using a simple prescription: replace $\tilde{u}$ by u_0, interpret u_0 and b as in (5.4) in the rest frame, and average over the weighted distribution of parallel velocities. This gives

$$\begin{aligned}
\alpha^{\mu\nu}(k) = {} & \frac{q^2 n}{m(\hat{d}_\perp)^r [K_{3/2}(\rho_\perp)/\rho_\perp^{1/2}]} \int_{-1}^{1} dv_0\, g(v_0) \\
& \times \int_0^\infty d\xi\, \xi\, (\hat{d}_\perp)^r \bigg\{ [ka_\perp(\xi)]^2\, a^{\mu\nu}\big(k, a_\perp(\xi)\big)\, \frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \\
& \qquad - [(k^2)_{3\perp} g^{\mu\nu} - k_{3\perp}^\mu k^\nu - k^\mu k_{3\perp}^\nu + k^2 g_{3\perp}^{\mu\nu}] \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} \bigg\},
\end{aligned} \tag{5.10}$$

with the dependence on $v_\parallel = v_0$ appearing through $g_{3\perp}^{\mu\nu}$, $k_{3\perp}^\mu$, $a_\perp^\mu(\xi)$, $(k^2)_{3\perp}$ and $r_\perp(\xi)$. Similarly, the form (4.8) may be generalized, and then the $v_\parallel$ dependence also appears through $k_\parallel b^\mu$ and $k_\parallel b^\nu$, which can be rewritten as $-kb\, b^\mu$ and $-kb\, b^\nu$ respectively. Then the explicit dependences on v_0 follow from (5.4)–(5.6), $g_{3\perp}^{\mu\nu} = g^{\mu\nu} + b^\mu b^\nu$, $k_{3\perp}^\mu = k^\mu + kb\, b^\mu$, $bu_0 = 0$, $kb = \gamma_0(k_\parallel - \omega v_0)$ and

$$a_\perp^\mu(\xi) = \big[\gamma_0[\rho - i\gamma_0(\omega - k_\parallel v_0)\xi], -i\mathbf{k}_\perp\xi + \gamma_0[\rho - i\gamma_0(\omega - k_\parallel v_0)\xi] v_0 \mathbf{b}\big], \tag{5.11}$$

$$(k^2)_{3\perp} = \gamma_0^2(\omega - k_\parallel v_0)^2 - k_\perp^2, \tag{5.12}$$

$$r_\perp(\xi) = \{[\rho - i\gamma_0(\omega - k_\parallel v_0)\xi]^2 + k_\perp^2\xi^2\}^{1/2}, \tag{5.13}$$

where ω and $\mathbf{k}$ are the components on k^μ in the rest frame. The generalization of the form (4.8) is similarly straightforward.

6. Discussion and conclusions

The primary objectives in the present paper and in Part 2 are

(*a*) to develop a covariant formalism that allows one to apply Trubnikov's method of calculation of the linear response tensor to strictly parallel and strictly perpendicular relativistic, thermal distributions;

(*b*) to identify a relativistic counterpart for the nonrelativistic bi-Maxwellian and DGH distributions for which the linear response tensor can be calculated by factorizing into integrals over the perpendicular and parallel distributions.

The methodology is developed in the present paper for an unmagnetized plasma, and is applied to the magnetized case in Part 2.

Exact expressions for the linear response tensor for the strictly parallel distribution (3.1) are given by (3.7), (3.9) and (A 2) (see Appendix A) and for the strictly perpendicular distribution (4.1) by (4.7), (4.8) and (A 3). These exact expressions involve relativistic plasma dispersion functions in the form of an integral over a Macdonald function with a complex argument. The relationship between these plasma dispersion functions and other relativistic and nonrelativistic plasma dispersion functions is summarized in Appendix B.

The counterpart of a DGH distribution for which the response tensor factorizes into separate integrals is given by (5.8). This distribution contains an arbitrary function, $g(v_0)$ of the parallel speed, $v_0 = v_\parallel$, of the particles. For the counterpart of a bi-Maxwellian distribution one requires that in the limit as $\rho_\perp \to \infty$, when there is nonperpendicular motion, (5.8) reduces to the strictly parallel distribution (3.1). This condition implies

$$g(v_0) = \frac{\gamma_0^2}{K_1(\rho_\parallel)} \exp(-\rho_\parallel\gamma_0). \tag{6.1}$$

On inserting (6.1) into (5.8) with $r = 0$, the counterpart of a bi-Maxwellian distribution is identified as

$$f(p_\parallel, p_\perp) = \frac{n\,\rho_\perp^{1/2}}{(2\pi)^{1/2}2\gamma_\perp m^3 K_1(\rho_\parallel)K_{3/2}(\rho_\perp)} \exp\!\left(-\frac{\rho_\parallel\gamma}{\gamma_\perp} - \rho_\perp\gamma_\perp\right). \tag{6.2}$$

where (5.6) is used with $\gamma_\perp = (m^2 + p_\perp^2)^{1/2}/m$.

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Appendix A. Alternative forms for response tensors

The explicit forms for the response tensor written down in the text are derived from (2.7). Alternative forms are obtained by starting from (2.6). The alternative forms for the isotropic, strictly parallel, strictly perpendicular and relativistic DGH distributions are respectively

$$\alpha^{\mu\nu}(k) = -\frac{q^2 n}{m}\left\{\frac{n_{\rm pr}}{n}\,g^{\mu\nu} + \frac{\rho}{K_2(\rho)}\int_0^\infty d\xi\left[-k^2\xi a^\mu(\xi)a^\nu(\xi)\,\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\right.\right.$$
$$\left.\left.\times[ia^\mu(\xi)k^\nu + ik^\mu a^\nu(\xi) + k^2\xi\,g^{\mu\nu}]\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)}\right]\right\}, \quad \text{(A 1)}$$

with $n_{\rm pr} = nK_1(\rho)/K_2(\rho)$,

$$\alpha^{\mu\nu}(k) = -\frac{q^2 n}{m}\left\{\frac{n_{\rm pr}}{n}\,g^{\mu\nu} + \frac{1}{K_1(\rho_\parallel)}\int_0^\infty d\xi\left[-k^2\xi a_\parallel^\mu(\xi)a_\parallel^\nu(\xi)\,\frac{K_2\big(r_\parallel(\xi)\big)}{r_\parallel^2(\xi)}\right.\right.$$
$$\left.\left.\times[ia_\parallel^\mu(\xi)k^\nu + ik^\mu a_\parallel^\nu(\xi) + k^2\xi\,g_\parallel^{\mu\nu}]\,\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)}\right]\right\}, \quad \text{(A 2)}$$

with $n_{\rm pr} = nK_0(\rho)/K_1(\rho)$, and

$$\alpha^{\mu\nu}(k) = -\frac{q^2 n}{m}\left\{\frac{n_{\rm pr}}{n}\,g^{\mu\nu} + \frac{\rho_\perp^{1/2}}{K_{3/2}(\rho_\perp)}\int_0^\infty d\xi\left[-k^2\xi a_\perp^\mu(\xi)a_\perp^\nu(\xi)\,\frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)}\right.\right.$$
$$\left.\left.\times[ia_\perp^\mu(\xi)k^\nu + ik^\mu a_\perp^\nu(\xi) + k^2\xi\,g_{3\perp}^{\mu\nu}]\,\frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)}\right]\right\}, \quad \text{(A 3)}$$

with $n_{\rm pr} = nK_{1/2}(\rho)/K_{3/2}(\rho)$.

Appendix B. Plasma dispersion functions

The relativistic plasma dispersion functions that appear in all three special cases, specifically (2.21) or (2.22) for the isotropic thermal distribution, (3.7) or (3.9) for the strictly parallel thermal distribution, and (4.7) or (4.8) for the strictly perpendicular thermal distribution, involve a ξ integral over a Macdonald function whose argument is a complex function of ξ. The reduction of these relativistic plasma dispersion functions to their nonrelativistic counterparts is outlined here.

B.1. Standard relativistic plasma dispersion functions

As shown by Melrose (1982), the relativistic dispersion functions can be rewritten by deforming the contour of ξ integration so that the argument of the Macdonald functions is real along the entire contour. For example, for the isotropic case, on writing $\xi = x + iy$, the function $r(\xi)$ in (2.21) becomes

$$r(\xi) = [\rho^2 - 2i\rho\,k\tilde{u}\,x + 2\rho\,k\tilde{u}\,y - k^2(x^2 + 2ixy - y^2)]^{1/2}. \quad \text{(B 1)}$$

For $k^2 < 0$, the deformed contour is along the imaginary axis, $0 < y < y_0 = -\rho\,k\tilde{u}/k^2$, and then parallel to the real axis at $y = y_0$. The first portion gives the reactive part of the response and the second portion gives the dissipative part of the response. For $k^2 > 0$ the contour can be rotated through $\frac{1}{2}\pi$ such that $r(\xi)$ is real for all $0 < y < \infty$; there is then no dissipative term, as is necessarily the case for $k^2 > 0$ in a collisionless unmagnetized plasma. The resulting expressions for the real parts can be rewritten in terms of other relativistic plasma dispersion functions. For example, the function introduced by Godfrey *et al.* (1975*a*) may be defined by writing

$$T(z,\rho) = \int_{-1}^{1} dv\, \frac{e^{-\rho\gamma}}{v-z} = \frac{2i\rho\, k\tilde{u}}{z} \int_0^\infty d\xi\, \frac{K_1\big(r(\xi)\big)}{r(\xi)}, \tag{B 2}$$

with $z = k\tilde{u}/[(k\tilde{u})^2 - k^2]^{1/2}$, and the foregoing procedure gives

$$T(z,\rho) = \begin{cases} -\dfrac{2\rho}{1-z^2} \displaystyle\int_0^z du\, \frac{K_1(\rho R)}{R} + i\pi\, e^{-\rho(1-z^2)^{-1/2}} & \text{for } z < 1, \\ \dfrac{2\rho}{1-z^2} \displaystyle\int_z^\infty du\, \frac{K_1(\rho R)}{R} & \text{for } z > 1. \end{cases} \tag{B 3}$$

with $R^2 = (1-u^2)/(1-z^2)$. Proceeding in this manner, and using the results given by Godfrey *et al.* (1975*a*), all the plasma dispersion functions that appear in the response tensors for the isotropic, strictly parallel and strictly perpendicular cases may be rewritten in terms of $T(z,\rho)$ and $\partial T(z,\rho)/\partial z$.

B.2. The nonrelativistic limit

The nonrelativistic approximation to the plasma dispersion functions that appear in the various response tensors here involves two steps. For example, for the isotropic case, the first step is to make the weakly relativistic approximation $\rho \gg 1$ by approximating the Macdonald functions by their asymptotic form:

$$\int_0^\infty d\xi\, \frac{K_\nu\big(r(\xi)\big)}{r^\nu(\xi)} \approx \left(\frac{\pi}{2}\right)^{1/2} \int_0^\infty d\xi\, \frac{e^{-r(\xi)}}{[r(\xi)]^{\nu+1/2}}, \tag{B 4}$$

which for half-integral ν follow directly from (4.9) for large $r(\xi)$. The other step is to approximate $r(\xi)$ by expanding up to second order in ξ,

$$r(\xi) \approx \rho - ik\tilde{u}\,\xi - \frac{k^2-(k\tilde{u})^2}{2\rho}\,\xi^2, \tag{B 5}$$

in the exponential function in (B 4), with $r(\xi)$ approximated by ρ in the denominator in (B 4). Then one evaluates the resulting integral in (B 4) as in (B 3) by integrating up the imaginary axis to $y = k\tilde{u}/[k^2 - (k\tilde{u})^2]$. In the rest frame, (B 4) reduces to

$$\int_0^\infty d\xi\, \frac{K_\nu\big(r(\xi)\big)}{r^\nu(\xi)} \approx \left(\frac{\pi}{\rho}\right)^{1/2} \frac{|\mathbf{k}|e^{-\rho}}{\omega^2 z}\, [\phi(w) - i\pi^{1/2} w\, e^{-w^2}], \tag{B 6}$$

with $w = (\frac{1}{2}\rho)^{1/2}\omega/|\mathbf{k}|$ and where $\phi(x) = 2xe^{-x^2}\int_0^x dt\, e^{t^2}$ is a form of the familiar nonrelativistic plasma dispersion function.

References

Bornatici, M., Cano, R., De Barbieri, O. and Engelmann, F. 1983 Electron cyclotron emission and absorption in fusion plasmas. *Nucl. Fusion* **23**, 1153–1257.

Dewar, R. L. 1977 Energy–momentum tensors for dispersive electromagnetic waves. *Aust. J. Phys.* **30**, 533–575.

Dory, R. A., Guest, G. E. and Harris, E. G. 1965 Unstable electrostatic plasma waves propagating perpendicular to a magnetic field. *Phys. Rev. Lett.* **14**, 131–133.

Godfrey, B. B., Newberger, B. S. and Taggart, K. A. 1975*a* A relativistic plasma dispersion function. *IEEE Trans. Plasma Sci.* **3**, 60–67.

Godfrey, B. B., Newberger, B. S. and Taggart, K. A. 1975*b* Relativistic linear theory in the absence of external fields. *IEEE Trans. Plasma Sci.* **3**, 68–75.

Melrose, D. B. 1973 A covariant formulation of wave dispersion. *Plasma Phys.* **15**, 99–106.

Melrose, D. B. 1982 Covariant description of dispersion in a relativistic thermal electron gas. *Aust. J. Phys.* **35**, 41–52.

Melrose, D. B. 1997*a* Covariant form of Trubnikov's response tensor for a relativistic magnetized thermal plasma. *J. Plasma Phys.* **57**, 479–488.

Melrose, D. B. 1997*b* Response of a relativistic anisotropic thermal plasma. Part 2. Magnetized particles. *J. Plasma Phys.* **58**, 721–734.

Silin, V. P. 1960 On the electromagnetic properties of a relativistic plasma. *Soviet Phys. JETP* **11**, 1136–1140.

Silin, V. P. 1961 Electromagnetic properties of a relativistic plasma, II. *Soviet Phys. JETP* **13**, 430–435.

Stix, T. H. 1962 *The Theory of Plasma Waves*. McGraw-Hill: New York.

Synge, J. L. 1957 *The Relativistic Gas*. North-Holland: Amsterdam.

Trubnikov, B. A. 1958 Magnetic emission of high temperature plasma. Thesis, Moscow Institute of Engineering and Physics; [English translation in AEC-tr-4073, US Atomic Energy Commission, Oak Ridge, Tennessee, 1960].

Trubnikov, B. A. and Yakubov, V. B. 1963 Cyclotron radiation of electrons having a two-dimensional Maxwellian distribution. *Plasma Phys. (J. Nucl. Energy C)* **5**, 7–14.