J. Plasma Physics (1997), *vol.* 58, *part* 4, *pp.* 721–734. Printed in the United Kingdom 721

Response of a relativistic anisotropic thermal plasma. Part 2. Magnetized particles

D. B. MELROSE

Research Centre for Theoretical Astrophysics, School of Physics, University of Sydney, NSW 2006, Australia

(Received 5 December 1996 and in revised form 17 March 1997)

The linear response tensor for a relativistic counterpart of a bi-Maxwellian distribution is calculated in closed form for a magnetized plasma. The method used involves the following steps: (1) a covariant form of the response tensor for an arbitrary distribution of particles is derived using both a forward-scattering method and a covariant version of Vlasov theory; (2) the response tensor is evaluated for a strictly perpendicular, thermal distribution using a method due to Trubnikov; (3) the parallel distribution is built up by applying a Lorentz transformation to sum over a weighted, continuous distribution of parallel speeds. A convenient starting point for a detailed investigation of waves in such a plasma is the small-gyroradius approximation to the general expression for the response tensor, and the relevant approximation is derived and discussed briefly. The expansion of the general expression for the response tensor in Bessel functions in given in an appendix.

1. Introduction

In an accompanying paper, Melrose (1997*b*, hereinafter referred to as Part 1), the response tensors are evaluated for unmagnetized relativistic thermal distributions in one (strictly parallel), two (strictly perpendicular) and three (isotropic) dimensions, and for a generalization of the strictly perpendicular distribution that is a relativistic counterpart of a DGH distribution (Dory *et al.* 1965). The calculation involves starting from an expression for the response tensor for an arbitrary distribution derived using a covariant forward-scattering method. In the present paper the results of Part 1 are generalized to a magnetized plasma.

Relativistic effects are known to be important in magnetized thermal plasmas near perpendicular propagation, even when they would otherwise be considered weak. Relativistic effects modify the properties of cyclotron harmonic waves in important ways, (see e.g. Shkarofsky 1966*b*; Bornatici *et al.* 1983; Robinson 1988). The usual starting point for the discussion of relativistic effects in cyclotron harmonic waves is an exact expression for the linear response tensor for a relativistic thermal plasma derived by Trubnikov (1958). However, Trubnikov's response tensor is for a strictly isotropic distribution, and cannot be used to discuss the effect of anisotropy of the distribution of particles. The inclusion of the effects of anisotropy on the properties of cyclotron harmonic waves is essential when discussing instabilities. Specific instabilities that lead directly to escaping radiation constitute electron cyclotron maser emission (ECME) mechanisms. A relativistic effect is an intrinsic ingredient in the most widely favoured theory for ECME, (see e.g. Twiss 1958; Wu and Lee 1979; Omidi and Curnett 1982; Melrose *et al.* 1982; Wu 1986; Melrose 1986). The relativistic effect in this version of ECME is included through a 'semirelativistic' approximation to the gyroresonance condition, in which the Lorentz factor in the gyrofrequency $\Omega = |q|B/m\gamma$ is approximated by $1/\gamma = 1 + \frac{1}{2}v^2$, and relativistic effects are otherwise ignored. A fully relativistic calculation of the response tensor for an anisotropic distributions of particles would provide an alternative starting point for a discussion of ECME. The objective in this paper is to generalize the results of Part 1 to a magnetized plasma with these applications in mind.

The outline of the present paper is as follows. In Sec. 2 the response tensor for an arbitrary distribution is derived using the covariant, forward-scattering method for a magnetized plasma. In Sec. 3 the results of Part 1 for the response tensor for the isotropic, strictly parallel and strictly perpendicular relativistic thermal distributions are generalized to the magnetized case. In Sec. 4 the response tensor is derived and discussed for the relativistic counterpart of a DGH distribution proposed in (1.5.9), where equation (a.b) of Part 1 is denoted by (1.a.b).

2. General expression for the response tensor

The starting point for the calculation in the next section is a covariant expression for the response tensor for an arbitrary distribution of particles. The appropriate general response is derived in this section using forward-scattering theory. The derivation is based on the current associated with a single particle, which is expanded in powers of a test field and then averaged over the distribution of particles. The resulting averaged current is identified as the induced current due to the distribution of particles. An alternative derivation that starts from a covariant version of Vlasov theory is presented in Appendix A, and an explicit form after expanding in Bessel functions is written down in Appendix B.

2.1. First-order single-particle current

The orbit of a particle in a magnetostatic field, described by its Maxwell tensor, $F_0^{\mu\nu} = Bf^{\mu\nu}$ with $B = (\frac{1}{2}F_0^{\mu\nu}F_{0\mu\nu})^{1/2}$, is found by solving the equation of motion

$$\frac{du^\mu(\tau)}{d\tau} = \frac{q}{m}F_0^{\mu\nu}u_\nu(\tau) + S^\mu(\tau), \tag{2.1}$$

$$S^\mu(\tau) = \frac{iq}{m}\int \frac{d^4k_1}{(2\pi)^4}\, e^{-ik_1X(\tau)}k_1u(\tau)\, G^{\mu\nu}\big(k_1, u(\tau)\big)A_\nu(k_1), \tag{2.2}$$

where the notation is that used in Melrose (1997*b*). The source term $S^\mu(\tau)$ in (2.1) is of first order in the amplitude $A(k)$ of the fluctuating electromagnetic field. A perturbation expansion of the orbit,

$$X^\mu(\tau) = X^{(0)\mu}(\tau) + X^{(1)\mu}(\tau) + \dots, \tag{2.3}$$

with $u^\mu(\tau) = dX^\mu(\tau)/d\tau$, is inserted into (2.1). The zeroth-order solution is

$$X^{(0)\mu}(\tau) = x_0^\mu + t^{\mu\nu}(\tau)u_{0\nu}, \tag{2.4}$$

$$t^{\mu\nu}(\tau) = g_\parallel^{\mu\nu}\tau + g_\perp^{\mu\nu}\frac{\sin\Omega_0\tau}{\Omega_0} - \eta f^{\mu\nu}\frac{\cos\Omega_0\tau}{\Omega_0}, \tag{2.5}$$

$$g_\perp^{\mu\nu} = -f^\mu{}_\alpha f^{\alpha\nu}, \qquad g_\parallel^{\mu\nu} = g^{\mu\nu} - g_\perp^{\mu\nu}, \tag{2.6}$$

where x_0 and u_0 are constant 4-vectors, and where $\Omega_0 = |q|B/m$ and $\eta = q/|q|$.

The first-order correction to the orbit is

$$X^{(1)\mu}(\tau) = \int^{\tau} d\tau'' \int^{\tau''} d\tau' \, \dot{t}^{\mu\nu}(\tau'' - \tau') S_\nu(\tau'), \tag{2.7}$$

$$\dot{t}^{\mu\nu}(\tau) = \frac{dt^{\mu\nu}(\tau)}{d\tau} = g_{\parallel}^{\mu\nu} + g_{\perp}^{\mu\nu} \cos \Omega_0 \tau + \eta f^{\mu\nu} \sin \Omega_0 \tau. \tag{2.8}$$

After a partial integration, (2.7) can be rewritten in the form

$$X^{(1)\mu}(\tau) = \int_0^{\infty} d\xi \, T^{\mu\nu}(\xi) S_\nu(\tau - \xi), \tag{2.9}$$

$$T^{\mu\nu}(\xi) = t^{\mu\nu}(\xi) - t^{\mu\nu}(0) = g_{\parallel}^{\mu\nu} \xi + g_{\perp}^{\mu\nu} \frac{\sin \Omega_0 \xi}{\Omega_0} + \eta f^{\mu\nu} \frac{1 - \cos \Omega_0 \xi}{\Omega_0}. \tag{2.10}$$

The first two terms in the perturbation expansion of the single-particle current

$$J_{\text{sp}}^{\mu}(k) = q \int d\tau \, u^{\mu}(\tau) \, e^{ikX(\tau)} \tag{2.11}$$

are the zeroth-order current, which is of no direct interest here, and the first-order current

$$J_{\text{sp}}^{(1)\mu} = iq \int d\tau \, ku^{(0)}(\tau) \, G^{\alpha\mu}\big(k, u^{(0)}(\tau)\big) \, X_{\alpha}^{(1)}(\tau) \, e^{ikX^{(0)}(\tau)}, \tag{2.12}$$

$$ku(\tau) G^{\mu\nu}\big(k, u(\tau)\big) = ku(\tau) \, g^{\mu\nu} - k^{\mu} u^{\nu}(\tau). \tag{2.13}$$

On inserting the first-order perturbation (2.9) into the orbit, (2.12) gives

$$\begin{aligned} J_{\text{sp}}^{(1)\mu}(k) = \frac{q^2}{m} \int d\tau \int^{\tau} d\xi \int \frac{d^4 k_1}{(2\pi)^4} \, e^{i[kX^{(0)}(\tau) - k_1 X^{(0)}(\tau - \xi)]} \, T_{\alpha\beta}(\xi) \\ \times ku^{(0)}(\tau) \, G^{\alpha\mu}\big(k, u^{(0)}(\tau)\big) \, k_1 u^{(0)}(\tau - \xi) \, G^{\beta\nu}\big(k_1, u^{(0)}(\tau - \xi)\big) \, A_\nu(k_1). \end{aligned} \tag{2.14}$$

The current (2.14) may be used as a source term in the inhomogeneous wave equation to treat Thomson scattering by the particle. Here we are interested in the collective response of the medium, and this is described in terms of the induced current, which is the ('forward-scattering') contribution for $k_1 = k$ when all the first-order, single-particle currents (2.14) have the same phase and sum to give the induced current.

2.2. *The forward-scattering method for a magnetized plasma*

The derivation of the linear response tensor using the forward scattering method involves averaging the first-order single-particle current over the distribution of particles. This average follows by noting that the distribution $F(x_0, p_0)$ in 8-dimensional phase space represents the number of world lines (one per particle) threading the 7-dimensional surface $d^4x_0 \, d^4p_0/d\tau$ (cf. Dewar 1977). Hence the appropriate average follows from (2.14) by replacing the integral over $d\tau$ by the integral over $d^4x_0 \, d^4p_0$ times $F(x_0, p_0)$. Assuming a uniform distribution in space and time implies that $F(p_0)$ does not depend on x_0. Then x_0 appears in (2.14) only in an exponential factor $\exp[i(k - k_1)x_0]$, where (2.4) is noted. Thus the x_0 integral gives $(2\pi)^4 \, \delta^4(k - k_1)$. The k_1 integral in (2.14) is performed over the resulting delta function.

2.3. General expression for the response tensor

The forward-scattering method applied to (2.14) leads to an expression of the form $J^{(1)\mu}(k) = \alpha^{\mu\nu}(k)A_\nu(k)$, from which one identifies a general expression for the linear response tensor. This is

$$\alpha^{\mu\nu}(k) = \frac{q^2}{m}\int d^4p(\tau)\,F(p)\int_0^\infty d\xi\, e^{ik[X(\tau)-X(\tau-\xi)]}T_{\alpha,\beta}(\xi) \times ku(\tau)\,G^{\alpha\mu}\big(k,u(\tau)\big)\,ku(\tau-\xi)\,G^{\beta\nu}\big(k,u(\tau-\xi)\big), \tag{2.15}$$

where the superscript (0) on $u(\tau)$ is now omitted, with $u(\tau) = \dot{t}^{\mu\nu}(\tau)u_{0\nu}$. Also in (2.15) it is noted that the initial value of τ is related to the initial gyrophase, ϕ_0 say, and one is free to choose $\phi_0 = \Omega_0\tau$ and to write the integral over d^4p_0 as an integral over $d^4p(\tau)$. The distribution $F(p)$ is assumed to be independent of gyrophase, so that it has no dependence on τ.

As an aside, it is noted that the anti-Hermitian part of the response tensor (2.15) is

$$\alpha^{A\mu\nu}(k) = \frac{q^2}{2m}\int d^4p(\tau)\,F(p)\int_{-\infty}^\infty d\xi\, e^{ik[X(\tau)-X(\tau-\xi)]}\,T_{\alpha\beta}(\xi) \times ku(\tau)\,G^{\alpha\mu}\big(k,u(\tau)\big)\,ku(\tau-\xi)\,G^{\beta\nu}\big(k,u(\tau-\xi)\big). \tag{2.16}$$

That is, the Hermitian and anti-Hermitian parts of (2.15) correspond to the parts of the integrand in (2.15) that are respectively odd and even in ξ.

The response tensor in the form (2.15) is the appropriate generalization of the unmagnetized form (1.2.7), and (2.15) is the starting point for the calculation for specific distributions in the next section. An alternative derivation of (2.15) is given in Appendix A.

3. Isotropic, strictly parallel and strictly perpendicular cases

The response tensors for the three special cases of isotropic, strictly parallel and strictly perpendicular relativistic thermal distributions are evaluated in Part 1 for unmagnetized particles. The generalizations to magnetized particles for these three special cases are derived in this section. The response tensor for an isotropic, relativistic, thermal distribution was derived by Melrose (1997*a*), and the treatment of this case by the alternative method developed here provides a consistency check on the two methods.

3.1. Response tensor for an isotropic distribution

In treating the isotropic (Jüttner–Synge) distribution (1.2.12), it is convenient to define the 4-vectors

$$R^\mu(\xi) = k_\alpha T^{\mu\alpha}(\xi), \qquad \tilde{R}^\nu(\xi) = k_\beta T^{\beta\nu}(\xi), \qquad a^\mu(\xi) = \rho\tilde{u}^\mu - iR^\mu(\xi). \tag{3.1}$$

Then the counterparts of equations (1.2.15)–(1.2.18) are

$$I(\rho,\xi,s+s') = \frac{n\rho}{K_2(\rho)}\,\frac{K_1\big(r(\xi)\big)}{r(\xi)}, \tag{3.2}$$

$$r(\xi) = \{[a^\mu(\xi) - s^\mu - s'^\mu][a_\mu(\xi) - s_\mu - s'_\mu]\}^{1/2}, \tag{3.3}$$

$$\hat{u}^\mu\,\frac{K_1\big(r(\xi)\big)}{r(\xi)} = a^\mu(\xi)\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)}, \tag{3.4}$$

$$\hat{u}^\mu \hat{u}^\nu \, \frac{K_1\big(r(\xi)\big)}{r(\xi)} = -g^{\mu\nu} \, \frac{K_2\big(r(\xi)\big)}{r^2(\xi)} + a^\mu(\xi) a^\nu(\xi) \, \frac{K_3\big(r(\xi)\big)}{r^3(\xi)} \tag{3.5}$$

respectively. To apply these to (2.15), write

$$ku \, G^{\alpha\mu}(k,u) = (k^\sigma g^{\alpha\mu} - k^\alpha g^{\sigma\mu}) u_\sigma, \qquad ku \, G^{\beta\nu}(k,u) = (k^\tau g^{\beta\nu} - k^\beta g^{\tau\nu}) u_\tau,$$
$$u^\tau(\tau - \xi) = \dot{t}_\tau{}^\eta(-\xi) u^\eta(\tau).$$

The resulting expression for the response tensor for the isotropic thermal distribution is

$$\begin{aligned} \alpha^{\mu\nu}(k) = \frac{q^2 n\rho}{mK_2(\rho)} \int_0^\infty d\xi \, T_{\alpha\beta}(\xi) \, (k^\sigma g^{\alpha\mu} - k^\alpha g^{\sigma\mu})(k^\tau g^{\beta\nu} - k^\beta g^{\tau\nu}) \\ \times \dot{t}_\tau{}^\eta(-\xi) \bigg[- g_{\sigma\eta} \, \frac{K_2\big(r(\xi)\big)}{r^2(\xi)} + a_\sigma(\xi) a_\eta(\xi) \, \frac{K_3\big(r(\xi)\big)}{r^3(\xi)} \bigg]. \end{aligned} \tag{3.6}$$

Explicit evaluation of the coefficients of the $K_2\big(r(\xi)\big)/r^2(\xi)$ and $K_3\big(r(\xi)\big)/r^3(\xi)$ terms gives

$$\begin{aligned} T_{\alpha\beta}(\xi) \, (k^\sigma g^{\alpha\mu} - k^\alpha g^{\sigma\mu})(k^\tau g^{\beta\nu} - k^\beta g^{\tau\nu}) \, \dot{t}_\tau{}^\eta(-\xi) g_{\sigma\eta} \\ = \frac{d}{d\xi} [T^{\mu\nu}(\xi) k_\alpha k_\beta T^{\alpha\beta}(\xi) - R^\mu(\xi) \tilde{R}^\nu(\xi)], \end{aligned} \tag{3.7}$$

$$\begin{aligned} & T_{\alpha\beta}(\xi) \, (k^\sigma g^{\alpha\mu} - k^\alpha g^{\sigma\mu})(k^\tau g^{\beta\nu} - k^\beta g^{\tau\nu}) \, a_\sigma(\xi) \tilde{a}_\tau(\xi) \\ & = \rho^2 (k\tilde{u})^2 \tilde{T}^{\mu\nu}(\xi) - [2i\rho \, k\tilde{u} + k_\sigma k_\tau T^{\sigma\tau}(\xi)][T^{\mu\nu}(\xi) k_\alpha k_\beta T^{\alpha\beta}(\xi) - R^\mu(\xi) \tilde{R}^\nu(\xi)], \end{aligned} \tag{3.8}$$

where $\dot{t}_\tau{}^\eta(-\xi) a_\eta(\xi) = \tilde{a}_\tau(\xi)$ is used, and with

$$\tilde{T}^{\mu\nu}(\xi) = T_{\alpha\beta}(\xi) \, G^{\alpha\mu}(k,\tilde{u}) G^{\beta\nu}(k,\tilde{u}). \tag{3.9}$$

The counterpart of the identity (1.2.23) is

$$f(0) \frac{K_\nu(\rho)}{\rho^\nu} + \int_0^\infty d\xi \left[\frac{df(\xi)}{d\xi} \, \frac{K_\nu\big(r(\xi)\big)}{r^\nu(\xi)} + i f(\xi) \, ka(\xi) \, \frac{K_{\nu+1}\big(r(\xi)\big)}{r^{\nu+1}(\xi)} \right] = 0. \tag{3.10}$$

Using (3.10), the form (3.6) with (3.7) and (3.8) may be reduced to

$$\begin{aligned} \alpha^{\mu\nu}(k) = i \frac{q^2 n \rho^2 \, k\tilde{u}}{mK_2(\rho)} \int_0^\infty d\xi \, [k_\alpha T^{\mu\alpha}(\xi) \, k_\beta T^{\beta\nu}(\xi) \\ - k_\alpha k_\beta T_{\alpha\beta}(\xi) \, T^{\mu\nu}(\xi) - i\rho \, k\tilde{u} \, \tilde{T}^{\mu\nu}(\xi)] \frac{K_3\big(r(\xi)\big)}{r^3(\xi)}. \end{aligned} \tag{3.11}$$

The form (3.11) is the generalization of (1.2.24) to the magnetized case. The form (3.11) with (3.9) also reproduces equations (37) and (38) of Melrose (1997*a*), where the response tensor is derived in a different way.

The form that is the most convenient starting point for making the weakly relativistic approximation is given by equation (25) of Melrose (1997*a*). In the notation used in the present paper this is

$$\begin{aligned} \alpha^{\mu\nu}(k) = -\frac{q^2 n\rho}{m} \bigg\{ \tilde{u}^\mu \tilde{u}^\nu - \frac{i\rho(k\tilde{u})}{K_2(\rho)} \int_0^\infty d\xi \, \bigg[\dot{t}^{\mu\nu}(\xi) \, \frac{K_2\big(r(\xi)\big)}{r^2(\xi)} \\ - a^\mu(\xi) \tilde{a}^\nu(\xi) \, \frac{K_3\big(r(\xi)\big)}{r^3(\xi)} \bigg] \bigg\}. \end{aligned} \tag{3.12}$$

The equivalence of (3.12) and (3.6) may be shown directly by writing the coefficient (3.8) in the form $ika(\xi)f(\xi) - i\rho\, k\tilde{u}\, a^\mu(\xi)\tilde{a}^\nu(\xi)$, with

$$f(\xi) = -ika(\xi)\, T^{\mu\nu}(\xi) - a^\mu(\xi)\tilde{a}^\nu(\xi) + \rho[\tilde{u}^\mu\tilde{a}^\nu(\xi) + a^\mu(\xi)\tilde{u}^\nu],$$

and using (3.10) with $\nu = 2$.

3.2. Response tensor for a strictly parallel distribution

The response tensor for any strictly parallel distribution in the presence of a magnetic field is similar to the response tensor for the strictly parallel distribution in the absence of a magnetic field in that it involves no spiralling motion of the particles. The unmagnetized counterpart of (2.15) follows on replacing the factor $T_{\alpha\beta}(\xi)$ by $g^{\mu\nu}\xi$. The differences between the magnetized and unmagnetized response tensors for a strictly parallel distribution involve only the tensor $T_{\alpha\beta}(\xi)$.

The counterpart of (1.3.7) may be derived straightforwardly:

$$\begin{aligned}\alpha^{\mu\nu}(k) = \frac{q^2 n}{mK_1(\rho_\parallel)} \int_0^\infty d\xi\, \bigg\{ &[ka_\parallel(\xi)]^2\, T_{\alpha\beta}(\xi) G^{\alpha\mu}\big(k, a_\parallel(\xi)\big) \\ &\times G^{\beta\nu}\big(k, a_\parallel(\xi)\big)\, \frac{K_2\big(r_\parallel(\xi)\big)}{r_\parallel^2(\xi)} \\ &-[(k^2)_\parallel T^{\mu\nu}(\xi) - k_\parallel^\mu T_\alpha{}^\nu(\xi) k^\alpha \\ &-T^\mu{}_\beta(\xi) k^\beta k_\parallel^\nu + k^\alpha k^\beta T_{\alpha\beta}(\xi) g_\parallel^{\mu\nu}]\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)} \bigg\}.\end{aligned} \tag{3.13}$$

The counterpart of the form (1.3.9) may be derived from (3.13) by using the identity (I.3.8):

$$\begin{aligned}\alpha^{\mu\nu}(k) = \frac{q^2 n}{m} \bigg\{ &-\frac{n_{\rm pr}}{n}\, g_\perp^{\mu\nu} - \rho_\parallel \tilde{u}_\parallel^\mu \tilde{u}_\parallel^\nu \\ &-\frac{i}{K_1(\rho_\parallel)} \int_0^\infty d\xi\, \Big[it^{\mu\nu}(\xi) K_0\big(r_\parallel(\xi)\big)[\rho_\parallel\, k\tilde{u}_\parallel - ik_\alpha k_\beta T_\perp^{\alpha\beta}(\xi)] \\ &\times\bigg(-g_\parallel^{\mu\nu}\xi\, \frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)} + a_\parallel^\mu(\xi) a_\parallel^\nu(\xi)\, \frac{K_2\big(r_\parallel(\xi)\big)}{r_\parallel^2(\xi)} \bigg) \\ &+[a_\parallel^\mu(\xi) T_\perp^{\alpha\nu}(\xi) k_\alpha + T_\perp^{\mu\beta}(\xi) k_\beta a_\parallel^\nu(\xi)]\, \frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)} \Big] \bigg\},\end{aligned} \tag{3.14}$$

with $t^{\mu\nu}(\xi)$ given by (2.5) and $\dot{T}^{\mu\nu}(\xi) = dT^{\mu\nu}(\xi)/d\xi$.

3.3. Response tensor for a strictly perpendicular distribution

The generalization of the results in Sec. 5 of Part 1 to the magnetized case for a strictly perpendicular thermal distribution, cf. (1.4.1), is as follows. Equations (I.4.3)–(1.4.6) are replaced by

$$I(\rho_\perp, \xi) = \frac{n\rho_\perp^{1/2}}{K_{3/2}(\rho_\perp)}\, \frac{K_{1/2}\big(r_\perp(\xi)\big)}{r_\perp^{1/2}(\xi)}, \tag{3.15}$$

$$r_\perp(\xi) = [a_\perp^2(\xi)]^{1/2} = \left[(\rho_\perp - ik_{3\perp}\tilde{u})^2 + 2\frac{k_\perp^2}{\Omega_0^2}(1 - \cos\Omega_0\xi) \right]^{1/2}, \tag{3.16}$$

$$\hat{u}^\mu \, \frac{K_{1/2}\big(r_\perp(\xi)\big)}{r_\perp^{1/2}(\xi)} = a_\perp^\mu(\xi) \, \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)}, \tag{3.17}$$

$$\hat{u}^\mu \hat{u}^\nu \, \frac{K_{1/2}\big(r_\perp(\xi)\big)}{r_\perp^{1/2}(\xi)} = -g_{3\perp}^{\mu\nu} \, \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} + a_\perp^\mu(\xi) a_\perp^\nu(\xi) \, \frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \tag{3.18}$$

respectively, with $a_\perp^\mu(\xi) = g_{3\perp}^{\mu\nu} a_\nu(\xi)$. The form (1.4.7) for the response tensor is replaced by its magnetized counterpart

$$\begin{aligned}\alpha^{\mu\nu}(k) = {} & \frac{q^2 n \rho_\perp^{1/2}}{m K_{3/2}(\rho_\perp)} \int_0^\infty d\xi \, T_{\alpha\beta}(\xi) \, (k_{3\perp}^\sigma g^{\alpha\mu} - k^\alpha g_{3\perp}^{\sigma\mu})(k_{3\perp}^\tau g^{\beta\nu} - k^\beta g_{3\perp}^{\tau\nu}) \\ & \times \dot{t}_\tau{}^\eta(-\xi) \left[(g_{3\perp})_{\sigma\eta} \, \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} - a_{\perp\sigma}(\xi) a_{\perp\eta}(\xi) \, \frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \right],\end{aligned} \tag{3.19}$$

where $T_\perp^{\alpha\beta}(\xi) = g_{3\perp}^{\alpha\mu} g_{3\perp}^{\beta\nu} T_{\mu\nu}(\xi)$ is the projection of $T^{\alpha\beta}(\xi)$ onto the 3-dimensional subspace orthogonal to the direction of the magnetic field.

The counterpart of the alternative form (1.4.8) for the magnetized case is obtained from (3.19) as follows. First, separate into component in the 3-dimensional subspace spanned by $g_{3\perp}^{\mu\nu}$ plus components that have nonzero projections along b^μ or b^ν. Secondly, rewrite the components in the 3-dimensional subspace following the same steps as in the derivation of (3.12) from (3.6). This gives

$$\begin{aligned}\alpha^{\mu\nu}(k) = -\frac{q^2 n}{m} \Bigg\{ & -\frac{n_{\rm pr}}{n} \, b^\mu b^\nu + \rho_\perp \tilde{u}^\mu \tilde{u}^\nu \\ & -i \frac{\rho_\perp^{1/2}}{K_{3/2}(\rho_\perp)} \int_0^\infty d\xi \, \Bigg[(\rho_\perp \, k\tilde{u} - i k_\parallel^2 \xi) \, \Bigg(t_\perp^{(1)\mu\nu}(\xi) \, \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} \\ & -a_\perp^\mu(\xi) \tilde{a}_\perp^\nu(\xi) \, \frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \Bigg) \\ & -k_\parallel [b^\mu \tilde{a}_\perp^\nu(\xi) + a_\perp^\mu(\xi) b^\nu] \, \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} \Bigg] \Bigg\},\end{aligned} \tag{3.20}$$

$$t_\perp^{(1)\mu\nu}(\xi) = g_{3\perp}^{\mu\alpha} g_{3\perp}^{\nu\beta} \dot{t}_{\alpha\beta}(\xi). \tag{3.21}$$

The form (3.20) is a covariant generalization of a result derived by Trubnikov and Yakubov (1963); cf. also Bornatici *et al.* (1983).

The simple form of the $\mu = \nu = 3$ term in (3.20) necessarily applies to any two-dimensional distribution in which the particles have no motion along the 3-axis. That is, one must have $\alpha^{33}(k) = q^2 n_{\rm pr}/m$, where $n_{\rm pr}$ is the proper number density, for any two-dimensional distribution. The 33-component of (3.19) may be written in the form

$$\begin{aligned}\alpha^{33}(k) = -\frac{q^2 n \rho_\perp^{1/2}}{m K_{3/2}(\rho_\perp)} \int_0^\infty d\xi \, \xi \, \Bigg[& -i \frac{d}{d\xi} [k a_\perp(\xi)] \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} \\ & + [k a_\perp(\xi)]^2 \frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \Bigg].\end{aligned} \tag{3.22}$$

The reduction of (3.22) to $\alpha^{33}(k) = q^2 n_{\rm pr}/m$ involves using (3.10) twice, first with $\nu = \frac{3}{2}$ and $f(\xi) = -i\xi \, k a_\perp(\xi)$, and then with $\nu = \frac{1}{2}$ and $f(\xi) = 1$, and then noting that the proper number density is $n_{\rm pr} = nK_{1/2}(\rho_\perp)/K_{3/2}(\rho_\perp)$. An identity equivalent to (3.22) is implicit in a result stated by Trubnikov and Yakubov (1963).

4. Response tensor for the relativistic DGH distribution

The response tensor for the relativistic DGH distribution (1.5.8), namely

$$f(p_\parallel, p_\perp) = \frac{\partial v_\parallel}{\partial p_\parallel}\, g(v_\parallel)\, \frac{1}{\gamma_\perp}\, \frac{n\,\hat{d}^r_\perp \exp(-\rho_\perp\gamma_\perp)}{(2\pi)^{1/2} 2m^2 \hat{d}^r_\perp K_{3/2}(\rho_\perp)/\rho_\perp^{1/2}}, \tag{4.1}$$

is written down in this section, and the reduction to the weakly relativistic, small-gyroradius and nonrelativistic limits is discussed.

4.1. Explicit form for the response tensor

The response for the relativistic DGH distribution (4.1) is

$$\begin{aligned}\alpha^{\mu\nu}(k) = &-\frac{q^2 n}{m\hat{d}^r_\perp [K_{3/2}(\rho_\perp)/\rho_\perp^{1/2}]} \int_{-1}^{1} dv_0\, g(v_0)\\ &\times \int_0^\infty d\xi\, \hat{d}^r_\perp \bigg\{ T_{\alpha\beta}(\xi)(k^\sigma_{3\perp} g^{\alpha\mu} - k^\alpha g^{\sigma\mu}_{3\perp})(k^\tau_{3\perp} g^{\beta\nu} - k^\beta g^{\tau\nu}_{3\perp})\, \dot{t}_\tau{}^\eta(-\xi)\\ &\times \bigg[-(g_{3\perp})_{\sigma\eta}\, \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} + a_{\perp\sigma}(\xi) a_{\perp\eta}(\xi)\, \frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \bigg] \bigg\}.\end{aligned} \tag{4.2}$$

The form (4.2) is the generalization of the result (1.5.10) to magnetized particles. It is the counterpart of (3.19) for the relativistic DGH distribution (4.1), and it is also straightforward to write down a counterpart of the form (3.20), which is, for the special case $r = 0$,

$$\begin{aligned}\alpha^{\mu\nu}(k) = -\frac{q^2 n}{m} \int_{-1}^{1} dv_0\, g(v_0) \bigg\{ &-\frac{n_{\rm pr}}{n}\, b^\mu b^\nu + \rho_\perp u_0^\mu u_0^\nu\\ &-i\frac{\rho_\perp^{1/2}}{K_{3/2}(\rho_\perp)} \int_0^\infty d\xi\, \bigg[[\rho_\perp\, ku_0 - i(kb)^2 \xi] \bigg(t_\perp^{(1)\mu\nu}(\xi)\, \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)}\\ &-a^\mu_\perp(\xi)\tilde{a}^\nu_\perp(\xi)\, \frac{K_{5/2}\big(r_\perp(\xi)\big)}{r_\perp^{5/2}(\xi)} \bigg)\\ &-kb\,[b^\mu \tilde{a}^\nu_\perp(\xi) + a^\mu_\perp(\xi) b^\nu]\, \frac{K_{3/2}\big(r_\perp(\xi)\big)}{r_\perp^{3/2}(\xi)} \bigg] \bigg\},\end{aligned} \tag{4.3}$$

with $u_0 = [\gamma_0, \gamma_0 v_0 \boldsymbol{b}]$ and $b = [\gamma_0 v_0, \gamma_0 \boldsymbol{b}]$, cf. (1.5.4).

4.2. The small-gyroradius approximation

The expression (4.2) is too cumbersome to be of direct use in applications, and approximations must be made to proceed. Relevant approximations are the small-gyroradius approximation and the weakly relativistic approximation. For simplicity, in the following discussion only the case $r = 0$ is considered; the generalization to $r = 1, 2, \ldots$ follows by differentiating, as in (4.2).

The small-gyroradius approximation involves assuming that the term proportional to $k_\perp^2$ in the expression (3.16) for $r_\perp(\xi)$ is smaller than the other term in

(3.16). This requires that

$$r_0(\xi) \gg 2k_\perp/\Omega_0, \qquad r_0(\xi) = \rho_\perp - iku_0\,\xi. \tag{4.4}$$

One ignores the term proportional to $k_\perp^2$ except in the exponential function (cf. (1.4.9)), which is expanded in Bessel functions. Writing

$$\Lambda(\xi) = \frac{k_\perp^2}{\Omega_0^2 r_0(\xi)}, \tag{4.5}$$

one has

$$e^{-r_\perp(\xi)} \approx e^{-r_0(\xi)} e^{-\Lambda(\xi)(1-\cos\Omega_0\xi)} = e^{-r_0(\xi)} \sum_{n=-\infty}^{\infty} e^{-\Lambda(\xi)} I_n\big(\Lambda(\xi)\big) e^{in\Omega_0\xi}. \tag{4.6}$$

Applying the small-gyroradius approximation to (4.3) gives

$$\begin{aligned}\alpha^{\mu\nu}(k) = -\frac{q^2 n}{m}\int_{-1}^{1} dv_0\, g(v_0)\bigg\{ &-\frac{n_{\rm pr}}{n}\, b^\mu b^\nu + \rho_\perp u_0^\mu u_0^\nu \\ &-i\frac{\rho_\perp^{1/2}}{K_{3/2}(\rho_\perp)}\int_0^\infty d\xi\,\bigg[[\rho_\perp\, ku_0 - i(kb)^2\xi]\bigg(t_0^{(1)\mu\nu}(\xi)\,\frac{K_{3/2}\big(r_0(\xi)\big)}{r_0^{3/2}(\xi)} \\ &-t_0^{(2)\mu\nu}(\xi)\,\frac{K_{5/2}\big(r_0(\xi)\big)}{r_0^{5/2}(\xi)}\bigg) \\ &-kb\,[b^\mu a_0^{*\nu}(\xi) + a_0^\mu(\xi) b^\nu]\,\frac{K_{3/2}\big(r_0(\xi)\big)}{r_0^{3/2}(\xi)}\bigg]\bigg\},\end{aligned} \tag{4.7}$$

$$a_0^\mu(\xi) = \sum_{n=0}^{\infty} e^{-\Lambda + in\Omega_0\xi}\bigg[\bigg(r_0 u_0^\mu - \frac{k_\perp^\mu}{\Omega_0}\frac{n}{\Lambda}\bigg) I_n - i\eta\frac{k_\alpha f^{\mu\alpha}}{\Omega_0}\,(I_n - I_n')\bigg], \tag{4.8}$$

$$t_0^{(1)\mu\nu}(\xi) = \sum_{n=0}^{\infty} e^{-\Lambda + in\Omega_0\xi}\bigg(u_0^\mu u_0^\nu I_n + g_\perp^{\mu\nu}\, I_n' - i\eta\frac{k_\alpha f^{\mu\alpha}}{\Omega_0}\,\frac{n}{\Lambda}\, I_n\bigg), \tag{4.9}$$

$$\begin{aligned}t_0^{(2)\mu\nu}(\xi) = \sum_{n=0}^{\infty} e^{-\Lambda + in\Omega_0\xi}\bigg\{ &r_0^2 u_0^\mu u_0^\nu I_n - \frac{r_0}{\Omega_0}(k_\perp^\mu u_0^\nu + u_0^\mu k_\perp^\nu)\,\frac{n}{\Lambda}\, I_n \\ &+\frac{k_\perp^\mu k_\perp^\nu}{\Omega_0^2}\bigg(\frac{n^2}{\Lambda^2} I_n - \frac{1}{\Lambda} I'n\bigg) + \eta\frac{k_\alpha f^{\mu\alpha} k_\beta f^{\nu\beta}}{\Omega_0^2}\,(I_n - 2I_n' + I_n'') \\ &-i\eta\frac{k_\perp^\mu k_\alpha f^{\nu\alpha} - k_\alpha f^{\mu\alpha} k_\perp^\nu}{\Omega_0^2}\bigg[\frac{n}{\Lambda}\,(I_n - I_n') + \frac{n}{\Lambda^2}\, I_n\bigg] \\ &-i\eta r_0 (k_\alpha f^{\mu\alpha} u_0^\nu - u_0^\mu k_\alpha f^{\nu\alpha})\,(I_n - I_n')\bigg\},\end{aligned} \tag{4.10}$$

where Λ and r_0 denote $\Lambda(\xi)$ and $r_0(\xi)$ respectively, and where the argument $\Lambda(\xi)$ of the modified Bessel functions is implicit.

The derivation of the expression (4.7) is the primary objective of the present paper. The result (4.7) is the response tensor for the distribution (4.1) with $r = 0$, with all relativistic effects included. The only approximation made in (4.7) is the small-gyroradius approximation: in the opposite case of large gyroradii the unmagnetized form (1.5.10) applies. Further simplifying approximations may be made to (4.7) by assuming $\Lambda \ll 1$ and retaining only the lowest-order terms in the power-series expansions of the modified Bessel functions, and by assuming the weakly relativistic approximation $\rho_\perp \gg 1$ and retaining only the lowest-order terms in an expansion in $1/\rho_\perp$.

4.3. The weakly relativistic approximation

The weakly relativistic approximation corresponds to $\rho_\perp \gg 1$. Before making this approximation, it is convenient to rewrite (4.7) using the explicit expressions (1.4.9) for the Macdonald functions. This gives

$$\begin{aligned}\alpha^{\mu\nu}(k) = -\frac{q^2 n}{m}\int_{-1}^{1} dv_0\, g(v_0)\Bigg(&-\frac{n_{\rm pr}}{n}\, b^\mu b^\nu + \rho_\perp u_0^\mu u_0^\nu \\ &-i\frac{\rho_\perp}{1+1/\rho_\perp}\int_0^\infty d\xi\, \frac{e^{i\,ku_0\,\xi}}{r_0^2}\Bigg\{[\rho_\perp\, ku_0 - i(kb)^2\xi]\bigg[t_0^{(1)\mu\nu}(\xi)\left(1+\frac{1}{r_0}\right) \\ &-\frac{t_0^{(2)\mu\nu}(\xi)}{r_0}\left(1+\frac{3}{r_0}+\frac{3}{r_0^2}\right)\bigg] \\ &-kb\,[b^\mu a_0^{*\nu}(\xi) + a_0^\mu(\xi) b^\nu]\left(1+\frac{1}{r_0}\right)\Bigg\}\Bigg).\end{aligned} \tag{4.11}$$

The weakly relativistic approximation then follows by expanding in powers of $1/\rho_\perp$ and retaining only the lowest-order terms.

Substantial simplification occurs when one makes the weakly relativistic approximation and takes the small-gyroradius limit $\Lambda \to 0$. Then (4.11) reduces to

$$\alpha^{\mu\nu}(k) = -\frac{q^2 n}{m}\int_{-1}^{1} dv_0\, g(v_0) G^{\alpha\mu}(0,k,u_0)\tau_{\alpha\beta}(ku_0)G^{*\beta\nu}(0,k,u), \tag{4.12}$$

which coincides with the small-gyroradius limit of the expression (B 5) derived in Appendix B by a different procedure.

The approximations made in deriving (4.12) are equivalent to assuming that the particles have no perpendicular motion. If one assumes that $g(v_0)$ corresponds to a relativistic thermal distribution then (4.12) leads to the response tensor for the strictly parallel distribution (1.3.1) for the choice (1.6.1), namely

$$g(v_0) = \frac{\gamma_0^2}{K_1(\rho_\parallel)}\, \exp(-\rho_\parallel \gamma_0). \tag{4.13}$$

5. Discussion and conclusions

The primary objective in the present paper is to derive the linear response tensor for a relativistic counterpart of the DGH distribution (4.1). The special case $r = 0$ corresponds to a relativistic counterpart of a bi-Maxwellian distribution. The nonrelativistic bi-Maxwellian distribution distribution has been used extensively to discuss instabilities in magnetized plasmas with streaming motions and temperature anisotropies, (see e.g. Stix 1962), and the more general DGH distribution allows one to discuss instabilities due to loss-cone anisotropies. The results derived here include a general expression (4.3) for the response tensor for the case $r = 0$ in (4.1), and the result for integral $r > 0$ follows from (4.3) by differentiation, as in its unmagnetized counterpart (1.5.2). The general expression is too cumbersome for most purposes, and the small-gyroradius approximation to it is given by (4.7) or (4.11). The latter expressions are a convenient starting point for an investigation of the properties of waves in anisotropic relativistic magnetized thermal-like plasmas. Relativistic effects are known to play an important role in determining wave properties near perpendicular propagation ($\theta = \frac{1}{2}\pi$). In a thermal plasma the cyclotron harmonic waves identified in a nonrelativistic plasma, (see e.g. Gross 1951; Bernstein 1958; Dnestrovskii and Kostomarov 1961, 1962; Puri *et al.* 1973, 1975) are modified in important ways by relativistic effects, (see e.g. Shkarofsky 1966*b*; Bornatici *et al.* 1983; Robinson 1987*a, b*, 1988). The response tensor for a relativistic magnetized thermal distribution, first calculated by Trubnikov (1958), and approximations made to it by Shkarofsky (1966*a, b*) have been used in the discussion of relativistic effects on the wave properties by the authors cited above and others. An analogous discussion of the anisotropic case is possible starting from the results of the present paper. Although such an investigation is the primary motivation for the analytical developments reported here, the detailed investigation has yet to be carried out.

The response tensor for a magnetized strictly parallel relativistic thermal distribution is given by either (3.13) or (3.14), which are the counterparts of the unmagnetized versions (1.3.7) and (1.3.9) respectively.

In conclusion, it is noted that the procedure developed in Part 1 and the present paper for calculating response tensors is based on an idea that has not been used previously in the literature. The idea is to use the freedom to make Lorentz transformations to build up a parallel distribution of particles for any given perpendicular distribution. The procedure is greatly facilitated by the use of a covariant formalism, but this is a convenient rather than essential feature of the method. In the present paper this idea is applied to the strictly perpendicular relativistic thermal distribution, for which the response tensor was calculated using a noncovariant formalism by Trubnikov and Yakubov (1963), and the same procedure may be applied to any strictly perpendicular distribution for which the integral over perpendicular momentum can be performed in evaluating the response tensor. In this way, starting from any strictly perpendicular distribution for which the integral over perpendicular momentum can be performed, one can write down the response tensor for a class of relativistic distributions built up from this strictly perpendicular distribution and an arbitrary distribution $g(v_0)$ of parallel velocities.

Acknowledgements

I thank Stephen Hardy, Neil Cramer and Sergei Vladimirov for helpful comments.

Appendix A. Covariant Vlasov method

A covariant version of Vlasov theory was used by Melrose (1997*b*) to derive the response tensor specific for an arbitrary distribution of magnetized particles. The first-order perturbation in the distribution function is

$$F^{(1)}\big(k, p(\tau)\big) = -iq\, A_\nu(k) \int_0^\infty d\xi\, e^{ik[X(\tau)-X(\tau-\xi)]}$$
$$\times ku(\tau-\xi)\, G^{\alpha\nu}\big(k, u(\tau-\xi)\big)\, \dot{t}_\alpha{}^\beta(\tau-\xi)\, \frac{\partial F(p)}{\partial p^\beta}, \tag{A 1}$$

$$\frac{\partial F(p)}{\partial p_\alpha(\tau')} = \frac{\partial u^\beta(0)}{\partial u_\alpha(\tau')} \frac{\partial F(p)}{\partial p^\beta} = \dot{t}^\beta{}_\alpha(-\tau')\, \frac{\partial F(p)}{\partial p^\beta}. \tag{A 2}$$

The resulting expression for the response tensor is

$$\alpha^{\mu\nu}(k) = -iq^2 \int d^4p(\tau) \int_0^\infty d\xi\, u^\mu(\tau)\, e^{ik[X(\tau)-X(\tau-\xi)]}$$
$$\times ku(\tau-\xi)\, G^{\alpha\nu}\big(k, u(\tau-\xi)\big)\, \dot{t}_\alpha{}^\beta(\tau-\xi)\, \frac{\partial F(p)}{\partial p^\beta}. \tag{A 3}$$

The expression (A 3) is equivalent to (2.15), as may be shown by partially integrating. Using the identity

$$\dot{t}_\alpha{}^\beta(\tau) \frac{\partial}{\partial p^\beta} [ku(\tau)\, G^{\alpha\nu}\big(k, u(\tau)\big)] = 0, \tag{A 4}$$

one finds

$$\alpha^{\mu\nu}(k) = \frac{iq^2}{m} \int d^4p(\tau)\, F(p) \int_0^\infty d\xi\, e^{ik[X(\tau)-X(\tau-\xi)]}$$
$$\times \dot{t}_{\beta\alpha}(\tau-\xi) \left\{ \dot{t}^{\mu\alpha}(\tau) + iu^\mu(\tau) k_\sigma \big[t^{\sigma\alpha}(\tau) - t^{\sigma\alpha}(\tau-\xi)\big] \right\} ku(\tau)$$
$$\times G^{\alpha\mu}\big(k, u(\tau)\big)\, ku(\tau-\xi)\, G^{\beta\nu}\big(k, u(\tau-\xi)\big), \tag{A 5}$$

which further simplifies using

$$\dot{t}_{\beta\alpha}(\tau-\xi) \dot{t}^{\mu\alpha}(\tau) = \dot{t}^\mu{}_\beta(\xi), \tag{A 6}$$

$$\dot{t}_{\beta\alpha}(\tau-\xi) \big(t^{\sigma\alpha}(\tau) - t^{\sigma\alpha}(\tau-\xi)\big) = T^\sigma{}_\beta(\xi). \tag{A 7}$$

The equivalence of (A 5) and (2.15) follows by inspection after expanding in Bessel functions and performing the integrals over proper time explicitly.

Appendix B. Expansion in Bessel functions

The generating function for Bessel functions is

$$e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} e^{in\phi} J_n(z). \tag{B 1}$$

The expansion needed here is

$$u^\mu(\tau) e^{ikX(\tau)} = e^{ikx_0} \sum_{s=-\infty}^{\infty} e^{-is\eta\psi} e^{i[(ku)_\parallel - s\Omega_0]\tau} U^\mu(s,k), \tag{B 2}$$

$$U^\mu(s,k) = \big(\gamma J_s(k_\perp R), \gamma \boldsymbol{V}(s,k)\big), \tag{B 3}$$

$$\boldsymbol{V}(s,k) = \big(\tfrac{1}{2} v_\perp [e^{-i\eta\psi} J_{s-1}(k_\perp R) + e^{i\eta\psi} J_{s+1}(k_\perp R)]$$
$$-\tfrac{1}{2} i\eta v_\perp [e^{-i\eta\psi} J_{s-1}(k_\perp R) - e^{i\eta\psi} J_{s+1}(k_\perp R)], v_\parallel J_s(k_\perp R)\big). \tag{B 4}$$

The ξ integrals in (2.15) and (A 5) are trivial after expanding in Bessel functions using (B 3). Both forms reduce to the same result

$$\alpha^{\mu\nu}(k) = -\frac{q^2}{m}\int d^4p\,F(p)\sum_{s=-\infty}^{\infty} G^{\alpha\mu}(s,k,u)\tau_{\alpha\beta}[(ku)_\parallel - s\Omega_0]G^{*\beta\nu}(s,k,u), \tag{B 5}$$

$$\tau^{\mu\nu}(\omega) = g_\parallel^{\mu\nu} + \frac{\omega}{\omega^2-\Omega_0^2}(\omega g_\perp^{\mu\nu} + i\eta\Omega_0 f^{\mu\nu}), \tag{B 6}$$

$$G^{\mu\nu}(s,k,u) = g^{\mu\nu}J_s(k_\perp R) - \frac{k^\mu U^\nu(s,k)}{(ku)_\parallel - s\Omega_0}, \tag{B 7}$$

where the subscript 0 on u and $p = mu$ is now redundant. In (A 5) the identity $k_\nu G^{\mu\nu}(k,u) = 0$ ensures that the charge-continuity and gauge-invariance conditions are manifestly satisfied.

References

Bernstein, I. B. 1958 Waves in a plasma in a magnetic field. *Phys. Rev.* **109**, 10–21.

Bornatici, M., Cano, R., De Barbieri, O. and Engelmann, F. 1983 Electron cyclotron emission and absorption in fusion plasmas. *Nucl. Fusion* **23**, 1153–1257.

Dewar, R. L. 1977 Energy–momentum tensors for dispersive electromagnetic waves. *Aust. J. Phys.* **30**, 533–575.

Dnestrovskii, Yu. N. and Kostomarov, D. P. 1961 Dispersion equation for an ordinary wave moving in a plasma perpendicular to an external magnetic field. *Soviet Phys. JETP* **13**, 986–990.

Dnestrovskii, Yu. N. and Kostomarov, D. P. 1962 Dispersion equation for an extraordinary wave moving in a plasma perpendicular to an external magnetic field. *Soviet Phys. JETP* **14**, 1089–1095.

Dory, R. A., Guest, G. E. and Harris, E. G. 1965 Unstable electrostatic plasma waves propagating perpendicular to a magnetic field. *Phys. Rev. Lett.* **14**, 131–133.

Gross E. P. 1951 Plasma oscillations in a static magnetic field. *Phys. Rev.* **82**, 232–242.

Melrose, D. B. 1986 *Instabilities in Space and Laboratory Plasmas*. Cambridge University Press.

Melrose, D. B. 1997*a* Covariant form of Trubnikov's response tensor for a relativistic magnetized thermal plasma. *J. Plasma Phys.* **57**, 479–488.

Melrose, D. B. 1997*b* Response of a relativistic anisotropic thermal plasma: Part 1. Unmagnetized particles. *J. Plasma Phys.* **58**, 705–719.

Melrose, D. B., Rönnmark, K. G. and Hewitt, R. G. 1982 Terrestrial kilometric radiation: the cyclotron theory. *J. Geophys. Res.* **87**, 5140–5150.

Omidi, N. and Gurnett, D. A. 1982 Growth rate calculations of auroral kilometric radiation using the relativistic resonance condition. *J. Geophys. Res.* **87**, 2377–2383.

Puri, S., Leuterer, F. and Tutter, M. 1973 Dispersion curves for the generalized Bernstein modes. *J. Plasma Phys.* **9**, 80–100.

Puri, S., Leuterer, F. and Tutter, M. 1975 The totality of waves in a homogeneous Vlasov plasma. *J. Plasma Phys.* **14**, 169–178.

Robinson, P. A. 1987*a* Dispersion of electron Bernstein waves including weakly relativistic and electromagnetic effects. Part 1. Ordinary modes. *J. Plasma Phys.* **37**, 435–447.

Robinson, P. A. 1987*b* Dispersion of electron Bernstein waves including weakly relativistic and electromagnetic effects. Part 2. Extraordinary modes. *J. Plasma Phys.* **37**, 449–465.

Robinson, P. A. 1988 Weakly relativistic dispersion of Bernstein waves. *Phys. Fluids* **31**, 107–114.

Shkarofsky, I. P. 1966*a* Dielectric tensor in Vlasov plasmas near cyclotron harmonics. *Phys. Fluids* **9** 561–570.

Shkarofsky, I. P. 1966*b* Dispersion of waves in cyclotron harmonic resonance regions in plasmas. *Phys. Fluids* **9**, 570–580.

Stix, T. H. 1962 *The Theory of Plasma Waves*. McGraw-Hill, New York.

Trubnikov, B. A. 1958 Magnetic emission of high temperature plasma thesis, Moscow Institute of Engineering and Physics; [English translation in AEC-tr-4073, US Atomic Energy Commission, Oak Ridge, Tennessee, 1960].

Trubnikov, B. A. and Yakubov, V. B. 1963 Cyclotron radiation of electrons having a two-dimensional Maxwellian distribution. *Plasma Phys.* (*J. Nucl. Energy C*) **5**, 7–14.

Twiss, R. Q. 1958 Radiation transfer and the possibility of negative absorption in radio astronomy. *Aust. J. Phys.* **11**, 564–579.

Wu, C. S. 1986 Kinetic cyclotron and synchrotron maser instabilities. Radio emission processes by direct amplification of radiation. *Space Sci. Rev.* **41**, 215–298.

Wu, C. S. and Lee, L. C. 1979 A theory of the terrestrial kilometric radiation. *Astrophys. J.* **230**, 621–626.