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Covariant Electromagnetic Forces in a Time-dependent and Inhomogeneous Medium

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Abstract

The force density acting on a non-stationary and inhomogeneous dispersive medium in the presence of a high-frequency electromagnetic field is calculated in a covariant formalism. It is demonstrated that the average Lorentz force density due to a wave in the medium can be written using only the Maxwell electromagnetic tensor.

The electromagnetic properties of a non-stationary and inhomogeneous medium are important for many applications when considering propagation of electromagnetic waves. It is well known that electromagnetic forces acting on the medium in the presence of wave fields are closely connected with the energy and momentum densities of the fields [1–3]. Therefore the problem of the definition of the ponderomotive force depends strongly on the way the total energy (and momentum) of the system (which includes the background medium and the waves) is separated into background and wave subsystems [4]. For a nonstationary and inhomogeneous system, e.g., as results from dissipation of the wave energy, the problem is even more complicated because the scale of the nonstationarity and inhomogeneity must also be determined.

Use of a covariant four-dimensional (4D) formalism [5, 6] to calculate the covariant electromagnetic force restricts the choice of the separation into subsystems because of Lorentz and gauge invariance. If in addition the quantum microscopic theory (e.g., covariant relativistic quantum theory for plasma [7] with the proper renormalization procedures) is employed, it is possible to demonstrate that the statistical average (over a distribution of waves) of the mass operator provides a relativistic quantum calculation of the ponderomotive force in the canonical separation [4]. The canonical separation is equivalent to assigning momentum k and energy ω_k to the wave [4, 8]. Thus the additional requirement on the separation into background and wave subsystems is that the 4-momentum of the waves is required to be proportional to the wave 4-vector k^μ. However, in general in the electrodynamics of continuous media the separation is ambiguous, which was the subject of the historical Abraham–Minkowsky controversy, see, e.g., [4].

In this paper, we demonstrate that in the covariant (classical) formalism, the average Lorentz 4-force density due to a wave in a time-dependent and inhomogeneous plasma can be written using the Maxwell tensor in a form which contains no explicit dependence on the properties (including the characteristic nonstationarity and inhomogeneity scales) of the (spatial and time) dispersive medium. The canonical covariant ponderomotive force density appears as a 4-gradient of the field Lorentz invariant $E^2 - B^2$ (this fact has also been noted in [7]). Furthermore, the difference between the averaged Lorentz force density and the ponderomotive force density in the alternative "physical" separation [4] into background and wave subsystems has the same form as in a stationary and homogeneous dispersive medium, with the corresponding (weak) x-dependence of the wave 4-vector and amplitude, as well as of the particles' averaged distribution function. Thus for the two most favored of the possible separations into background and wave subsystems the definition of the characteristic scale is not needed to write the ponderomotive force density due to a wave in the dispersive medium.

Calculations here are similar to those in [6] (additionally taking into account the slow nonstationarity and weak inhomogeneity of the system) where we refer the reader for details. Plasma particles are described by a covariant distribution function

$$\mathscr{F}(x, p) = 2m\delta(p^2 - m^2)H(p^0)f(\boldsymbol{p}, \boldsymbol{x}, t), \tag{1}$$

where natural units ($c = 1$ and $\varepsilon_0 \mu_0 = 1$) are used, $H(p^0)$ is the step function of the time component of the 4-momentum p^0; the metric tensor is $g_{\alpha\beta} = \mathrm{diag}\,(1, -1, -1, -1)$. Furthermore, in (1) $f(\boldsymbol{p}, \boldsymbol{x}, t)$ is the conventional distribution function in the 6-dimensional phase space. Other notations are standard.

The wave is described in the WKB approximation by

$$A^\mu(x) = A^\mu(x, k) \exp[i\Theta(x)] + c.c., \tag{2}$$

where $A^\mu(x, k)$ is the slowly varying amplitude of the wave 4-potential, $\Theta(x)$ is the wave eikonal, and *c.c.* is for complex conjugate. The contravariant wave 4-vector is defined by $k^\mu(x) = -\partial^\mu \Theta(x)$ where $\partial^\mu \equiv \partial/\partial x_\mu$.

The canonical energy-momentum tensor for the background system is given by

$$T_b^{\mu\nu}(x) = \int \mathrm{d}p F(x, p) u^\mu P_c^\nu, \tag{3}$$

where $F(x, p)$ is the averaged over the fast scale oscillation centre (OC) distribution function [4, 6] depending on the OC variables x and p, u^μ is the 4-velocity of the OC, and P_c^μ is the covariant canonical momentum

$$P_c^\mu = -(g^{\mu\nu} - u^\mu u^\nu)\frac{\partial R}{\partial u^\nu} - u^\mu R. \tag{4}$$

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In eq. (4), R is the Lagrangian for the OC orbit which can be approximated by

$$R \approx -m + \frac{q^2}{m} a^{\alpha\beta} A_\alpha A_\beta^*, \tag{5}$$

where we assume that the average wave field is zero $\langle A \rangle = 0$, and the tensor $a^{\mu\nu}$ is given by ($ku = k^\mu u_\mu$)

$$a^{\mu\nu} = G^{\rho\mu} G^\nu_\rho, \qquad G^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu u^\nu}{ku} \tag{6}$$

Using the OC equation of motion

$$\frac{dP_c^\mu}{d\tau} + \partial^\mu R = 0, \tag{7}$$

where $d\tau$ is the OC distance along a world line in 4-space, together with the identity (cf. eq. (7) of [6])

$$\partial^\mu \int dp F(x, p)(R g^{\mu\nu} + u^\mu P_c^\nu) = \int dp R \partial^\nu F(x, p), \tag{8}$$

which has been found using the covariant Vlasov equation for the OC distribution function F, we obtain the canonical ponderomotive force density

$$\begin{aligned} f_b^\nu(x) &= \partial_\mu T_b^{\mu\nu}(x) \\ &= -\int dp F(x, p) \partial^\nu R \\ &\approx -\frac{q^2}{m} \int dp F(x, p) \partial^\nu (a^{\alpha\beta} A_\alpha A_\beta^*). \end{aligned} \tag{9}$$

In the physical separation [4] into background and wave subsystems, the ponderomotive force density can be written as (cf. eq. (18) of Ref. [6])

$$f_b^\nu = \partial_\mu T_Q^{\mu\nu} + f_L^\nu. \tag{10}$$

Here, $T_Q^{\mu\nu}$ is the energy-momentum tensor for the quiver motion which appears after separation of the 4-velocity into average and quiver parts, and subsequent averaging over the fast oscillations of the quiver 4-velocity, see Ref. [7], eqs (15)–(17); it is given by

$$T_Q^{\mu\nu} = -\frac{q^2}{m} \int dp F(x, p)(G^{\mu\alpha} G^{\nu\beta} + G^{\nu\alpha} G^{\mu\beta} - a^{\alpha\beta} u^\mu u^\nu) A_\alpha A_\beta^* \tag{11}$$

Furthermore, f_L^ν in eq. (10) is the average Lorentz force density imposed on the particles by the wave.

The expression for the averaged Lorentz force density (taking into account the slow nonstationarity and weak inhomogeneities of the system) is obtained from the exact equation of motion [6]

$$f_L^\mu = \left\langle q, \int dp \mathscr{F}(x, p) m u^\mu u^\nu \right\rangle = \langle F^{\mu\nu}(x) J_\nu(x) \rangle, \tag{12}$$

where we introduce the Maxwell tensor

$$F^{\mu\nu}(x) = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{13}$$

Next, we need to assume a specific relation between the 4-current J_μ and the wave potential A^ν, which relation defines the polarization 4-tensor, $\alpha^{\mu\nu}$, of the medium.

In the linear approximation, the general relation between the current and the wave potential is given by the non-local expression

$$J^{(1)}(x) = \int \hat{\alpha}(x, x') A(x') \, dx', \tag{14}$$

where $\hat{\alpha}(x, x')$ is the polarization operator. In a stationary and homogeneous medium $\hat{\alpha}(x, x') = \alpha(x - x')$, which yields the well known simple local dependence between the current and the wave potential amplitude. In a slowly nonstationary and weakly inhomogeneous system, the (local) linear relation includes additional terms, in particular, proportional to time and space derivatives of the polarization operator. In this case, the dependence of the polarization operator on the nonstationarity and inhomogeneity scale must be defined since the non-local operator $\hat{\alpha}(x, x')$ is a rapidly changing function of $\xi = x - x'$ and a slowly changing function of another argument which depends linearly on x and x' and can be written in the most general form as $x - \beta\xi$, where β is a constant, $0 \leqslant \beta \leqslant 1$. Thus we have

$$\hat{\alpha}(x, x') \approx \alpha(\xi, x) - \beta(\xi \cdot \partial)\alpha(\xi, x). \tag{15}$$

The exact value of the scale parameter β is not obvious a priori, moreover, it may be dependent on various processes, e.g., energy and momentum exchange with the surrounding environment in open systems.

Phenomenologically, since the work [9] (see also [1]), it has been customary to assume $\beta = 1/2$. This choice corresponds to conservation of the wave action in the canonical separation into background and wave subsystems. Thus the conservation laws (of the wave action and the wave energy) provide arguments for the above particular choice of β [10]. However, the definition of the wave action depends on how the separation into background and wave subsystems is made. Therefore the conventional choice of $\beta = 1/2$ in (15) depends on a specific separation into background and wave subsystems. Moreover, for an open system, when the effective inhomogeneity scale can be determined by the energy and momentum exchange with external sources and sinks, the value of β appears to be ill-defined if we have no exact (microscopic) information on the character of the energy exchange. Fortunately, as we now show, the average Lorentz force density can be written in a form containing only slowly changing wave amplitudes and having no explicit dependence on β. Thus we do not need to specify the scale β in our calculations.

Substituting the linear response current (14) into (12) and using (15), we find

$$\begin{aligned} f_L^\mu(x) &= \partial^\mu(\alpha_{\nu\rho}^{(0)} A^\nu A^{*\rho}) - A^\nu A^{*\rho} \partial^\mu \alpha_{\nu\rho}^{(0)} \\ &\quad - \partial^\nu(\alpha_{\nu\rho}^{(0)} A^\mu A^{*\rho} + \alpha_{\nu\rho}^{(0)} A^{*\mu} A^\rho) \\ &\quad + 2k^\mu \alpha'_{\nu\rho} A^\nu A^{*\rho} - k^\mu \partial^\gamma \left(\frac{\partial \alpha_{\nu\rho}^{(0)}}{\partial k^\gamma} A^\mu A^{*\rho} \right) \\ &\quad - (2\beta - 1) k^\mu \frac{\partial^2 \alpha_{\nu\rho}^{(0)}}{\partial k \cdot \partial x} A^\mu A^{*\rho}, \end{aligned} \tag{16}$$

where $\alpha^{(0)}$ is the real part of the polarization tensor

$$\alpha_{\mu\nu}^{(0)} = -\mathrm{Re} \frac{q^2}{m} \int dp F(x, p) a_{\mu\nu}(x), \tag{17}$$

and α' is its imaginary part, The latter leads to wave dissipation in the system and is therefore responsible for the dissipative part of the force density f_L. If we neglect the x-dependence of the polarization function a and wave vectors k, eq. (16) coincides with eq. (29) of [6].

The dispersion equation, taking into account (15) and the weak x-dependence of the wave amplitude and phase, is given by

$$(-ik + \partial) \cdot (-ik + \partial)A_\mu - (-ik_\mu + \partial_\mu)(-ik_\nu + \partial_\nu)A^\nu$$
$$= \alpha^{(0)}_{\mu\nu} A^\nu + i\alpha'_{\mu\nu} A^\nu - i\partial^\gamma\left(\frac{\partial \alpha^{(0)}_{\mu\nu}}{\partial k^\gamma} A^\nu\right) - i(\beta - 1)\frac{\partial^2 \alpha^{(0)}_{\mu\nu}}{\partial k \cdot \partial x} A^\nu. \quad (18)$$

We see that the right hand side of eq. (18) contains terms analogous to those in the expression for the Lorentz force density (16). Next, we use the definition of the Maxwell tensor (13) to rewrite (16) with the help of the dispersion equation (18) in the form

$$f^\mu_L = -\tfrac{1}{2}\partial^\mu(F^{\nu\rho}F^*_{\nu\rho}) + \partial^\nu(F^{\mu\rho}F^*_{\nu\rho} + F^{*\nu\rho}F_{\nu\rho}). \quad (19)$$

In deriving (19), we use the identity $\partial^\mu k^\nu = \partial^\nu k^\mu$, which follows directly from the definition of the wave 4-vector k.

The result (19) is the main point of our paper. Note that all the explicit dependencies on the medium characteristics such as the polarization tensor a and its derivatives, as well as derivatives of the wave 4-vector, k, are removed through use of the dispersion relation (18). As a consequence, only x-derivatives of the wave field amplitude are needed to define the Lorentz force density. It is also worth noting that although we start from the equation for wave potentials which are not observable quantities, the final result (19) contains only the observable wave fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and their derivatives.

From eq. (9), we easily establish that the canonical ponderomotive force density can also be written in a simple form containing only derivatives of the squared wave field amplitudes:

$$f^\mu_b = -\tfrac{1}{2}\partial^\mu(F^{\nu\rho}F^*_{\nu\rho}). \quad (20)$$

Note that the tensor $T^{\mu\nu}_Q$ in the physical split up into background and wave subsystems (10) has the same form (11) in time-dependent and inhomogeneous systems as in a stationary and homogeneous medium. Therefore for this particular separation the ponderomotive force density also has no irrelevant scale parameters, such as β in (15).

The 3-force density implied by the 4-force density (19) is

$$\boldsymbol{f}_L = -\varepsilon_0\left[\nabla(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2) + \nabla \cdot (\boldsymbol{E}\boldsymbol{E}^* + \boldsymbol{B}\boldsymbol{B}^* + c.c.) - \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{E} \times \boldsymbol{B}^* + c.c.)\right]. \quad (21)$$

The 3-force density corresponding to the canonical ponderomotive force density (20) is given by

$$\boldsymbol{f}_b = -\varepsilon_0\nabla(|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2). \quad (22)$$

A similar expression to (22) is obtained as a result of the statistical averaging over a distribution of waves of the mass operator in relativistic quantum calculation [7] thus providing a quantum microscopic argument in favor of the canonical separation.

In summary, we have demonstrated that the Lorentz force due to the high-frequency wave field acting on a time-dependent and inhomogeneous dispersive medium can be written in a form that contains only derivatives of observable wave field amplitudes. The canonical ponderomotive force density can be written especially simply being just a 4-gradient of the electromagnetic field invariant $|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2$; this result has also been obtained in a QED microscopic consideration of a relativistic quantum plasma [7].

The fact that the force due to the wave field in a nonstationary and inhomogeneous dispersive medium can be written in a way similar to that in a vacuum (i.e. as a divergence of the vacuum Maxwell stress tensor) is the main new result of the paper. We note that previously the main problem when calculating the ponderomotive force has been a proper calculation of the stress tensor in the medium see, e.g., [1, 4]). We emphasize that although the properties of the medium do not appear explicitly in our final result (20), they do appear implicitly through the dispersion relation for the waves and through the slow time and space dependences of the wave amplitude due to the changes in the medium.

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