

## STOCHASTIC FARADAY ROTATION

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### ABSTRACT

Different ray paths through a turbulent plasma can produce stochastic Faraday rotation leading to depolarization of any linearly polarized component. Simple theory predicts that the average values of the Stokes parameters decay according to  $\langle Q \rangle$  and  $\langle U \rangle \propto \exp(-\delta_i)$ , with  $\delta_i \propto \lambda^4$ . It is pointed out that a definitive test for such depolarization is provided by the fact that  $\langle Q^2 + U^2 \rangle$  remains constant while  $\langle Q \rangle^2 + \langle U \rangle^2$  decreases  $\propto \exp(-2\delta_i)$ . The averages to which this effect, called polarization covariance, should apply are discussed; it should apply to spatial averages over a polarization map or temporal averages over a data set, but not to beamwidth and bandwidth averages that are intrinsic to the observation process. Observations of depolarization would provide statistical information on fluctuations in the turbulent plasma along the line of sight, specifically, the variance of the rotation measure. Other effects that can also cause depolarization are discussed. Favorable data sets to which the test could be applied are spatial averages over polarization maps of the lobes of radio galaxies and temporal averages for the binary pulsar PSR B1259–63 as it goes into and comes out of eclipse by the wind of its companion.

*Subject headings:* magnetic fields — MHD — plasmas — polarization — pulsars: individual (PSR B1259–63) — turbulence

### 1. INTRODUCTION

Faraday rotation can cause depolarization of an astrophysical synchrotron source in three ways: internal depolarization, bandwidth depolarization, and external depolarization (e.g., Burn 1966). Internal depolarization is due to differential Faraday rotation when a ray emitted at the far side of the source has its plane of polarization rotated through  $\gtrsim \pi/2$  before reaching the near side of the source. Bandwidth depolarization occurs because of differential rotation of the plane of polarization across the bandwidth of observation (e.g., Simonetti, Cordes, & Spangler 1984; Lazio, Spangler, & Cordes 1990). External depolarization is due to a random component of the magnetic field along the ray path between the source and the observer through the intervening medium (interplanetary, interstellar, intergalactic, or intracluster). Two different treatments of external depolarization are available. One treatment, referred to here as stochastic Faraday rotation, is based on the assumption that the (Faraday) angle through which the plane of polarization is rotated contains a random component (e.g., Burn 1966; Simonetti et al. 1984; Tribble 1991). The other treatment of external depolarization involves including the effect of the magnetic field in scintillation theory (Erukhimov & Kirsh 1973; Tamoikin & Zamek 1974; Melrose 1993a, 1993b, hereafter papers A and B, respectively). Simonetti et al. (1984) established that these two treatments are equivalent for the mean values of the Stokes parameters, as shown in the Appendix below. Such stochastic Faraday rotation is of practical interest in determining the properties of the medium through which the radiation propagates: measurement of the frequency at which depolarization occurs provides information on the fluctuations in the magnetic field along the line of sight, which complements information from the dispersion measure (DM), rotation measure (RM), and scattering measure (SM; e.g., Taylor & Cordes 1993). However, in cases where depolarization is observed, it is not necessarily

clear that it is due to stochastic Faraday rotation rather than to one of the other depolarizing mechanisms.

In the present paper, it is shown that a definitive test for stochastic Faraday rotation follows by measuring the mean square values of the Stokes parameters, specifically,  $\langle Q^2 \rangle$ ,  $\langle U^2 \rangle$ , and  $\langle QU \rangle$ . These mean values can be derived in a very simple way by assuming that the Faraday rotation angle can be treated as a random variable. Let the Faraday angle be

$$\phi_F = \phi_0 + RM\lambda^2, \quad (1)$$

where  $\phi_0$  is the angle at the source and  $\lambda$  is the wavelength. The statistical average is performed by assuming that  $\phi_F$  has a random component that obeys Gaussian statistics. This simple model is justified in the Appendix, where it is shown that the theory of scintillations (e.g. Narayan 1992) in a magnetized plasma leads to the same result for  $\langle Q^2 \rangle$ ,  $\langle U^2 \rangle$ , and  $\langle QU \rangle$  as this model. The model implies that  $\langle Q^2 + U^2 \rangle$  is a constant; that is, although  $\langle Q \rangle$  and  $\langle U \rangle$  decay,  $\langle Q^2 + U^2 \rangle$  does not. This simple but surprising result is referred to here as polarization covariance; it distinguishes depolarization due to stochastic Faraday rotation from other depolarization mechanisms.

In § 2 a model for stochastic Faraday rotation is used to derive formulae that describe the depolarization and the polarization covariance. In § 3 the nature of the averaging process is discussed. In § 4.5 some specific examples of possible depolarization are considered. The conclusions are summarized in § 6.

### 2. RANDOM VARIATIONS IN THE ROTATION MEASURE

Faraday rotation results from the difference between the refractive indices of the two natural modes of a magnetized plasma. In the simplest approximation the polarization of the natural modes is assumed circular, and then Faraday rotation affects only the linear polarization. Here the polar-

ization is described in terms of the Stokes parameters,  $I$ ,  $Q$ ,  $U$ , and  $V$ . Only  $Q$  and  $U$  are affected by Faraday rotation;  $I$  and  $V$  are not mentioned further in this paper.

2.1. Faraday Rotation

Faraday rotation causes the Stokes parameters  $Q$  and  $U$  to vary with distance  $z$  along the ray path according to

$$\frac{\partial}{\partial z} \begin{bmatrix} Q(z) \\ U(z) \end{bmatrix} = \rho_V(z) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} Q(z) \\ U(z) \end{bmatrix}, \tag{2}$$

$$\rho_V = \frac{e^3}{8\pi^2 \epsilon_0 m_e^2 c^3} n_e B_{\parallel} \lambda^2. \tag{3}$$

It is convenient to write equation (2) in the form

$$\frac{\partial}{\partial z} \mathcal{P}(z) = i\rho_V(z)\mathcal{P}(z), \quad \mathcal{P} = Q + iU \tag{4}$$

(e.g., Spangler 1982). The solution of equation (4) is

$$\mathcal{P}(z) = e^{i\phi_F(z)} \mathcal{P}_0, \quad \phi_F(z) = \int_0^z dz' \rho_V(z'), \tag{5}$$

where  $\mathcal{P}_0 = Q_0 + iU_0$  is determined by the intrinsic polarization at the source (at  $z = 0$ ). In the following it is assumed that the axes are oriented such that  $U_0 = 0$ , so that one has  $\mathcal{P}_0 = Q_0$ .

2.2. Fluctuations in the Faraday Angle

Fluctuations in the plasma parameters along the ray path may be taken into account by separating the Faraday angle,  $\phi_F$ , into an average value,  $\langle \phi_F \rangle$ , and a fluctuating part,  $\delta\phi_F$ :

$$\phi_F = \langle \phi_F \rangle + \delta\phi_F, \quad \langle \phi_F \rangle = \langle \rho_V \rangle L, \tag{6}$$

where  $L$  is the length of the ray path. It is convenient to incorporate the average Faraday rotation into modified Stokes parameters, denoted by tildes (paper A):

$$\tilde{\mathcal{P}}(z) = e^{-i\langle \phi_F \rangle} \mathcal{P}(z) = e^{i\delta\phi_F(z)} Q_0, \tag{7}$$

where the final equality follows from equation (5). One has  $\tilde{\mathcal{P}} = \tilde{Q} + i\tilde{U}$ , with  $\tilde{Q}$  and  $\tilde{U}$  real. The fluctuating part of the Faraday angle,  $\delta\phi_F$ , on the right-hand side of equation (7), is now assumed to be a random variable.

2.3. The Mean Stokes Parameters

The mean values of the Stokes parameters follow from the statistical average of equation (7), which may be evaluated using the property,  $\langle \exp(i\phi) \rangle = \exp(-\langle \phi^2 \rangle/2)$  for a Gaussian random variable,  $\phi$ . [The derivation of this property follows from the power series expansion of the exponential and  $\langle \phi^{(2n+1)} \rangle = 0$ ,  $\langle \phi^{2n} \rangle = 3 \cdot 5 \dots \cdot (2n-1) \langle \phi^2 \rangle^n$  for  $n \geq 0$  and integer.] This gives

$$\langle \tilde{\mathcal{P}} \rangle = Q_0 \exp \left[ -\frac{1}{2} \langle (\delta\phi_F)^2 \rangle \right]. \tag{8}$$

It follows that the Stokes parameters vary according to

$$\langle \tilde{Q} \rangle = Q_0 e^{-\delta_l}, \quad \langle \tilde{U} \rangle = 0, \tag{9}$$

where equation (1) is used and  $\delta_l = \langle (\delta\phi_F)^2 \rangle/2$ . The decay of  $\langle \tilde{Q} \rangle$  described by equation (9) corresponds to a depolarization of the radiation. This result has been derived both by the foregoing method (Burn 1966; Tribble 1991) and also from scintillation theory for a magnetized plasma in the special case of axial rays (Erukhimov & Kirsh 1973;

Tamoikin & Zamek 1974; Simonetti et al. 1984; paper A). The exponent of the decay factor in equation (9) is determined by

$$\langle (\delta\phi_F)^2 \rangle = \int_0^L dz \int_0^L dz' \langle \delta\rho_V(z) \delta\rho_V(z') \rangle, \tag{10}$$

with  $\delta\rho_V = \rho_V - \langle \rho_V \rangle$ . Equation (10) may be rewritten by analogy with equation (6) in the form  $\langle (\delta\phi_F)^2 \rangle = \langle \delta\rho_V^2 \rangle L^2$ .

2.4. The Mean Square Stokes Parameters

The mean values of the squares of the Stokes parameters and of the products of the Stokes parameters follow from the statistical average of the modulus squared of equation (7) and of the product of equation (7) and its complex conjugate. Under the assumption that  $\delta\phi_F$  is a Gaussian random variable, these give

$$\begin{aligned} \langle \tilde{\mathcal{P}}^2 \rangle &= Q_0^2 \exp(-4\delta_l), \\ \langle \tilde{\mathcal{P}} \tilde{\mathcal{P}}^* \rangle &= Q_0^2. \end{aligned} \tag{11}$$

In terms of the Stokes parameters, equation (11) implies

$$\begin{aligned} \langle \tilde{Q}^2 \rangle &= Q_0^2 e^{-2\delta_l} \cosh(2\delta_l), \\ \langle \tilde{U}^2 \rangle &= Q_0^2 e^{-2\delta_l} \sinh(2\delta_l), \\ \langle \tilde{Q} \tilde{U} \rangle &= 0. \end{aligned} \tag{12}$$

The result from equation (12) was derived in paper B by solving the fourth-order moment equation for a magnetized plasma for axial rays. The fact that the foregoing theory reproduces the result of the more general theory confirms the equivalence of the two theories for the higher order moments for axial rays.

The covariances of the polarized component of the radiation are given by

$$\begin{aligned} \frac{\langle \tilde{Q}^2 \rangle}{\langle \tilde{Q} \rangle^2} &= \cosh(2\delta_l), \\ \frac{\langle \tilde{U}^2 \rangle}{\langle \tilde{Q} \rangle^2} &= \sinh(2\delta_l). \end{aligned} \tag{13}$$

It follows that for small  $\delta_l$  the mean square of  $\tilde{Q}$  is equal to the square of its mean; however, as  $\delta_l$  increases, the ratio of the mean square fluctuations to the square of the mean increases, becoming arbitrarily large for  $\delta_l \gg 1$ . The mean of  $\tilde{U}$  is zero by definition, and the mean square of  $\tilde{U}$  is small compared to the mean square of  $\tilde{Q}$  for small  $\delta_l$ , but one has  $\langle \tilde{Q}^2 \rangle \approx \langle \tilde{U}^2 \rangle$  for  $\delta_l \gg 1$ . This somewhat surprising prediction may be attributed to the second expression in equation (11): because the Faraday angle does not appear, the value of  $Q^2 + U^2$  is independent of the statistical averaging. Whether or not this is the case for averaging processes in practice is discussed in the next section.

2.5. Consequences of Polarization Covariance

Equations (8) and (11) may be rewritten in a variety of ways. The tildes on the Stokes parameters correspond to

$$\begin{aligned} \tilde{Q} &= Q \cos \langle \phi_F \rangle + U \sin \langle \phi_F \rangle, \\ \tilde{U} &= -Q \sin \langle \phi_F \rangle + U \cos \langle \phi_F \rangle \end{aligned} \tag{14}$$

(compare eq. [7]). The relations in equation (12) involve the parameters  $\langle\phi_F\rangle$  and  $\delta_i$ , but not  $Q_0$ . Various relations between the averages of the Stokes parameters (without the tildes) may be derived by using equation (14) and eliminating one or more of these three parameters. For example, relations that involve  $Q_0$  and  $\delta_i$  but not  $\langle\phi_F\rangle$  are

$$\langle Q^2 \rangle + \langle U^2 \rangle - \langle Q \rangle^2 - \langle U \rangle^2 = Q_0^2(1 - e^{-2\delta_i}), \quad (15)$$

$$\langle Q^2 \rangle \langle U^2 \rangle - \langle QU \rangle^2 = \frac{1}{4} Q_0^4 (1 - e^{-8\delta_i}). \quad (16)$$

More generally, there are five observable parameters in this theory,  $\langle Q \rangle$ ,  $\langle U \rangle$ ,  $\langle Q^2 \rangle$ ,  $\langle U^2 \rangle$ , and  $\langle QU \rangle$ , and three unknowns,  $\langle\phi_F\rangle$ ,  $\delta_i$ , and  $Q_0$ . Thus measurement of all five observables would provide two constraints on the theory. One of these is equivalent to  $\langle\tilde{U}\rangle = 0$ , implying  $\langle\tilde{Q}\tilde{U}\rangle = 0$ , and the other is a definitive test for stochastic Faraday rotation.

### 3. CONDITIONS FOR POLARIZATION COVARIANCE TO BE OBSERVED

The conditions under which polarization covariance should be observed may be identified by considering the manner in which the observational data are recorded and the nature of the averaging process intrinsic in, or applied to, the data. For a given radio telescope observing at a given frequency, data are recorded over an integration time,  $\Delta t$ , and over a number of frequency channels, each of bandwidth,  $\Delta\nu$ , centered on the given frequency. In principle, the basic bits of information are the values of  $I$ ,  $Q$ , and  $U$  (and maybe  $V$ ) for each  $\Delta t$  and  $\Delta\nu$ . Over an observation time, the averages of  $I$ ,  $Q$ , and  $U$  are simply the averages of the values for each  $\Delta t$ , and similarly the values of  $I^2$ ,  $Q^2$ ,  $U^2$ ,  $QU$ , and so on are to be obtained by storing the appropriate products for each  $\Delta t$  and then averaging each of them over the observation time. The data may also be averaged in frequency over all the channels. Furthermore, the telescope has a characteristic beamwidth,  $\Delta\theta$ , and the bits of data are already averaged over this beamwidth. Hence, one may identify three types of average: intrinsic averages in the data over  $\Delta t$ ,  $\Delta\nu$ , and  $\Delta\theta$ ; the averages applied to the data over the observation time and bandwidth; and temporal averaging over times long compared with a given observation or spatial averages on scales large compared with the resolution of the telescope.

#### 3.1. Averages over Space or Time

The interpretation of an ensemble average, denoted by the angle brackets above, is not specified a priori. It should apply to the averages over space, time, or beamwidth. As the spatial scale or timescale over which the average is performed is increased, the value of  $\delta_i$  should increase (or remain constant). In principle, measurement of  $\delta_i$  provides information on the fluctuations over the relevant spatial scale or timescale.

Spatial averaging requires a map showing the variation of RM (e.g., Simonetti et al. 1984; Simonetti & Cordes 1986, 1988). To simplify the discussion, suppose that the Faraday rotation all occurs in a screen at a distance  $L$ . An angular separation,  $\delta\theta$ , corresponds to a transverse displacement  $r = L\delta\theta$  at the screen, and hence an average over  $\delta\theta$  corresponds to an average over transverse displacement at the screen. Simonetti et al. (1984) introduced a structure func-

tion for RM:

$$\kappa_{\text{RM}}(\delta\theta) = \langle [\text{RM}(\theta) - \text{RM}(\theta + \delta\theta)]^2 \rangle, \quad (17)$$

and they showed that it consists of a statistical part and a geometric part. The statistical part for a power-law spectrum of fluctuations is of the form

$$\kappa_{\text{RM}}(\delta\theta) \propto (\delta\theta)^\beta \quad (18)$$

for  $0 < \beta < 2$ , with  $\beta = 5/3$ , corresponding to a Kolmogorov spectrum. The geometric part is due to the different path lengths through the random medium, and it appears to be unimportant in practice.

The condition for the theory of stochastic Faraday rotation to apply is that the variations in the phases of the components in the two modes be large, so that their difference, and hence the Faraday angle, may be treated as a random variable (Lee & Jokipii 1975; Simonetti et al. 1984). The phase difference between different points would be known for an ideal map, but in practice the sampling is coarse and the separation over which phase coherence is lost is not known a priori. The theory for depolarization, as described by equations (8) and (12), applies only over separations large compared with that over which phase coherence is lost.

Temporal variations in the Faraday angle can arise from two effects: intrinsic temporal changes in the screen (e.g., due to wave motions therein) and convective motions, involving relative motions between the screen and the observer. Only convective motions are considered here. An angular displacement,  $\Delta\theta$ , of the image then occurs in a time interval,  $\Delta t$ , related by

$$L \Delta\theta = v \Delta t \quad (19)$$

as the pattern sweeps across the observer at speed  $v$ .

For an unresolved source with an apparent size  $\theta_{\text{eff}}$ , the minimum time on which different ray paths are sampled is

$$t_{\text{min}} = \min[\theta_{\text{eff}}, \theta_B]L/v, \quad (20)$$

and temporal averaging is relevant only over times  $\gg t_{\text{min}}$ . For a resolved source that fills the beam, the minimum time for temporal averaging is set by the beamwidth of the telescope,  $t_{\text{min}} = L\theta_B/v$ . By way of illustration, for a screen at a distance of several kiloparsecs and with  $v \sim 100 \text{ km s}^{-1}$  (of order the velocity of the Earth through the Galaxy), even for a source of apparent angular size  $\theta_{\text{eff}} \sim 1 \text{ mas}$ , equation (20) implies that  $t_{\text{min}} \sim 2 \text{ months}$  is the minimum time for a temporal average to show the effect of stochastic Faraday rotation for observations with milliarcsecond resolution. The timescale for temporal variations is shorter for observations with higher angular resolution and for closer screens. The effect is of no practical interest for extragalactic screens.

#### 3.2. Beamwidth Averaging

Beamwidth averaging is relevant for an extended, resolved source, that is, for a source with angular size,  $\theta_S$ , satisfying  $\theta_S > \theta_B$ . A source with  $\theta_S \ll \theta_B$  is unresolved, and beamwidth averaging is not relevant.

The observed intensity for a resolved source is the actual intensity convolved with a point-spread function,  $p(r)$ , where  $r$  is the perpendicular displacement on an image

screen (e.g., Rohlfs & Wilson 1996, their eq. [5.64]). For a Gaussian beam one has

$$p(r) = \frac{1}{2\pi r_0^2} \exp\left(-\frac{r^2}{2r_0^2}\right), \quad (21)$$

where  $r_0$  is a constant.

Consider an idealized model in which  $\delta\phi_F(r, z)$  varies because of a wave with wavenumber  $K$  and amplitude  $\delta B$ , propagating along the  $x$ -axis with magnetic fluctuations along the  $z$ -axis. Specifically, assume  $\delta\phi_F(r, z) = f_0 \sin Kx$ , with  $f_0 = (\delta B/B_{\parallel}) \text{RM} \lambda^2$ . If the beamwidth were zero, one would observe  $\mathcal{P} = Q_0 \exp(if_0 \sin Kx)$ . On convolving this with the point-spread function (eq. [21]), one finds

$$\langle \tilde{\mathcal{P}} \rangle_b = Q_0 \sum_{n=-\infty}^{\infty} J_n(f_0) e^{inKx} e^{-n^2 K^2 r_0^2 / 2}, \quad (22)$$

where the subscript  $b$  denotes the average over beamwidth. In the limit  $r_0 \rightarrow 0$ , the final exponential in equation (22) goes to unity, and the sum gives  $\mathcal{P} = Q_0 \exp(if_0 \sin Kx)$ , as required. For  $Kr_0 \gg 1$ ,  $\langle \tilde{\mathcal{P}} \rangle_b$  decreases sharply, and only the term  $n = 0$  contributes to the sum in equation (22).

Although this model is highly idealized, it suffices to illustrate that significant depolarization occurs for  $Kr_0 \gg 1$ , that is, when the fluctuations in RM occur over a scale  $\sim 1/K \ll r_0$ , i.e., small compared with the resolution of the telescope.

For an unresolved source,  $\theta_S < \theta_B$ , the beamwidth average is replaced by an average over the apparent source. The apparent size cannot be smaller than the scatter-image size of a point source affected by scattering in the interstellar medium (ISM) or other turbulent medium through which the radiation passes (e.g., Rickett 1990). A simple model for the angular broadening due to such scattering (e.g., Blandford & Narayan 1985) involves a Gaussian profile similar to that assumed above in performing the beamwidth average. Now  $I(r) \propto \exp(-r^2/2r_1^2)$  is interpreted as the brightness distribution as a function of  $r$  for an apparent source of angular size  $\sim r_1/L$ . Thus, for a point source the average over beamwidth is effectively that over the angular size of the scatter image, rather than over the beamwidth of the radio telescope.

### 3.3. Average over Bandwidth

An average over bandwidth causes depolarization, because  $\phi_F \propto \nu^{-2}$  is a function of frequency. Over  $\Delta\nu$  this causes a change  $|\Delta\phi_F| = |2\phi_F \Delta\nu/\nu|$  and hence a fractional reduction in the degree of linear polarization by  $2\Delta\nu/\nu$ . Depolarization due to bandwidth averaging is qualitatively different from the other averages considered above, because it affects the amplitude rather than the intensity. Polarization covariance applies to the statistics of the measurements of  $Q$  and  $U$ , and bandwidth averaging limits the ability to measure values of  $Q$  and  $U$ .

## 4. LINEAR DEPOLARIZATION MEASURE (LDM)

Assuming that depolarization is observed because of stochastic Faraday rotation in the ISM, the wavelength at which the depolarization becomes important provides information on the statistical properties of the ISM. This may be described in terms of a ‘‘linear depolarization measure,’’ (LDM) defined (see eq. [9]) as

$$\delta_l = \frac{1}{2} \langle (\delta\phi_F)^2 \rangle = \frac{1}{2} \text{LDM} \lambda^4, \quad \text{LDM} = \langle (\text{RM} - \langle \text{RM} \rangle)^2 \rangle, \quad (23)$$

where equation (1) is used. The quantity LDM is determined by the variance in RM and may be measured by any observation of stochastic depolarization.

Four observationally determined parameters have been used to describe the statistical properties of the ISM: the DM, the emission measure (EM), the SM, and the RM (e.g., Cordes & Lazio 1991; Cordes et al. 1991). These parameters determine the integral along the line of sight of, respectively,  $n_e$ ,  $n_e^2$ ,  $C_N^2$ , and  $n_e B_{\parallel}$ , where  $C_N^2$  is the structure constant for the electron density fluctuations. The additional parameter, LDM, is determined by the integral along the line of sight of  $(n_e B_{\parallel} - \langle n_e B_{\parallel} \rangle)^2$ .

To illustrate how the LDM could be used to infer information concerning fluctuations in the magnetic field, consider two extreme cases (e.g., Simonetti & Cordes 1986, 1988). On the one hand, if the magnetic field in the ISM is relatively uniform, then the fluctuations in  $(n_e B_{\parallel})^2$  are dominated by those in  $n_e^2$ , and one would expect the ratio  $\text{LDM}/\text{RM}^2$  to vary across the Galaxy in the same way as does the ratio  $\text{SM}/\text{DM}^2$ . On the other hand, if the magnetic field is sufficiently random, or if regions of opposite sign of  $B_{\parallel}$  nearly cancel, then one can have  $\langle B_{\parallel}^2 \rangle \gg \langle B_{\parallel} \rangle^2$ , implying that  $\text{LDM}/\text{RM}^2$  could exceed unity because of near cancellation of contributions to RM of opposite sign along the line of sight.

Existing data suggest that the variance in RM in the ISM is of the same order but not much larger than  $\langle \text{RM} \rangle^2$  (Simonetti et al. 1984; Simonetti & Cordes 1986, 1988). The implication is that  $B_{\parallel}$  does not change sign many times along these lines of sight. Further data confirming (or otherwise) this result is clearly important for our understanding of the structure of the magnetic field in the ISM. In principle, polarization covariance can be used to estimate  $\langle \text{RM} \rangle^2$  with much finer resolution than can be achieved by averaging over a polarization map.

## 5. OBSERVATIONS OF DEPOLARIZATION

Depolarization observed in some sources has been interpreted in terms of stochastic Faraday rotation, but the definitive test for polarization covariance has not been performed. In this section, some specific examples are discussed.

### 5.1. Depolarization in the ISM

Lazio et al. (1990) used polarization data for eight double-lobed radio galaxies to discuss variations in RM in the Cygnus region. They reported significant depolarization for several of these sources at  $\lambda \gtrsim 10\text{cm}$ . For sources for which RM could be determined, values in the range  $\sim 250$ – $800 \text{ rad m}^{-2}$  were found. The data were compared with a model that gives a relation of the form (see eq. [18])  $\kappa(\theta) = \kappa_0(\theta/\theta_0)^\beta$ , with  $\kappa_0 \sim (10 \text{ rad m}^{-2})^2$  for  $\theta_0 \sim 10''$  and with  $\beta = 5/3$ . The data are described moderately well by the model, but a smaller value of  $\beta \sim 1$  would appear to give a better fit.

The wavelength,  $\lambda_0$ , at which depolarization due to stochastic Faraday rotation should occur follows from equations (8) and (9):

$$\lambda_0 \approx 30 \text{ cm} \left( \frac{\kappa_0^{1/2}}{10 \text{ rad m}^{-2}} \right) \left( \frac{\theta_B}{10''} \right)^{-\beta/2}, \quad (24)$$

where  $\theta_B$  is the beamwidth of the telescope. With  $\kappa_0^{1/2} = 10 \text{ rad m}^{-2}$  for a beamwidth of  $\approx 10''$ , equation (24) implies

depolarization at  $\lambda \gtrsim 30$  cm. Thus, stochastic Faraday rotation is plausible for the depolarization reported by Lazio et al. (1990). Such depolarization would be due to variations in RM on a scale  $r \lesssim L\theta_B$ , below the resolution of the telescope.

### 5.2. Depolarization in Intergalactic Media

It has been suggested that differences in the degree of polarization in the two lobes of radio galaxies might be due to either internal depolarization or due to stochastic Faraday rotation in a circumgalactic medium (Strom & Jägers 1988; Laing 1988; Garrington et al. 1988). The observed depolarization occurs typically at  $\lambda$  between 6 and 20 cm. An example of an extra-Galactic source for which detailed polarization data are available is the inner 2 kpc of M87 (Owen, Eilek, & Keel 1990). This source has relatively high polarization,  $\gtrsim 10\%$ , in the range 4.6–4.9 GHz, with large values of RM  $\sim 1000$ – $8000$  rad  $m^{-2}$  that vary over scales  $\lesssim 1$  kpc across the source.

Suppose that there are fluctuations in RM on a scale smaller than can be observed, with variance LDM. Then the frequency at which the depolarization should become significant is  $\nu_0 \sim 10$  GHz ( $LDM^{1/2}/1000$  rad  $m^{-2}$ ). The absence of substantial depolarization indicates that  $LDM^{1/2}$  cannot be much larger than  $1000$  rad  $m^{-2}$ . However, it would be surprising if the magnetic field were so uniform that one had  $LDM^{1/2} \ll RM$ , and hence one should expect depolarization to become important at frequencies of order of those already observed. Tribble (1991) used the theory for beamwidth depolarization to place a limit on the value of  $B$  in the circumgalactic medium.

The data for M87 should allow one to perform spatial averages of  $Q$ ,  $U$ , and  $Q^2 + U^2$  over maps at the available frequencies, and this would provide a test for the interpretation of the depolarization, which should affect  $Q$  and  $U$  but not  $Q^2 + U^2$ . However, this test has yet to be made.

### 5.3. The Snake

The Snake (Gray et al. 1991) is a long ( $\sim 20'$ ), thin ( $\lesssim 10''$ ) structure roughly perpendicular to the Galactic plane at  $\sim 1^\circ$  from the Galactic center. It is highly polarized,  $\sim 60\%$ , at 10.55 GHz and less polarized at 4.79 GHz (Nicholls & Gray 1992). There is an abrupt decrease in polarization roughly at its midpoint (Gray 1993).

For the decrease in polarization to be due to stochastic Faraday rotation requires an abrupt increase in the fluctuations in RM at the point where the polarization decreases. One possibility is that there is an interstellar cloud along the line of sight covering only the weakly polarized portion of the Snake. Testing this suggestion requires multi-frequency observations to determine RM along the Snake.

### 5.4. Pulsars

Most pulsars are highly polarized (e.g., Lyne & Manchester 1988), implying that stochastic Faraday rotation is normally unimportant. However, there are exceptions that could provide useful tests for the theory of stochastic Faraday rotation.

First consider the frequency at which depolarization might be expected because of variations in RM in the ISM. The relevant variations are over the apparent angular size, which along with the time delay and the pulse width may be

derived from a simple model for refractive scattering (Blandford & Narayan 1985; Rickett 1990). Gwinn, Bartel, & Cordes (1993) found that the angular size is reasonably well fitted by an estimate based on the temporal broadening,  $\tau$ , through  $\theta_\tau = 3.3(c\tau/L)^{1/2}$ . At 327 MHz angular sizes ranged from the detection limit ( $\sim 2$  mas) to  $\sim 65$  mas. Using equation (24) with  $\theta_B$  replaced by the apparent angular size,  $\lesssim 100$  mas, one concludes that depolarization for most pulsars should occur only at unobservably low frequencies, consistent with the observations.

One exceptional case where depolarization of pulsars might be observable is a region around a supernova remnant (SNR) where Alfvén waves are excited in association with shock acceleration of relativistic particles (Spangler et al. 1986). Gwinn et al. (1993) argued that such turbulence may account for strong scattering around young SNR. These authors noted that there are variations in DM with time up to 0.01–0.06 pc  $cm^{-3}$ . Assuming a magnetic field  $\sim 3$   $\mu G$ , this suggests the contribution to RM from the turbulent region around the SNR could be comparable to the total RM  $\sim 100$ – $1000$  rad  $m^{-2}$ . In a model for SNR with sharp edges, Achterberg, Blandford, & Reynolds (1994) suggested that the required Alfvén waves imply fluctuations in the magnetic field  $\delta B/B \sim 3 \times 10^{-3}$  on a scale  $\sim 10^{10}$  m, which is smaller than the scattering disc. This model implies fluctuations in RM across the scattering disc as large as  $\sim 1$  rad  $m^{-2}$ , which suggests that depolarization should occur at  $\lambda \gtrsim 1$  m.

A clear example where depolarization does occur is for the binary pulsar PSR B1259–63, which is eclipsed by the wind of its companion for several weeks near periastron. Large fluctuations in DM and RM were observed in the 1994 eclipse (Johnston et al. 1996), and, in the following eclipse in 1996, depolarization was observed over a few days (around day  $-80$ ) before the pulsar disappeared and for several days as it came out of eclipse (S. Johnston 1998, private communication). This depolarization was clearly associated with fluctuations in RM on a timescale of minutes. This system provides the possibility of testing the theory of stochastic Faraday rotation in detail. Multi-frequency observations of the polarization on a short timescale in the epochs where the variations in RM occur would provide a data set allowing the averages of  $Q$ ,  $U$ , and  $Q^2 + U^2$  to be performed over various timescales as the level of fluctuations in RM changes. It is hoped that such a data set will be recorded around the next periastron.

## 6. CONCLUSIONS

It has long been known that stochastic Faraday rotation can lead to depolarization (Burn 1966). Writing  $Q = Q_0 \cos \phi_F$  and  $U = Q_0 \sin \phi_F$  for the observed Stokes parameters in terms of  $Q_0$  at the source and assuming  $\phi_F = \phi_{F0} + \delta\phi_F$  with  $\delta\phi_F$  obeying Gaussian statistics, implies  $\langle Q \rangle = Q_0 \cos \phi_{F0} \exp(-\delta_i)$  and  $\langle U \rangle = Q_0 \sin \phi_{F0} \exp(-\delta_i)$ . It is pointed out in the present paper that this depolarization does not affect  $\langle Q^2 + U^2 \rangle$ , which should remain constant while  $\langle Q \rangle^2 + \langle U \rangle^2$  decreases. This effect, which is referred to as polarization covariance, was first identified within the framework of a more general theory for scintillations in an anisotropic medium (paper B), and stochastic Faraday rotation provides an obvious interpretation of it. The interpretation is that depolarization attributable to stochastic Faraday rotation is due simply to random rotation of the Stokes vector. In principle, whether depolarization is due to

stochastic Faraday rotation or to some other effect may be tested by recording data on  $Q$  and  $U$  on the shortest spatial scales and timescales available and calculating  $Q^2 + U^2$  and  $\phi_F$  from these data. In stochastic Faraday rotation,  $\phi_F$  varies randomly while  $Q^2 + U^2$  remains constant. The variance in the rotation measure is proportional to  $\langle \phi_F^2 \rangle$ , but the same information is provided by comparison of  $\langle Q \rangle^2 + \langle U \rangle^2$  and  $\langle Q^2 + U^2 \rangle$ . The latter quantities could be recorded automatically on the shortest timescale available on a telescope by modifying the software in polarization measurements to record the value of  $Q^2 + U^2$ , as well as the values of  $Q$  and  $U$ , so that the integrating over the time during the observation gives  $\langle Q^2 + U^2 \rangle$ ,  $\langle Q \rangle$ , and  $\langle U \rangle$  directly.

In principle, polarization covariance provides a definitive test for depolarization due to stochastic Faraday rotation, and the conditions under which this test should apply are discussed in § 3. There are at least five relevant averages:

1. Beamwidth averaging: the average is over the beamwidth of the observation. If the source is larger than the scattering disc but smaller than the telescope beam, this average is over the unresolved source.
2. Averaging of a scattering-broadened image: the average is over the scattering disc.
3. Bandwidth averaging: the average is with respect to frequency over the bandwidth of the observation.
4. Spatial averaging: the average is with respect to position over a polarization map.

5. Temporal averaging: the average is over a temporal sequence of observations.

Of these five, the first three are intrinsic to the collection of the data, and polarization covariance does not apply to them. However, for spatial and temporal averages, stochastic Faraday rotation leads to a depolarization in which  $\langle Q \rangle^2 + \langle U \rangle^2$  decreases and  $\langle Q^2 + U^2 \rangle$  remains constant.

Any observation of such depolarization provides information on the statistical properties of fluctuations in the magnetized plasma. Specifically, the wavelength at which the depolarization sets in may be used to define an LDM (see eq. [23]), which provides a measure of the variance in RM and hence on the fluctuations in  $(n_e B_{\parallel})^2$  along the ray path. Available data on the variation in RM in the ISM suggest that  $\langle B_{\parallel}^2 \rangle / \langle B_{\parallel} \rangle^2$  is slightly greater than unity (Simonetti et al. 1984; Simonetti & Cordes 1986, 1988).

The definitive test for depolarization due to stochastic Faraday rotation has yet to be applied to any specific data set. Possible systems for which depolarization is plausibly due to stochastic Faraday rotation are discussed in § 5. Polarization maps of the lobes of radio galaxies could be used to test the theory for spatial averaging. The next eclipse of the binary pulsar PSR B1259–63 should provide an almost ideal test of the theory for temporal averaging.

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## APPENDIX A

The theory of strong scintillations (e.g., Ishimaru 1978) may be developed in terms of equations for the moments of the wave amplitude. The wave amplitude,  $u(z, r)$ , is described in the parabolic approximation, with the paraxial direction along the  $z$  axis and with  $r$  the two-dimensional vector perpendicular to this direction. The equation for the  $n$ th-order moment involves a first derivative with respect to  $z$  and second derivatives with respect to the  $n$   $r$ s. The statistical average over the fluctuations is based on an ensemble average assuming Gaussian statistics. The generalization to an anisotropic medium was made by Kukushkin & Ol'yak (1990, 1991) and in paper A and paper B. In a simple plasma in which the modes are circularly polarized, the dielectric tensor,  $K_{ij}$ , may be separated into an isotropic part,  $K_I$ , and a part,  $K_V$ , associated with the anisotropy. Three correlation functions are needed to describe the fluctuations,  $\delta K_I$  and  $\delta K_V$ : these are written  $A_{XY}(z, r) = A_{YX}(z, r)$ , which are proportional to  $\langle \delta K_X(z, r) \delta K_Y(z, 0) \rangle$ , with  $X, Y = I, V$ . Alternatively, the fluctuations may be described in terms of the phase structure function,  $D(z, r) = \langle [\phi(z, r) - \phi(z, 0)]^2 \rangle$ , with  $D(z, r) = 2 \Delta z [A(z, 0) - A(z, r)]$  for a “thin screen” of thickness  $\Delta z$ . The generalization to an anisotropic medium involves three such functions,  $D_{XY}(z, r)$ , constructed from the three  $A_{XY}(z, r)$ .

When considering the effects of Faraday rotation on the correlation functions of  $Q$  and  $U$ , one is interested in the case where all the  $r$ s are zero. In paper A it was shown that the nontrivial second-order moments then give

$$\frac{\partial}{\partial z} \begin{pmatrix} \langle \tilde{Q} \rangle \\ \langle \tilde{U} \rangle \end{pmatrix} + \frac{\omega^2}{2c^2} A_{VV} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \langle \tilde{Q} \rangle \\ \langle \tilde{U} \rangle \end{pmatrix} = 0. \quad (25)$$

Integration of equation (25) over a slab of thickness  $\Delta z$  reproduces equation (9) with

$$2\delta_l = \frac{\omega^2}{c^2} A_{VV} \Delta z. \quad (26)$$

The fourth-order moments include the correlation functions between the Stokes parameters. In paper B it was shown that, when all the  $r$ s are zero, the nontrivial fourth-order moment equations then reduce to

$$\frac{\partial}{\partial z} \begin{pmatrix} \langle \tilde{Q}\tilde{Q} \rangle \\ \langle \tilde{U}\tilde{U} \rangle \\ \langle \tilde{Q}\tilde{U} \rangle \end{pmatrix} + \frac{\omega^2}{c^2} A_{VV} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \langle \tilde{Q}\tilde{Q} \rangle \\ \langle \tilde{U}\tilde{U} \rangle \\ \langle \tilde{Q}\tilde{U} \rangle \end{pmatrix} = 0, \quad (27)$$

where the tildes have the same meaning as in § 2, and where  $A_{VV} = A_{VV}(z, 0)$ . Integration of equation (25) over a slab of thickness  $\Delta z$  reproduces equation (12) with  $\delta_l$  given by equation (26). This formally justifies the treatment of the Faraday angle  $\phi_F$  as a random variable in § 2 in discussing the correlation functions between the Stokes parameters.

## REFERENCES

- Achterberg, A., Blandford, R. D., & Reynolds, S. P. 1994, *A&A*, 281, 220  
Blandford, R., & Narayan, R. 1985, *MNRAS*, 213, 591  
Burn, B. J. 1966, *MNRAS*, 113, 67  
Cordes, J. M., & Lazio, T. J. 1991, *ApJ*, 376, 123  
Cordes, J. M., Weisberg, J., Frail, D. A., Spangler, S. R., & Ryan, M. 1991, *Nature*, 354, 121  
Erukhimov, L. M., & Kirsh, P. I. 1973, *Izv. Vyssh. Uchebn. Zaved. Radiofiz.*, 16, 1783  
Garrington, S. T., Leahy, J. P., Conway, R. G., & Laing, R. A. 1988, *Nature*, 331, 149  
Gray, A. D. 1993, Ph.D. thesis, Univ. Sydney  
Gray, A. D., Cram, L. E., Ekers, R. D., & Goss, W. M. 1991, *Nature*, 353, 237  
Gwinn, C. R., Bartel, N., & Cordes, J. M. 1993, *ApJ*, 410, 673  
Ishimaru, A. 1978, *Wave Propagation and Scattering in Random Media*, Vol. 2 (New York: Acad. Press)  
Johnston, S., Manchester, R. N., Lyne, A. G., D'Amico, N., Bailes, M., Gaensler, B. M., & Nicastro, L. 1996, *MNRAS*, 279, 1026  
Kukushkin, A. V., & Ol'yak, M. R. 1990, *Radiophys. Quantum Electron.*, 33, 1002  
———. 1991, *Radiophys. Quantum Electron.*, 34, 612  
Laing, R. A. 1988, *Nature*, 331, 147  
Lazio, T. J., Spangler, S. R., & Cordes, J. M. 1990, *ApJ*, 363, 515  
Lee, L. C., & Jokipii, J. R. 1975, *ApJ*, 196, 695  
Lyne, A. G., & Manchester, R. N. 1988, *MNRAS*, 234, 477  
Melrose, D. B. 1993a, *J. Plasma Phys.*, 50, 267  
———. 1993b, *J. Plasma Phys.*, 50, 283  
Narayan, R. 1992, *Philos. Trans. R. Soc. London, A*, 341, 151  
Nicholls, J., & Gray, A. D. 1992, *Proc. Astron. Soc. Australia*, 10, 233  
Owen, F. N., Eilek, J. A., & Keel, W. C. 1990, *ApJ*, 362, 449  
Rickett, B. J. 1990, *ARA&A*, 28, 561  
Rohlf, K., & Wilson, T. L. 1996, *Tools of Radio Astronomy* (Berlin: Springer)  
Simonetti, J. H., & Cordes, J. M. 1986, *ApJ*, 310, 160  
———. 1988, in *AIP Conf. Proc. 174, Radio Wave Scattering in the Interstellar Medium*, ed. J. M. Cordes, B. J. Rickett, & D. C. Backer (New York: AIP), 134  
Simonetti, J. H., Cordes, J. M., & Spangler, S. R. 1984, *ApJ*, 284, 126  
Spangler, S. R. 1982, *ApJ*, 261, 310  
Spangler, S. R., Mutel, R. L., Benson, J. M., & Cordes, J. M. 1986, *ApJ*, 301, 312  
Strom, R. G., & Jägers, W. J. 1988, *A&A*, 194, 79  
Tamoikin, V. V., & Zamek, I. G. 1974, *Izv. Vyssh. Uchebn. Zaved. Radiofiz.*, 17, 31  
Taylor, J. H., & Cordes, J. M. 1993, *ApJ*, 320, L35  
Tribble, P. C. 1991, *MNRAS*, 250, 726