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Generalized Trubnikov functions for unmagnetized plasmas

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Abstract. A class of relativistic dispersion functions for unmagnetized thermal plasmas is defined by generalizing functions first defined by Trubnikov in 1958. Recursion relations are derived that allow one to generate explicit expressions for the class of functions in terms of the relativistic plasma dispersion function $T(z, \rho)$ introduced by Godfrey et al. in 1975. These functions are relevant to the description of the response of a weakly mangetized, highly relativistic, thermal plasma.

1. Introduction

In a covariant generalization of Trubnikov's (1958) derivation of the linear response tensor for an unmagnetized, relativistic, thermal electron gas, Melrose (1997a) found that, in order for the charge-continuity and gauge-invariance conditions to be satisfied, the functions defined by Trubnikov must satisfy certain identities. These identities allow one to derive a variety of superficially different forms for the response tensor, in both the isotropic and anisotropic cases (Melrose 1997b). There are several different methods for calculating the response tensor for an unmagnetized, relativistic thermal electron gas, and these lead to a variety of different relativistic plasma dispersion functions (RPDFs). These different forms imply inter-relations between the different RPDFs, and such relations can be inferred by comparing Trubnikov's (1958) results with the results of other methods (cf. Melrose 1997a).

In this paper a general class of Trubnikov functions is defined and evaluated. In particular, recursion relations are derived and used to construct explicit expressions for these functions in terms of the RPDF $T(z, \rho)$ introduced by Godfrey et al. (1975). The motivation for this investigation is twofold. First, the equivalences derived here have hitherto been inferred only indirectly in special cases from known results in the literature, and it is clearly desirable to identify a systematic procedure for establishing the equivalences directly. Secondly, the Trubnikov functions are particularly important for a magnetized, relativistic, thermal electron gas, where they provide a starting point for detailed calculations, specifically in the mildly relativistic limit (see e.g. Shkarofsky 1966). Trubnikov's result may also be used to treat the weakly magnetized case (for arbitrarily temperature) by expanding in powers of the magnetic field. However, the usefulness of this procedure is severely limited by the fact that the additional Trubnikov functions that arise in this expansion do not have known equivalences. The functions defined and evaluated here include the additional Trubnikov functions that arise in this way.

2. Generalized Trubnikov functions

A class of generalized Trubnikov functions is defined by writing

$$t_\nu^n(z,\rho) = (k\tilde{u})^{n+1} \int_0^\infty d\xi\, \xi^n \, \frac{K_\nu\big(r(\xi)\big)}{r^\nu(\xi)}, \tag{2.1}$$

with ν and n non-negative integers, and where K_ν is a Macdonald function of order ν with argument

$$r(\xi) = \left[\rho^2 - 2i\rho\, k\tilde{u}\, \xi + \frac{1-z^2}{z^2}\,(k\tilde{u})^2\xi^2\right]^{1/2}. \tag{2.2}$$

The covariant notation of Melrose (1997a,b) is used: ρ is the inverse temperature in units of the rest energy; in the rest frame of the plasma, $k\tilde{u} = \omega$ is the frequency and $z = \omega/|\mathbf{k}|$ is the phase speed.

Identities may be derived by partially integrating, by differentiating with respect to ρ and by differentiating with respect to z; these involve differentiating $r(\xi)$ with respect to ξ, ρ and z respectively. One also uses the identity

$$\frac{1}{r}\,\frac{d}{dr}\left(\frac{K_\nu(r)}{r^\nu}\right) = -\frac{K_{\nu+1}(r)}{r^{\nu+1}}. \tag{2.3}$$

Partially integrating in (2.1) leads to the identity

$$t_{\nu+1}^{n+1}(z,\rho) = \frac{i\rho z^2}{1-z^2}\, t_{\nu+1}^n(z,\rho) + \frac{z^2}{1-z^2} \begin{cases} \dfrac{K_\nu(\rho)}{\rho^\nu} & \text{for } n = 0, \\ n t_\nu^{n-1}(z,\rho) & \text{for } n > 0. \end{cases} \tag{2.4}$$

Differentiating (2.1) with respect to ρ leads to

$$t_{\nu+1}^{n+1}(z,\rho) = -i\rho\, t_{\nu+1}^n(z,\rho) - i\frac{\partial t_\nu^n(z,\rho)}{\partial\rho}. \tag{2.5}$$

Equations (2.4) and (2.5) may be combined to give an alternative identity

$$t_{\nu+1}^{n}(z,\rho) = -\frac{1-z^2}{\rho}\,\frac{\partial t_\nu^n(z,\rho)}{\partial\rho} + \frac{iz^2}{\rho} \begin{cases} \dfrac{K_\nu(\rho)}{\rho^\nu} & \text{for } n = 0, \\ n t_\nu^{n-1}(z,\rho) & \text{for } n > 0, \end{cases} \tag{2.6}$$

Equations (2.4)–(2.6) are useful in constructing all the functions in terms of any given one.

A separate identity follows by differentiating (2.1) with respect to z:

$$t_{\nu+1}^{n+2}(z,\rho) = z^3\,\frac{\partial t_\nu^n(z,\rho)}{\partial z}. \tag{2.7}$$

This identity is not found particularly useful in the following discussion.

The generalized Trubnikov functions are related to other RPDFs. A convenient choice for RPDF is that introduced by Godfrey et al. (1975):

$$T(z,\rho) = \int_{-1}^{1} dv\, \frac{e^{-\rho\gamma}}{v - z}. \tag{2.8}$$

Godfrey et al. wrote down a set of three partial differential equations satisfied by $T(z,\rho)$, and two of these are used here:

$$z\frac{\partial T(z,\rho)}{\partial\rho} = 2K_1(\rho) + \frac{(1-z^2)}{\rho}\,\frac{\partial T(z,\rho)}{\partial z}, \tag{2.9}$$

$$(1-z^2)\,\frac{\partial^2 T(z,\rho)}{\partial\rho^2} = 2zK_0(\rho) + T(z,\rho). \tag{2.10}$$

The problem addressed in the present paper is to express $t^n_\nu(z,\rho)$ in terms of $T(z,\rho)$ and $T'(z,\rho) = \partial T(z,\rho)/\partial z$.

3. Explicit expressions

To construct explicit expressions for the Trubnikov functions, one needs one independent calculation. The simplest is for a strictly parallel (one-dimensional) distribution (Melrose 1997b, Melrose et al. 1999). The simplest Trubnikov function arises from

$$\begin{aligned}\int \frac{d^4p\,F(p)}{ku} &= \frac{n}{K_1(\rho)}\int d^4p\,\frac{\delta(p^2-m^2)\,\delta^2(p_\perp)\,e^{-\rho(u\tilde{u})}}{ku}\\ &= \frac{n}{2K_1(\rho)}\int_{-\infty}^{\infty}\frac{d(\gamma v)}{\gamma}\,\frac{e^{-\rho\gamma}}{\gamma\omega(1-v/z)},\end{aligned} \tag{3.1}$$

where the integral is evaluated by choosing the rest frame, $\tilde{u} = [1,\mathbf{0}]$, writing $d^4p = d\varepsilon\, d^2\mathbf{p}_\perp\, dp_\parallel$ and $p^2 = \varepsilon^2 - \mathbf{p}_\perp^2 - p_\parallel^2$, where $\mathbf{p}_\perp$ and $p_\parallel$ are the components of the 3-momentum perpendicular and parallel respectively to the one-dimensional axis, and then writing $p_\parallel = m\gamma v$ and $ku = \omega\gamma(1-v/z)$. Trubnikov's procedure applied to this integral involves writing the resonant denominator as an exponential, using

$$\frac{1}{ku} = -i\int_0^\infty d\xi\, e^{iku\,\xi}, \tag{3.2}$$

then writing $\gamma = \cosh\chi$ and $v = \tanh\chi$, so that (3.1) becomes

$$\begin{aligned}\int \frac{d^4p\,F(p)}{ku} &= -\frac{in}{2K_1(\rho)}\int_0^\infty d\xi \int_{-\infty}^{\infty} d\chi\, e^{-[(\rho-i\omega)\cosh\chi + (i\omega\xi/z)\sinh\chi]}\\ &= -\frac{in}{2K_1(\rho)}\int_0^\infty d\xi\, K_0\big(r(\xi)\big).\end{aligned} \tag{3.3}$$

In the final step in (3.3), following Trubnikov, the exponent is written as

$$-r(\xi)\cosh(\xi+\phi), \qquad \text{with} \quad r^2(\xi) = (\rho-i\omega)^2 + \left(\frac{i\omega\xi}{z}\right)^2$$

and with ϕ the appropriate (complex) phase, and then the integral representation

$$K_0(r) = \int_0^\infty d\chi\, e^{-r\cosh\chi} = \frac{1}{2}\int_{-\infty}^{\infty} d\chi\, e^{-r\cosh\chi} \tag{3.4}$$

is used. Thus Trubnikov's procedure leads to an integral proportional to $t^0_0(z,\rho)$. The integral (3.1) may also be evaluated directly by writing $d(\gamma v) = dv\,\gamma^3$ and comparing the result with the ρ-derivative of (2.8). In this way, one finds

$$t^0_0(z,\rho) = \frac{iz}{2}\frac{\partial T(z,\rho)}{\partial\rho}. \tag{3.5}$$

Starting from (3.5), an arbitrary $t^n_\nu(z,\rho)$ may be generated using the recursion formulae.

The Trubnikov functions with $n = 0$ follow from (3.5) using (2.6). For $n = 1$, 2, 3, one finds

$$t_1^0(z,\rho) = -\frac{iz}{2\rho}\,T(z,\rho), \tag{3.6}$$

$$t_2^0(z,\rho) = \frac{i}{2\rho^3}\,\{2\rho K_1(\rho) - (1-z^2)[zT(z,\rho) - (1-z^2)T'(z,\rho)]\}, \tag{3.7}$$

$$t_3^0(z,\rho) = -\frac{iz}{2\rho^3}\,[-2zK_2(\rho) + (1-z^2)T(z,\rho)] \\ -\frac{3i(1-z^2)^2}{2\rho^5}\,\{-2\rho K_1(\rho) + (1-z^2)[zT(z,\rho) - (1-z^2)T'(z,\rho)]\}. \tag{3.8}$$

In constructing $t_{\nu+1}^0(z,\rho)$, one requires the ρ-derivative of $t_\nu^0(z,\rho)$, and because these derivatives are also required for $n > 0$, it is appropriate to note their explicit forms:

$$\frac{\partial t_1^0(z,\rho)}{\partial\rho} = \frac{iz}{2(1-z^2)}\,\big[2zK_0(\rho) + T(z,\rho)\big], \tag{3.9}$$

$$\frac{\partial t_1^0(z,\rho)}{\partial\rho} = \frac{i}{2\rho^2}\,[-2\rho K_1(\rho) + zT(z,\rho) - (1-z^2)T'(z,\rho)], \tag{3.10}$$

$$\frac{\partial t_2^0(z,\rho)}{\partial\rho} = \frac{izT(z,\rho)}{2\rho^2} \\ +\frac{3i(1-z^2)^2}{2\rho^4}\,\{-2\rho K_1(\rho) + (1-z^2)[zT(z,\rho) - (1-z^2)T'(z,\rho)]\}. \tag{3.11}$$

To construct the functions with $n = 1$, it is convenient to use (2.4):

$$t_{\nu+1}^1(z,\rho) = -i\rho t_{\nu+1}^0(z,\rho) - i\frac{\partial t_\nu^0(z,\rho)}{\partial\rho}. \tag{3.12}$$

To construct the functions with $n = 2$, it is convenient to use (2.4):

$$t_{\nu+1}^2(z,\rho) = \frac{i\rho z^2}{1-z^2}\,t_{\nu+1}^1(z,\rho) + \frac{z^2}{1-z^2}t_\nu^0(z,\rho), \tag{3.13}$$

with $t_{\nu+1}^1(z,\rho)$ given by (3.12).

By way of illustration, Melrose (1982) wrote down various explicit expressions for the longitudinal ($\alpha^L(k)$) and transverse ($\alpha^T(k)$) parts of the response tensor for an unmagnetized, isotropic, thermal electron gas. The unmagnetized limit of Trubnikov's (1958) result implies

$$\alpha^L(k) \propto T_2^0(z,\rho) - \frac{T_3^2(z,\rho)}{z^2}, \qquad \alpha^T(k) \propto T_2^0(z,\rho).$$

The equivalence of these expressions and the corresponding results derived by Godfrey et al. (1975) in terms of $T(z,\rho)$ and $T'(z,\rho)$ follow directly from the results derived above.

4. Discussion and conclusions

As mentioned in Sec. 1, the motivation for this investigation is twofold. The results presented here establish directly the inter-relation between the Trubnikov functions and other RPDFs for an unmagnetized plasma, and thus fills a long-standing gap in the literature.

The results obtained here are relevant to the treatment of a weakly magnetized, relativistic, thermal electron gas. Trubnikov's (1958) expression for the response tensor involves functions of the form (2.1) with

$$r(\xi) = \left[(\rho - i\omega\xi)^2 + k_\parallel^2\xi^2 + \frac{2k_\perp^2}{\Omega_e^2}\,(1 - \cos\Omega_e\xi)\right]^{1/2}, \tag{4.1}$$

with $\Omega_e = eB/m$. Trubnikov's tensor is used (see e.g. Shkarofsky 1966; Bornatici et al. 1983; Robinson 1986; Melrose 1997a) to treat the mildly relativistic ($\rho \gg 1$), small-gyroradius ($k_\perp^2 \ll \Omega_e^2\rho$) limit in which $r(\xi)$ is assumed large so that the Macdonald functions may be approximated by their asymptotic forms. The weakly magnetized limit is obtained by expanding in powers of Ω_e, both in (4.1) and elsewhere in the relevant integrands. To zeroth order, the known unmagnetized limit is reproduced, and the results obtained here allow one to express the magnetized corrections in terms of the functions $T(z,\rho)$ and $T'(z,\rho)$, whose properties are known in some detail (see e.g. Godfrey et al. 1975). Thus the results derived here are a necessary preliminary for the derivation of properties of a weakly magnetized thermal plasma for temperatures between the known mildly relativistic limit $\rho \gg 1$ (Shkarofsky 1966) and the known extremely relativistic limit $\rho \to 0$ (Melrose 1997c).

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