

University of Sydney
School of Physics

SUPPLEMENTARY NOTES
for
Senior Experimental Physics Handbook

These notes are for students enrolled in any of the following units of study. The credit point number indicates the number of credit points allocated to the Physics lab component of the unit:

Semester 1

- PHYS 3040, 3940** Electromagnetism & Lab (4 credit points)
- PHYS 3050, 3950** Nanoscience/Optics & Lab (2 credit points)
- PHYS 3051, 3951** Thermodynamics/Biol. Physics & Lab (2 credit points)
- PHYS 3052, 3952** Nanoscience/Thermodynamics & Lab (2 credit points)
- PHYS 3053, 3953** Thermodynamics/Optics & Lab (2 credit points)

Semester 2

- PHYS 3060, 3960** Quantum Mechanics & Lab (4 credit points)
- PHYS 3070, 3970** Plasma Physics/Cond. Matter & Lab (2 credit points)
- PHYS 3071, 3971** High Energy/Astrophysics & Lab (2 credit points)
- PHYS 3072, 3972** Plasma Physics/Astrophysics & Lab (2 credit points)
- PHYS 3073, 3973** Plasma Physics/High Energy & Lab (2 credit points)
- PHYS 3074, 3974** High Energy/Cond. Matter & Lab (2 credit points)
- PHYS 3075, 3975** Cond. Matter/Astrophysics & Lab (2 credit points)

Contents

1	THE WIB SMITH PRIZE FOR EXPERIMENTAL PHYSICS	4
2	LABORATORY RULES AND ADVICE	4
2.1	Laboratory equipment	5
2.2	Laboratory library	5
3	LABORATORY SAFETY	5
3.1	Fire: whole laboratory complex	5
3.2	Toxic Materials e.g. X-Ray Diffraction and Holography experiments	5
3.3	Microwave Radiation: Wave Propagation and Microwave experiments	5
3.4	Laser Radiation: Optical Images and Holography experiments	6
3.5	Ionising Radiation X-Ray Diffraction, Alpha Particles, Mossbauer, Cosmic Rays, Nuclear Lifetimes, Gamma Rays, Beta Particles and X-Rays experiments	6
4	GUIDELINES FOR RECORDING EXPERIMENTAL WORK	7
4.1	INTRODUCTION	7
4.2	GENERAL	7
4.3	LOGBOOKS	7
4.3.1	Description of Experimental Work	7
4.3.2	Records of Observations and Data	8
4.3.3	Tables	8
4.3.4	Graphs	8
4.3.5	Analysis of data	9
4.3.6	Results	9
4.3.7	Conclusions	9
5	REPORT WRITING	10
6	DATA ANALYSIS	13
6.1	Introduction	13
6.2	Confidence intervals	13
6.3	Systematic Errors	14
6.3.1	Sources of error - meter and oscilloscope errors	15
6.3.2	Combining systematic errors	16
6.4	Random errors	18
6.5	Probability distributions	19
6.5.1	The normal distribution	19
6.5.2	The Poisson distribution	20
6.6	The statistics of data samples	21
6.6.1	The sample mean	21
6.6.2	The sample standard deviation	21
6.6.3	Standard error of the mean	21
6.7	Confidence intervals and statistical tests	22
6.7.1	The confidence interval for the mean	22
6.7.2	The confidence interval for the standard deviation	24
6.7.3	Statistical test	24

6.7.4	Tests for the mean and standard deviation	25
6.7.5	Rejection of data	25
6.7.6	The χ^2 goodness-of-fit test	25
6.8	Combining statistical errors	28
6.9	Errors and graphs	29
6.9.1	The method of least squares	30
6.9.2	Confidence intervals for the slope and intercept	31
6.9.3	A word of warning	31
6.10	Slopes and intercepts	32
7	SOME USEFUL PHYSICAL CONSTANTS	34
8	SI PREFIXES	34

1 THE WIB SMITH PRIZE FOR EXPERIMENTAL PHYSICS

WIB Smith, known to all as WIBS, was a senior lecturer in Physics from 1963 to 1981 and was an exceptionally good tutor in the Senior Physics laboratory. He believed that (nearly) all physics could be described in a simple manner if some effort was put into it. He had a great talent for communicating with students in this way.

After WIBS' tragic death in 1983, his friends and colleagues contributed a substantial sum of money to establish a prize in his memory. It is awarded to the student who best combines the characteristics of experimental skill, proficiency and exceptional motivation in the Senior Physics laboratory, providing that person shows sufficient merit. The tutors become aware of a student's motivation during discussions about experiments, particularly of the physics involved in an experiment.

This prize is normally awarded to a student taking at least 8 credit points of experimental physics units in the course of the year.

2 LABORATORY RULES AND ADVICE

- (a) At the beginning of the semester you will be supplied with general introductory notes. Notes for each experiment will be supplied as required throughout the year. It is essential to read carefully the notes on an experiment before you start work - the night before is suggested. Do not expect every step in the procedure to be described. However, if you understand the physics in advance, your activities in the laboratory will run smoothly and efficiently.
- (b) No smoking, eating or drinking except in the area allocated for tea or coffee. Leave wet clothing etc. outside the laboratory.
- (c) Turn off the power from main sockets when you leave the laboratory at the end of an afternoon's work. You can leave plugs in plugboard sockets. Turn off all battery operated equipment.
- (d) It helps to set the apparatus out neatly.
- (e) Equipment is not to be moved from one bench to another. Please report any deficiencies to a staff member.
- (f) A reasonable standard of dress is expected; in particular, shoes must be worn. Although all the apparatus is safe, with any mains operated electrical equipment there could be an extremely small but finite risk of electric shock. Non-conducting shoes are a definite advantage in this respect. They also protect your toes in the unlikely event of a heavy piece of equipment being tipped off a bench! Handle the high voltage equipment with care. If a malfunction occurs do not remove protective covers, call a staff member.
- (g) A student who is sick may be allowed to skip an experiment if a medical certificate is provided within a reasonable time. Please check out individual cases with the laboratory supervisor. In such cases a *Request for Special Consideration* should be submitted.

- (h) Do not remove bench notes for an experiment from their proper place i.e. on the table for that experiment.

2.1 Laboratory equipment

Check that all the equipment for your experiment is available before you start (The notes of most experiments are headed by an equipment list).

There are a number of calculators in the laboratory that are available for use, but it would be clearly convenient to have your own.

There are many computers in the laboratory. Some of the laboratory's experiments have sections where the data is fed directly to one of the computers. In other experiments a computer program is needed to process manually entered data. Further information on the use of the computers is given in a bench file near the computers.

2.2 Laboratory library

A small number of books are kept in a cupboard near the entrance to the laboratory. Apart from the exception below, books may be borrowed overnight or over the weekend. Longer borrowing is not allowed - the books could be needed by other students. Enter your name, the date and book's author in the "library borrowing" pad located on the front bench. For obvious reasons we do not allow reference handbooks to leave the laboratory. The main physics library has copies of these handbooks for the use of students who wish to work up there.

3 LABORATORY SAFETY

The Hazards are:-

3.1 Fire: whole laboratory complex

The presence of fire will normally be indicated by automatic activation of an alarm. Leave the laboratory in an orderly but rapid manner, helping disabled colleagues if necessary. Assemble on Physics Road near the main door of the main physics building. Obey directions of fire wardens (persons wearing green plastic hats).

3.2 Toxic Materials e.g. X-Ray Diffraction and Holography experiments

Do not taste or ingest any samples used in the experiments. (None are really poisonous but the principle must be followed of course.)

3.3 Microwave Radiation: Wave Propagation and Microwave experiments

The power flux (watt per sq. metre) of this radiation is below the Australian standard limit of 0.2 mW per sq. cm (for the public, 24 hour exposure). Exposure to

radiation should be kept As Low As is Reasonably Achievable (the ALARA principle): do not sit or stand so your eyes are less than 30 cm from the transmitting antenna or the waveguide of the Wave Propagation experiment.

3.4 Laser Radiation: Optical Images and Holography experiments

Do not place eyes so that the beam can enter and possibly focus on the retina. Do not place specularly reflecting surfaces in the laser beam; the reflected radiation could enter someone's eye. The lasers give about 1 mW which means that diffusely reflected radiation is safe.

3.5 Ionising Radiation X-Ray Diffraction, Alpha Particles, Mossbauer, Cosmic Rays, Nuclear Lifetimes, Gamma Rays, Beta Particles and X-Rays experiments

The absorbed dose is measured in gray (J/Kg of irradiated tissue). Heavily ionising radiation (like alpha particles) causes much more radiation damage than gamma rays or beta particles. This is allowed for by using the dose equivalent measured in sievert. For betas and gammas the two measures are the same. All accessible alpha sources in the laboratory are very weak and there is no significant danger from them.

The maximum limit for the public is 5 millisievert in one year (for radiation workers it is 50 mSv in one year, but we use the more stringent limit here). This compares with the natural radiation background of 1 to 2 mSv/yr. Radiation protection agencies recommend that one attempts to hold the dose equivalent to be less than 1 mSv plus the background in one year and that the ALARA principle be followed (see note about microwave radiation above).

The 1 mSv plus background rule can easily be met in the laboratory by not handling the sources unnecessarily or for unnecessarily long times and by keeping the stronger sources in the shielding containers where provided. The strongest source in the laboratory is a 40 GBq Americium Beryllium neutron source in the red metal box located in the south east corner of the laboratory. Other strong sources are kept in a lead lined radioactive safe located below the red box. The 5 mSv/yr limit has been marked on the floor with a red band. Follow the ALARA principle: do not go into this area unless you need to. Note that a person spending 15 minutes on every day of the year, standing on the red band would receive 0.05 mSv in addition to the greater than 1 mSv background.

4 GUIDELINES FOR RECORDING EXPERIMENTAL WORK

4.1 INTRODUCTION

The written record is a vital part of experimental work and it is inevitable that the assessment of experimental work in Physics Laboratories is largely based on it. When reading your record the physics staff are trying to assess the quality of the practical work as well as that of the record itself.

The following guidelines set out what is expected in the written record of experimental work in physics.

4.2 GENERAL

- A written record should be made directly in a permanently bound LOGBOOK while an experiment is actually being carried out.
- If an experiment is written up after it has been completed, using the LOGBOOK, the LABORATORY NOTES (and other references if appropriate), the result is generally called a REPORT.
- In Junior, Intermediate, and Senior years your LOGBOOK will generally contain the required complete record of each experiment and you will only be expected to write REPORTS for a limited number of experiments. All of the following sections apply for LOGBOOKS in Junior, Intermediate, and Senior years.

4.3 LOGBOOKS

A LOGBOOK is a working record of experimental work and should contain sufficient information such that, at any time in the future, a REPORT (see Section 1.2) could be written. It should be possible for a 'scientifically literate person' to understand what was done by reading the logbook record in conjunction with the laboratory notes for the experiment.

4.3.1 Description of Experimental Work

- The date should be recorded at the beginning of each session in the laboratory.
- Each experimental topic should be given a descriptive heading and, where appropriate, the identifying number of the apparatus should be recorded.
- A brief introduction, description of apparatus and experimental procedures should be included. This should amount to only a few sentences. A reference can be given to the notes for more details.
- Schematic or block (rather than pictorial) diagrams should be included where appropriate.
- Circuit diagrams should be included.

4.3.2 Records of Observations and Data

- The LOGBOOK must contain the original record of all observations and data - including mistakes! Never erase or obliterate “incorrect” readings. Simply cross them out in such a way that they can be read if need be.
- Type and serial numbers of instruments used in addition to the standard apparatus should be recorded whenever there is the possibility that measurements may have to be repeated or the calibration checked.
- Zero readings, calibration factors, resolution limits and rated accuracies of instruments should be recorded whenever they are relevant.
- Particular points regarding your experimental technique such as, for example, precautions taken to avoid backlash or the effects of temperature changes should be noted.
- Other factors such as temperature and pressure should be recorded if there is a possibility that they may affect the observational data or the outcome of the experiment.
- All relevant non-numerical observations should be clearly described. A sketch should be used whenever it would aid the description.
- Each reading should be identified by name or defined symbol together with its numerical value and unit.

4.3.3 Tables

- Observations and data should be tabulated whenever appropriate.
- Each table should have an identifying caption.
- Columns in tables should be labelled with the names or symbols for both the variable and the units in which it is measured. If a symbol is used it should be defined.

4.3.4 Graphs

- In some experiments you will get a graphical printout of data from the computer used to run the experiment. While tabulated data can be graphed by hand, it is preferable to plot data using *Excel* or preferably *Origin*. Both applications are available on the computers at the front of the laboratory. Graphs should be securely fastened in the LOGBOOK.
- Each graph should have a descriptive caption, which refers to any relevant distinguishing parameters, and/or conditions, which do not appear in the graph itself.
- Axes of graphs should be labelled with the names or symbols for the quantity and its unit. Numerical values should be written along each axis. Scales should be chosen which make it easy for intermediate values to be read quantitatively.

- Data points on graphs should be clearly identified. If more than one set of points is plotted each set should be plotted using a different symbol. Error bars should be drawn wherever appropriate to indicate the uncertainty in each data point except in the case of a large number of data points when an indication of typical error bars is acceptable.
- Construction points for theoretical curves should be erased or distinguished from experimental points.
- Each trend line should be drawn smoothly without sharp discontinuities.

4.3.5 Analysis of data

- Calculations for each section of an experiment should be done as soon as relevant measurements have been completed so that it can be seen whether some measurements should be repeated.
- The organisation of calculations should be sufficiently clear for mistakes (if any) to be easily found.
- In cases where intermediate results of calculations need to be written down they should be recorded in the LOGBOOK. Repeated calculations of the same kind should be tabulated.
- The procedures used for estimating and combining uncertainty estimates should be clearly specified. Non-standard procedures should be described in detail.

4.3.6 Results

- Results should be given with an estimate of their uncertainty wherever possible. The type or nature of each uncertainty estimate should be specified unambiguously.
- Results should be compared whenever possible with accepted values or with theoretical predictions.
- Serious discrepancies in results should be examined and every effort made to locate the reason.

4.3.7 Conclusions

- A conclusion should be written for every topic or experiment.
- The conclusion should make sense when read in isolation from the rest of the record.
- The conclusion should be consistent with the results and contain a comparison with accepted values or with theoretical predictions (see Section 5.2). Any discrepancies should be discussed.
- If there are specific comments to be made about the experiment or apparatus it is appropriate to include them in the Conclusions.

5 REPORT WRITING

A Report should be written in the style of a scientific paper and should not be simply a tidy version of your logbook. It should be a self-contained document understandable to a scientifically literate person not in possession of the laboratory manual.

In preparing a report particular attention should be paid to the following points.

DO's

- You should divide your report into sections using headings such as INTRODUCTION, THEORY, EXPERIMENTAL PROCEDURE, RESULTS, DISCUSSION, CONCLUSION, REFERENCES. This will help you to organise your material and also help the reader. If an experiment has several distinct parts, it is preferable to identify each part by a heading and include the results, discussion and perhaps specific experimental procedure relevant to that part under this heading, rather than lumping all of the results together and all of the discussion.
- Write in clear, concise English, presented coherently and logically. Explain carefully what you did, why you did it, what you found out, and what your conclusions are.
- The INTRODUCTION should be brief but should include the aim or purpose of the experiment. It must relate directly to the subject of the report and set the scene.
- The EXPERIMENTAL PROCEDURE section must contain enough specific detail for the reader to assess your experimental technique; it may well be crucial for your later analysis. The RESULTS section should not just list or tabulate the experimental data without any words of explanation. Use complete English sentences to describe your results, e.g. "The Hall voltage for the p -type semiconductor was measured as a function of B at $I = 30$ mA. The results are shown in Fig. 3."
- The DISCUSSION section should be one of the main sections of the report, dealing with the significance of your results. It can include, if appropriate, discussion of:
 - agreement or otherwise between your results and theory or accepted results.
 - whether your procedures or techniques were adequate (or not), and/or how they might be improved.
 - the historical or technical significance of the experiment (but do not waffle - one or two sentences is usually sufficient).
 - the main sources of error, although it is often more appropriate to discuss errors when they are first mentioned in the PROCEDURE or RESULTS sections.

- an explanation or hypothesis regarding any obvious discrepancy. It is not acceptable to make wild guesses like “... the difference was probably caused by the humidity”. If you suspect some possible source of error, explain or estimate how it would affect your results (it might make them worse!).
- Your report must have a CONCLUSION. State briefly the main results and indicate the extent to which you achieved the purpose of the experiment as described in your INTRODUCTION.
- Though the report should be self-contained you should not reproduce large slabs of the laboratory manual in your report. Relevant information should be summarised in a few sentences. This applies particularly to the THEORY section. The manual should be given as one of your REFERENCES for further details.
- Define all symbols.
- Include circuit diagrams and Figure captions, e.g. “Fig.3. Temperature T vs time t at $x = 0.5$ m”. Refer to each Figure by its number in the text of your report.
- Describe any limitations of your experimental technique. For example, you may have forgotten to check for a zero error ... be honest!

DON'Ts

- Do not exceed the suggested length.
- Do not derive formulae and do not include intermediate steps in calculations; just include the main results. Important equations should be given a number in brackets at the end of the line, and referred to in the text by its number.

$$y = ax + bx + c \tag{3}$$

- Do not use abbreviated English.
- Do not give tables of results if a graph is more appropriate. Give one or the other but not both.
- Do not give instructions to the reader like “Measure V at $B = 0.5$ T”. Say “ V was measured at $B = 0.5$ T”.
- Do not write rubbish like “the results were pretty good considering the apparatus we had to use”. There is no such thing as “perfect” apparatus. An instrument with an error of 10% might be perfectly adequate for the purpose of the experiment: for example, your result may be $2.5 \pm 10\%$ and theory says it should be 1.5. Most students would then conclude that the apparatus was at fault. It is equally valid to conclude that the theory is at fault. You must distinguish carefully between the two alternatives:

(a) is the aim to see whether the theory fits your experimental results, or

(b) is the aim to see if your results support the theory ?

Note that there are always certain assumptions in theories, such as “ignoring friction” etc, which cannot be met experimentally. If you get a result, which is close to but does not agree with theory within the experimental errors, then you should indicate possible reasons for the discrepancy and suggest whether it might be possible to include this effect in the theory or eliminate it from the experiment.

6 DATA ANALYSIS

6.1 Introduction

When you do an experiment in the laboratory, making observations and gathering data is only half of the job. The proper analysis of your results is equally important. In analysing your data you should try to achieve two aims:

- To present your results in a convenient and simple way, so that others - and you in three months time! - can quickly understand what you did.
- To assess your confidence in the results by analysing the errors which are present and determining how they affect your conclusions.

By now you should be familiar with the basic skills of tabulating and graphing data, determining slopes, etc., so we will be chiefly concerned with the second of these two points - error analysis. It should be emphasised that there is no magic formula which can always be used, but it is difficult to proceed without some knowledge of statistical methods, and when they may and may not be used. The purpose of these notes is to introduce some of the formal techniques for dealing with experimental data while still retaining an overall emphasis on commonsense and logic as the principal ingredients of “good” data analysis.

You should regard data analysis as an integral part of your experimental work, not something that is tacked on at the end. Good experimental scientists always analyse their results as they go along. There are many reasons for this, but perhaps the most important are:

- checking your work as you progress through an experiment is an excellent way to detect and correct mistakes at an early stage, thereby minimising time lost in repeating work; and
- it is easy to overlook some vital piece of information (e.g. a voltage setting, the range multiplier switch on a digital multimeter, etc.) and by working up your data immediately, while the apparatus is still set up, you can catch these oversights and record the necessary information.

6.2 Confidence intervals

We express our confidence in results by using “confidence intervals. For example, an energy difference might be given as $(2.57 \pm 0.14) \times 10^{-3}$ J.

The range 2.43 mJ to 2.71 mJ is the confidence interval for this quantity; the limits ± 0.14 mJ are often called the confidence limits. What this means is that the experimenter is reasonably sure that the “true” value of the energy difference lies within the stated range.

Confidence intervals go by many names: limits of accuracy, tolerance bands, error limits, etc., are all used. In any case, the general idea is always the same. The interval represents some kind of estimate of the confidence that the true value lies within the indicated range.

There are two pitfalls to be avoided. The optimistic physicist implicitly trusts the equipment and always quotes small confidence limits, and is also invariably

wrong. The pessimist takes the opposite view, giving 3 ± 3 mJ as the answer to our example. These limits are so large as to be useless for most purposes. It is therefore important to know how best to estimate the errors in your results. In the following pages we hope to make this clearer.

6.3 Systematic Errors

There are two kinds of errors - random and systematic.

- Random errors appear as fluctuations or “noise” in a series of measurements and will be considered in detail in section 3.
- Systematic errors, on the other hand, result in observations being consistently higher or lower than the “actual” value. They can be detected only by making measurements under different conditions (e.g. using a different set of apparatus or having another person repeat the observations).

“Accuracy” is the term we use to describe how well observations agree with the “actual” value. A good scientist always strives for the best accuracy possible, and if a discrepancy occurs, he will attempt to find the cause and eliminate it. Students often think that physicists are paranoid about accuracy and data analysis, but a few examples will illustrate the importance of accuracy in all the physical sciences.

- A student builds a simple audio oscillator, but is dismayed to find that the actual frequency of the circuit differs by 20% from the calculated value.
- A forensic chemist is called upon to give evidence about blood samples in court. The outcome of the case will depend on the accuracy of the results.
- The planets move about the Sun according to Kepler’s laws. However, each planet affects the orbits of the others very slightly; this is called “perturbation”. In the 19th Century, Adams and LeVerrier found that there were discrepancies in the planetary positions even when all the known perturbations were taken into account. They inferred the existence of a new planet, which was soon discovered (Neptune).

The problem in the first example is that the student didn’t appreciate the tolerances in electronic components. To get the desired results there are several possibilities:

- (a) Hand-select components with the “exact” values called for. This of course requires sufficiently accurate test equipment.
- (b) Use components with tighter tolerances.
- (c) Re-design the circuit so that it is less sensitive to the values of critical components.
- (d) Include an adjustable component (potentiometer, etc.) and calibrate the circuit.

Note that calibration is an important and useful way of reducing systematic errors. We check the equipment using instruments of known (high) accuracy, or use the equipment to measure “known” quantities. The systematic errors can be then corrected by a suitable adjustment (often it suffices to note the error and use a calibration factor rather than attempt to adjust the equipment). For work of the highest accuracy, reference sources of voltage, current, etc., are available which have calibration certificates tracing their accuracy back to the fundamental standards maintained by the national standards authorities.

The student’s situation is not much different from that of a production engineer who has to meet a product specification at the lowest cost. In this case a very careful analysis of the accuracy of supplied components is essential. The same options listed above are available, but cost must also be considered: (a) is usually the most expensive and (d) the least.

The second example illustrates an important aspect of accuracy: in most cases the “true” answer is not known and the scientist must determine it as best he or she can. In order to establish the reliability of the work, all the calibration procedures etc. must be carefully recorded in logbooks.

In the absence of properly documented measurements it becomes very difficult to assess the quality of any experimental work. Of course, in Senior Physics lab you will normally be doing experiments which have known results, but this is not a reason to be sloppy. The laboratory work should encourage you to develop good professional habits.

Finally the third example illustrates the most interesting feature of “systematic errors”. The “error” in this case is actually a new phenomenon. Many discoveries have been made by careful observations that reveal systematic discrepancies with theory.

6.3.1 Sources of error - meter and oscilloscope errors

It is impossible to give a complete list of all the possible sources of systematic errors that you might encounter, but one of the most common is the humble moving-coil voltmeter or ammeter.

Any voltmeter when calibrated against an accurately known voltage source will have a calibration curve similar to that shown in figure 1(a) or (b).

Sometimes a meter will read consistently high or low, but some meters will be high over part of the range and low over the other part (a comment applicable to all measuring instruments). All instruments exhibit non-linear behaviour to a certain extent, which means that the calibration curve (i.e. measured value vs actual value) is not a straight line. There are two ways of specifying the accuracy of a meter. In Fig. 1(a) we see that the meter reading is accurate to within $\pm 2\%$ of the full scale reading (i.e. within ± 0.2 V) over the entire range. Fig. 1(b) shows a more accurate meter in which the meter reading lies within $\pm 2\%$ of the “actual” value, regardless of the magnitude of the meter reading.

A reading of say 5.0 volts on meter (a) should be quoted as 5.0 ± 0.2 or on meter (b) as 5.0 ± 0.1 . Most of the moving coil meters in the Senior Physics laboratory are of type (a) ($\pm 2\%$ of full scale reading) and the oscilloscopes are of type (b) - i.e. $\pm 2\%$ of reading for both voltage (vertical deflection) and time (horizontal deflection) readings. These errors are usually larger than errors caused by interpolating scale

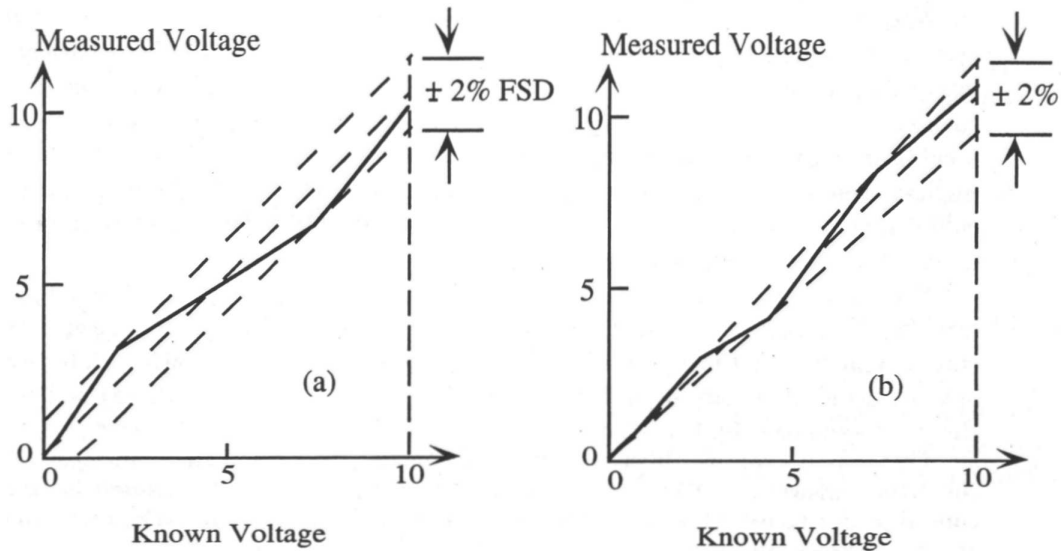


Figure 1: Typical meter calibration curves

graduations or errors due to the thickness of a meter needle or oscilloscope trace. These latter errors can therefore usually be neglected, but if not they should be added to the 2% calibration error.

A very common source of systematic error is the “zero error” of a meter. If the meter does not read zero when it is disconnected then we can either (a) subtract the “zero” reading from other readings or preferably (b) set the meter to read zero by means of a “zero adjust” knob or screw.

6.3.2 Combining systematic errors

Errors of addition and subtraction

If we measure two quantities $x \pm \Delta x$ and $y \pm \Delta y$, where Δx and Δy are our estimates of the systematic errors in x and y , then the sum of x and y may lie anywhere in the range $x + y + \Delta x + \Delta y$ to $x + y - \Delta x - \Delta y$. The maximum (worst possible) error in $f = x + y$ is therefore $\Delta f = \pm(\Delta x + \Delta y)$. Similarly if $\Delta f = x - y$, the maximum error in f is also $\Delta f = \pm(\Delta x + \Delta y)$.

This type of error analysis can be applied for example when we obtain the difference between two separate voltage measurements. A problem arises, however, when a small voltage difference is being measured. The voltage drop across a diode, for example, may rise from 0.60V to 0.61V when the current increases by a factor of 10. The values 0.60 and 0.61 may each be in error by ± 0.05 . The difference is not 0.01 ± 0.1 because the meter will be in error by practically the same amount for each reading - i.e. both values will be too high or too low. The error in each value is unimportant and the difference is $0.01 \pm$ the error in interpolating scale graduations.

Combining percentage errors

If we measure the voltage $V \pm \Delta V$ across a resistor and the current $I \pm \Delta I$ through the resistor, then we can obtain an estimate for the resistance as

$$\frac{V \pm \Delta V}{I \pm \Delta I} = R \pm \Delta R$$

To find R , we take the worst case with $V + \Delta V$ and $I - \Delta I$ where (expanding the denominator, and ignoring terms of order ΔI^2 and higher)

$$\frac{V + \Delta V}{I - \Delta I} \equiv \frac{V}{I} + \frac{V\Delta I}{I^2} + \frac{\Delta V}{I}$$

and therefore

$$R + \Delta R = \frac{V}{I} \left(1 + \frac{\Delta I}{I} + \frac{\Delta V}{V} \right)$$

Since $R = V/I$ we obtain

$$\frac{\Delta R}{R} = \frac{\Delta V}{V} + \frac{\Delta I}{I}$$

The fractional error in R is therefore equal to the sum of the fractional errors in V and I or, alternatively, the percentage error in R (i.e. $100(\Delta R/R)\%$) is therefore equal to the sum of the percentage errors in V and I . In general, the actual error is over-estimated by this procedure, since worst case errors rarely occur in practice. A less pessimistic method of combining errors is outlined in section 7.

Exercises

1. Show that the percentage errors in x and y must be added to obtain the percentage error in xy .
2. If $x = 2.0 \pm 0.1$ and $y = 4.0 \pm 0.4$, use the rules for adding maximum or percentage errors to determine the errors in (a) $x+y$; (b) $3x-y$; (c) x^2/y ; (d) $(x-1)/y$.

[Ans: (a) 6.0 ± 0.5 ; (b) 2.0 ± 0.7 ; (c) 1.0 ± 0.2 ; (d) 0.25 ± 0.05]

3. By differentiating the function $f = x^n$, find an expression for the percentage error in f in terms of the percentage error in x . Hence determine the maximum error in $f = \sqrt{2xy}$ where $x = 2.0 \pm 0.1$ and $y = 4.0 \pm 0.4$.

[Ans: 4.0 ± 0.3].

Significant figures

When quoting the error in a quantity x or a function of several quantities such as $2xy$, we usually round off the error to one, or at most two, significant figures e.g. $\pm 5\%$ (not $\pm 5.32\%$) or $\pm 2.3 \times 10^{-3}$ (not $\pm 2.312 \times 10^{-3}$). We do this because error estimates themselves are never determined accurately. The number of figures quoted for a quantity should then be consistent with the quoted error. For example, if we found that $f = 2xy = 1.5340621$ using a pocket calculator and the experimental error in f was ± 0.05 , then we should quote f as 1.53 ± 0.05 .

Exercise

If $x = 1$ and $y = 3$, quote a value for $f = x/y$ if the percentage errors in x and y are (a) 0.1% (b) 0.5% (c) 10%.

[Ans: (a) 0.3333 ± 0.0007 (b) 0.333 ± 0.003 (c) 0.33 ± 0.07]

6.4 Random errors

If you measure it once you know what you've got. If you measure it again and don't get the same answer, you don't know where you are.
- Failed Intermediate Physics student

Random errors ultimately limit any physical measurement (indeed this follows from quantum mechanics), but unlike "systematic" errors it is possible to treat random errors by the techniques of mathematical statistics. In the following we review the basic statistical methods that you will need to know.

In statistics, a fundamental distinction is made between the population and the sample. The population may be, for example, all eligible voters, and the sample a particular group selected for a survey. From the answers given by the voters in the (small) sample the opinion poll people make predictions as to the voting trends for the whole population. Of course they can and do get it wrong. The reason is that their samples are not always representative - in the jargon of statistics, the sample is "biased".

In the physics lab the "population" usually means "all the possible results for the experiment", while the sample is the small set of observations that we have actually made. Again there is always the chance that our results are "biased". This is just another name for systematic error.

Even if you were to do an experiment under ideal conditions, and were to repeat it as many times as you wished, you would find that the results do not all agree, but cluster around some mean value. The scatter is ultimately caused by the basic randomness of nature on microscopic scales, but in practice will usually be dominated by our inability to read scales accurately, etc. A notable exception occurs in some of the Physics III experiments where random quantum processes are observed directly.

It is important to know when and when not to apply statistical analysis to your data. Consider, for example, an oscilloscope used to measure the amplitude of a voltage source. If the source produces a "clean" sine wave, the random errors in reading the voltage will be approximately $\pm 1/10$ scale division, while the calibration error may be $\pm 5\%$. Provided the deflection is $\gg 2$ scale divisions, the calibration error will be much larger than the random errors associated with reading the graticule so we can safely ignore reading errors. On the other hand, suppose that the signal is very noisy, with fluctuations of 10%. In this case the random noise dominates and repeated measurements should be taken to reduce this contribution to the error.

If you are in doubt as to the correct way to analyse your errors, try the following. Repeat your measurement several times and calculate the standard deviation (see below). Compare the standard deviation with the estimated accuracy limits of the apparatus. If the standard deviation is significantly greater than the accuracy limits, then random errors will dominate.

Whenever random errors become significant, several questions arise immediately:

- How can we describe the population as a whole?
- What can we learn about the population from a small sample?
- What are the confidence limits of our results?
- Are our results consistent with the accepted or theoretical value?
- How do we cope with random errors when graphing our data?

These questions will be looked at in the following sections.

6.5 Probability distributions

Suppose we were to measure a physical quantity x very many times. We could then make a graph or histogram displaying the number of times that each particular value x_i occurred. The resulting normalised graph $P(x)$ is the population distribution. (Normalization simply involves scaling the bin values so that the area under the histogram is unity.) It is also called the probability distribution, since the probability of measuring $x = x_i$ in a single experiment is just $P(x_i)$.

The probability distribution contains all the information about the statistics of the population. In general distributions can be characterised by relatively few population parameters, which often are of physical interest. Thus the population mean is assumed to be the “true” value of the physical quantity being measured.

Obviously the probability distribution can come in very many shapes, but fortunately in most physical situations only a few distributions occur. In practice the most important in the Senior Physics laboratory are the normal or Gaussian distribution and the Poisson distribution.

6.5.1 The normal distribution

The normal or Gaussian distribution is the familiar “bell-shaped curve” that one obtains if random quantities are plotted. It occurs whenever there are a large number of small random effects which add up to affect the final result. A good example is the scatter in resistance values in a batch of resistors from a particular supplier. In this case small variations in the manufacturing process cause a scatter in the actual resistances which is well described by a Gaussian distribution.

Mathematically the Gaussian distribution can be described by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right) \quad (4)$$

where μ is the population mean and σ is the standard deviation of the population. The standard deviation is a measure of the dispersion or scatter of the population around the mean value. The function is shown in figure 2, and it can be seen that $f(x)$ is symmetrical about μ .

The probability of obtaining a result which lies between x and $x + dx$ is $f(x)dx$, so the probability that a measured value of x lies between x_1 and x_2 is found by integrating $f(x)$ between the limits of x_1 and x_2 . In particular, for a normal distribution, the probability is 0.68 that a single measurement of x will lie within $\pm 1\sigma$ of

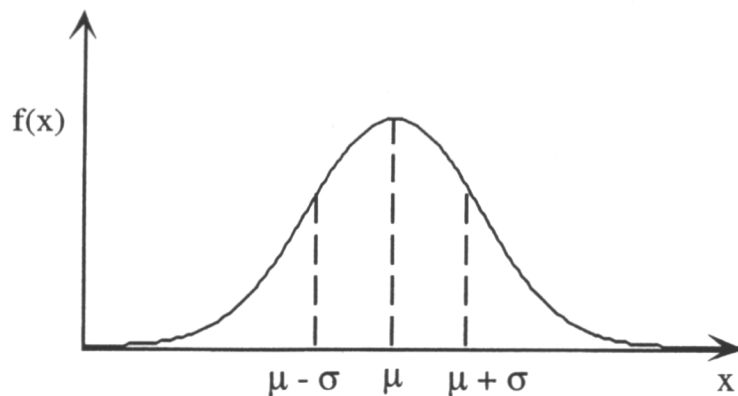


Figure 2: The normal distribution

Interval		Probability	Expectation (sample: 100)
x_1	x_2		
$-\infty$	$\mu - 2\sigma$	0.023	2
$\mu - 2\sigma$	$\mu - \sigma$	0.136	14
$\mu - \sigma$	μ	0.341	34
μ	$\mu + \sigma$	0.341	34
$\mu + \sigma$	$\mu + 2\sigma$	0.136	14
$\mu + 2\sigma$	$+\infty$	0.023	2
$-\infty$	$+\infty$	1.000	100

Table 1: The Normal Distribution

μ , while the probability is 0.95 (i.e. 20:1 on) that it will be within 2 standard deviations of the mean. Table 1 gives the probability of obtaining a value of x between various limits and the expected number of occurrences for a random sample of 100 values.

6.5.2 The Poisson distribution

The Poisson distribution is the other probability distribution that you will need in the Senior Physics Laboratory. It describes situations in which discrete “events” can occur at random intervals. For instance, it allows us to determine the probability of obtaining a given number of radioactive decays per second in a radioactive material, or a given number of telephone calls in one minute through an exchange, etc.

The Poisson distribution has only one parameter - the mean - and is given by the expression:

$$P(n) = \frac{1}{n!} \mu^n e^{-\mu} \quad (5)$$

Note that $P(n)$, unlike the normal distribution, is asymmetric or “skewed”. This is a natural consequence of the fact that n can never be negative. When the mean

value is large, the Poisson distribution can be approximated by the normal distribution with mean μ and a standard deviation $\sqrt{\mu}$.

6.6 The statistics of data samples

Suppose that the results of a series of experimental observations are x_1, x_2, \dots, x_n . The two most important 'statistics' for this set of data are the sample mean and sample standard deviation.

6.6.1 The sample mean

The sample mean \bar{x} is just the familiar average value:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad (6)$$

The sample mean \bar{x} approximates to the true population mean but of course will not in general be equal to it, due to random noise.

6.6.2 The sample standard deviation

The sample standard deviation, s is a measure of the scatter of the observed data points around the measured mean \bar{x} . It is found from the formula:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (7)$$

The quantity s^2 is called the sample variance.

N.B. The sample standard deviation uses “ $n-1$ weighting”, in the sense that $(n-1)$ appears where you might have expected n . If you calculate s directly, you should use this formula; if you use a calculator with a standard deviation key, check to make sure that it uses $(n-1)$ weighting (many calculators provide both n and $n-1$ weighting). The reason for using $n-1$ is that the sample mean appears on the right hand side of equation 7. The mean and standard deviation are therefore not independent and this can be taken into account by reducing n by 1.

If your calculator does not have a standard deviation key, the following alternative formula may be easier to use than equation 7:

$$(n-1)s^2 = \sum x_i^2 - \frac{1}{n}(\sum x_i)^2 \quad (8)$$

The sample standard deviation is an approximation to the true population standard deviation. Thus if we were to take another measurement, very roughly we would expect it to be within $\pm s$ of \bar{x} , 68% of the time.

6.6.3 Standard error of the mean

Your observed sample mean \bar{x} is itself a random variable, and an important statistical theorem states that it will have a normal distribution with a (population) mean and standard deviation σ/\sqrt{n} . Thus there is a 95% chance that \bar{x} lies within $\pm 2\sigma/\sqrt{n}$ of the “actual” value, and it follows that we can improve the precision of a result by increasing the number of data points n . The standard deviation of the sample is

usually known, so it is quite common to estimate the precision of \bar{x} by the standard error of the mean:

$$E = \frac{s}{\sqrt{n}} = \left(\frac{\sum (x_i - \bar{x})^2}{n(n-1)} \right)^{1/2} \quad (9)$$

For large values of n , $E \approx \sigma/\sqrt{n}$ but for small data sets the standard error of the mean significantly underestimates the effects of random errors. A better way to estimate the precision of \bar{x} is described in the next section, but E is easy to calculate, and is often used to give a quick estimate of the precision.

Exercises

1. A student measures a current with a nanoammeter. She obtains the following results: 6.1, 4.7, 6.7, 5.4 and 6.0 nA. What are the mean, standard deviation and standard error of the mean?

[Ans: 5.8 nA, 0.8 nA, 0.3 nA]

2. The meter has an error of 2% FSD and readings were made on the 10 nA range. Will the overall accuracy be improved significantly by taking further readings? Approximately how many readings should she take?

[Ans: yes, 16]

6.7 Confidence intervals and statistical tests

We cannot, of course, exactly determine the population parameters from a small set of observations. We can, however, estimate the probability $1 - \alpha$ that a parameter lies within a particular range. This range is called the $100(1 - \alpha)\%$ confidence interval. Thus α is the probability that the quantity lies outside the interval, and is often chosen to be 0.05 or 0.10 (corresponding to 95% and 90% confidence intervals).

6.7.1 The confidence interval for the mean

The confidence interval for the mean \bar{x} is

$$\bar{x} - \frac{s}{\sqrt{n}} \left(t_{n-1}; \frac{\alpha}{2} \right) < \mu < \bar{x} + \frac{s}{\sqrt{n}} \left(t_{n-1}; \frac{\alpha}{2} \right) \quad (10)$$

which is usually abbreviated to

$$\bar{x} \pm E \left(t_{n-1}; \frac{\alpha}{2} \right) \quad (11)$$

The quantity $t_{m;\alpha}$ is known as Student's¹ t -distribution and can be regarded as a correction factor to the standard error E . The t -distribution $t_{m;\alpha}$ is given in table 2. Note that t increases rapidly when we only have a few data points.

¹“Student” was the nom-de-plume of W.S. Gossett, who introduced the t -distribution in 1908.

a/2 m=n-1	.25	.1	.05	.025	.01	.005
1	1.000	3.078	6.314	12.706	31.821	63.657
2	.816	1.886	2.920	4.303	6.965	9.925
3	.765	1.638	2.353	3.182	4.541	5.841
4	.741	1.533	2.132	2.776	3.747	4.604
5	.727	1.476	2.015	2.571	3.365	4.032
6	.718	1.440	1.943	2.447	3.143	3.707
7	.711	1.415	1.895	2.365	2.998	3.499
8	.706	1.397	1.860	2.306	2.896	3.355
9	.703	1.383	1.833	2.262	2.821	3.250
10	.700	1.372	1.812	2.228	2.764	3.169
11	.697	1.363	1.796	2.201	2.718	3.106
12	.695	1.356	1.782	2.179	2.681	3.055
13	.694	1.350	1.771	2.160	2.650	3.012
14	.692	1.345	1.761	2.145	2.624	2.977
15	.691	1.341	1.753	2.131	2.602	2.947
16	.690	1.337	1.746	2.120	2.583	2.921
17	.689	1.333	1.740	2.110	2.567	2.898
18	.688	1.330	1.734	2.101	2.552	2.878
19	.688	1.328	1.729	2.093	2.539	2.861
20	.687	1.325	1.725	2.086	2.528	2.845
21	.686	1.323	1.721	2.080	2.518	2.831
22	.686	1.321	1.717	2.074	2.508	2.819
23	.685	1.319	1.714	2.069	2.500	2.807
24	.685	1.318	1.711	2.064	2.492	2.797
25	.684	1.316	1.708	2.060	2.485	2.787
26	.684	1.315	1.706	2.056	2.479	2.779
27	.684	1.314	1.703	2.052	2.473	2.771
28	.683	1.313	1.701	2.048	2.467	2.763
29	.683	1.311	1.699	2.045	2.462	2.756
30	.683	1.310	1.697	2.042	2.457	2.750
40	.681	1.303	1.684	2.021	2.423	2.704
60	.679	1.296	1.671	2.000	2.390	2.660
120	.677	1.289	1.658	1.980	2.358	2.617
∞	.674	1.282	1.645	1.960	2.326	2.576

Table 2: Student's t -distribution

Example

A student makes 4 independent determinations of the Rydberg constant for hydrogen using 4 lines of the Balmer series. The values obtained are:

$$\begin{aligned}1.097099 \times 10^7 \text{ m}^{-1} \\1.097046 \times 10^7 \text{ m}^{-1} \\1.097060 \times 10^7 \text{ m}^{-1} \\1.097107 \times 10^7 \text{ m}^{-1}\end{aligned}$$

What is the 95% confidence interval for the Rydberg constant?
From the data,

$$\begin{aligned}\bar{x} &= 1.097078 \times 10^7 \text{ m}^{-1} \\s &= 296.1 \text{ m}^{-1} \\E &= 148.0 \text{ m}^{-1}\end{aligned}\tag{12}$$

For a 95% confidence interval, take $\alpha/2 = 0.025$. From Table 2 we find $t_{3;0.025} = 3.182$. Hence the 95% confidence interval is

$$x = (1.097078 \pm 0.000047) \times 10^7 \text{ m}^{-1}$$

6.7.2 The confidence interval for the standard deviation

Similarly, the confidence interval for the standard deviation can be calculated. It is

$$s \left(\frac{n-1}{\chi_{n-1; \alpha/2}^2} \right)^{1/2} < \sigma < s \left(\frac{n-1}{\chi_{n-1; -\alpha/2}^2} \right)^{1/2}\tag{13}$$

Note that this interval is asymmetric. The quantity $\chi_{m; \alpha}^2$ is called the χ^2 -distribution (χ is the Greek letter “chi”, pronounced KY to rhyme with MY and $m = n - 1$ is the degrees of freedom) and is given in table 3.

6.7.3 Statistical test

Most of the experiment notes in the Senior Physics Laboratory include a variation on the statement:

Compare your results with the theoretical value ...

We can turn such instructions into propositions or “hypotheses”:

My data are consistent with the theoretical value (YES or NO — cross out whichever does not apply).

Because of the random errors which are present, we cannot state categorically that the results do/do not support the hypothesis, but often there are statistical tests which allow us to accept or reject the hypothesis with a certain level of significance. The level of significance, α , is defined to be the probability that we mistakenly reject the hypothesis as false when in fact it is true. Obviously we want α to be small and typically it is taken to be 0.05 or 0.10 (corresponding to 5% and 10% levels of significance).

6.7.4 Tests for the mean and standard deviation

The proposition that the population mean (or standard deviation) equals a particular value X (or S) is easy to test. We accept the hypothesis that $\mu = X$ (or $\sigma = S$) at the α level of significance if X or S lie in the confidence intervals given by equations 10 and 13 respectively.

The terminology can unfortunately be confusing! We use, for example, a 95% confidence interval to test at the 5% level of significance (blame the statisticians!).

Example

Does the student's data set in the example in section 6.7.1 agree with theory? The Rydberg constant for hydrogen is known to be $1.09766 \times 10^7 \text{ m}^{-1}$. This lies outside of the confidence interval so we can state that the data do not agree with theory at the 5% level of significance. There is clearly a systematic error present.

Other tests are not as straightforward. Two tests that are frequently needed are discussed in the following sections.

6.7.5 Rejection of data

Students are quick to reject data points as "discrepant". There are really only two ways of deciding whether a dubious point should be discarded.

- (a) Strong suspicion of systematic error. If the discrepant point is the only one your partner took, or if there is some other obvious problem (recorded, one hopes, in your log book), then you have grounds for rejecting the data.
- (b) Statistical testing. Let \bar{x} and s be the mean and standard deviation of the n data points excluding the possible discrepant point x' . Then x' is discrepant at the $100\alpha\%$ level of significance and should be rejected if the following inequality holds:

$$|x' - \bar{x}| > t_{n-1; \alpha/2} \left(\frac{n+1}{n} \right)^{1/2} s \quad (14)$$

If equation 14 is not satisfied, x' should be included with the other data.

6.7.6 The χ^2 goodness-of-fit test

It is often necessary to decide whether our data have a particular probability distribution. Again we cannot say for sure, but can only indicate the probability that the data fit the distribution.

We must first construct some "test statistic" which measures the deviation of the data from the theoretical distribution. If this test statistic is small we accept the hypothesis; otherwise we reject it.

Suppose that we have a number of data points, which we can graph as a histogram or tabulate. Let o_i be the number of points falling in the interval x_i to x_{i+1} (o is for "observed"). We can also calculate the expected number e_i for each interval using the theoretical distribution. For example, if the total number of points is 100 and we are testing a fit to the normal distribution, the numbers in table 3 can be used.

The test statistic is

$$\chi^2 = \sum_{i=1}^k \frac{(e_i - o_i)^2}{e_i} \quad (15)$$

where k is the total number of intervals.

Clearly if our data agree closely with the theoretical distribution χ^2 will be small. Thus we accept the hypothesis that our data come from the distribution. Specifically, we accept the hypothesis at the 100α level of significance if

$$\chi^2 < \chi_{\nu;\alpha}^2 \quad (16)$$

where $\chi_{\nu;\alpha}^2$ is the χ^2 -distribution with $\nu = k - p - l$ “degrees of freedom” and

k = number of terms in the sum of equation 15

p = number of parameters estimated from the data.

Tabulated values of χ^2 as a function of ν and α are given in table 3.

Note that there is always a risk that equation 16 will not be satisfied when in fact the data do come from the theoretical distribution. For example, if we test at the 5% level of significance a series of data sets drawn at random from a normal distribution, we would expect 1 in 20 sets to give a value of χ^2 which violates equation 16.

The number p will change depending on the distribution and on how we define the distribution parameters. For example, if we wish to test our data to see if they fit to a normal distribution having a mean and standard deviation equal to our measured \bar{x} and s , then $p = 2$. If instead we want to test the data to see if they come from a normal distribution with a given mean and standard deviation, then $p = 0$. For the Poisson distribution, if we take the mean equal to the observed mean, then $p = 1$.

There is one important point to note. The test works only if all the e_i are ≥ 5 . If this is not the case, the data should be regrouped.

Example

A batch of 100 resistors is checked to see if the resistors are normal distributed. We use the χ^2 test to see if the values are normally distributed with mean and standard deviation equal to the measured values.

It is found that $\bar{x} = 10.22 \text{ k}\Omega$ and $s = 0.29 \text{ k}\Omega$. We group the data into the intervals $(-\infty, \bar{x} - 2s)$, $(\bar{x} - 2s, \bar{x} - s)$, \dots , $(\bar{x} + 2s, +\infty)$. The theoretically expected count in each interval can be obtained from Table 3. The results are given below.

The number of degrees of freedom, ν , is $4 - 2 - 1 = 1$ and from Table 3 the value of χ^2 at the 5% level of significance is 3.84. Note that the highest and lowest groups had to be combined with the adjoining groups as they had fewer than 5 expected occurrences. The observed χ^2 is less than the critical value of 3.84 so the data support the hypothesis that the resistors come from a normal distribution.

ν	$\alpha = 0.30$	$\alpha = 0.20$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.02$	$\alpha = 0.01$	$\alpha = 0.001$
1	1.07	1.65	2.71	3.84	5.41	6.64	10.83
2	2.41	3.22	4.60	5.99	7.82	9.21	13.82
3	3.66	4.64	6.25	7.82	9.84	11.34	16.27
4	4.88	5.99	7.78	9.49	11.67	13.28	18.46
5	6.06	7.29	9.24	11.07	13.39	15.09	20.52
6	7.23	8.56	10.64	12.59	15.03	16.81	22.46
7	8.38	9.80	12.02	14.07	16.62	18.48	24.32
8	9.52	11.03	13.36	15.51	18.17	20.09	26.12
9	10.66	12.24	14.68	16.92	19.68	21.67	27.88
10	11.78	13.44	15.99	18.31	21.16	23.21	29.59
11	12.90	14.63	17.28	19.68	22.62	24.72	31.26
12	14.01	15.81	18.55	21.03	24.05	26.22	32.91
13	15.12	16.98	19.81	22.36	25.47	27.69	34.53
14	16.22	18.15	21.06	23.68	26.87	29.14	36.12
15	17.32	19.31	22.31	25.00	28.26	30.58	37.70
16	18.42	20.46	23.54	26.30	29.63	32.00	39.25
17	19.51	21.62	24.77	27.59	31.00	33.41	40.79
18	20.60	22.76	25.99	28.87	32.35	34.80	42.31
19	21.69	23.90	27.20	30.14	33.69	36.19	43.82
20	22.78	25.04	28.41	31.41	35.02	37.57	45.32
21	23.86	26.17	29.62	32.67	36.34	38.93	46.80
22	24.94	27.30	30.81	33.92	37.66	40.29	48.27
23	26.02	28.43	32.01	35.17	38.97	41.64	49.73
24	27.10	29.55	33.20	36.42	40.27	42.98	51.62
25	28.17	30.68	34.38	37.65	41.57	44.31	52.62
26	29.25	31.80	35.56	38.88	42.86	45.64	54.05
27	30.32	32.91	36.74	40.11	44.14	46.96	55.48
28	31.39	34.03	37.92	41.34	45.42	48.28	56.89
29	32.46	35.14	39.09	42.56	46.69	49.59	58.30
30	33.53	36.25	40.26	43.77	47.96	50.89	59.70

Table 3: The χ^2 distribution

Interval		Expected counts	Observed counts	$(e_i - o_i)^2/e_i$
x_i	$x_i + 1$	e_i	o_i	
$-\infty$	$\bar{x} - 2s$	14	16	0
		} 16		
$\bar{x} - 2s$	$\bar{x} - s$	2		
$\bar{x} - s$	\bar{x}	34	31	0.265
\bar{x}	$\bar{x} + s$	34	36	0.118
$\bar{x} + s$	$\bar{x} + 2s$	14	17	0.0625
		} 16		
$\bar{x} + 2s$	∞	2		
TOTAL:		100	100	0.445

Table 4: Using the χ^2 test

6.8 Combining statistical errors

The rules for combining statistical errors are different from those we use when combining maximum errors. Suppose for example we obtain a set of readings x_i and y_i which can be quoted in terms of the maximum errors as $\bar{x} \pm \Delta x$, $\bar{y} \pm \Delta y$ or in terms of the standard errors as $\bar{x} \pm E_x$, $\bar{y} \pm E_y$ where

$$E_x = \frac{s_x}{\sqrt{n_x}}$$

$$E_y = \frac{s_y}{\sqrt{n_y}}$$

The absolute maximum error in $f = x + y$ is $\Delta f = \Delta x + \Delta y$. The standard error in f is given by

$$E_f = (E_x^2 + E_y^2)^{1/2} \tag{17}$$

This procedure for combining errors can also be applied to maximum errors to obtain a more realistic estimate of the maximum error in f . In either case, the quadratic combination of errors is valid only if E_x , E_y (or Δx , Δy) are independent. The following table summarises the rules for combining standard errors.

Function f	Standard Error in f
$\left. \begin{array}{l} f = x + y \\ f = x - y \end{array} \right\}$	$E_f = (E_x^2 + E_y^2)^{1/2}$
$\left. \begin{array}{l} f = xy \\ f = x/y \end{array} \right\}$	$\frac{E_f}{f} = \left[\left(\frac{E_x}{x} \right)^2 + \left(\frac{E_y}{y} \right)^2 \right]^{1/2}$
$f = x^n$	$\frac{E_f}{f} = n \left(\frac{E_x}{x} \right)$

Since these rules apply only when the errors in x are independent of the errors in y , it may be necessary to arrange the formula for f before hand so that it only contains quantities whose errors are independent.

Example

Measurements of voltage (V) and current (I) are used to calculate resistance (R) and power (P). Errors in V do not depend on errors in I . We find

$$R = \frac{V}{I}$$

and

$$\frac{E_R}{R} = \left[\left(\frac{E_V}{V} \right)^2 + \left(\frac{E_I}{I} \right)^2 \right]^{1/2}$$

We cannot use $P = RI^2$ since the error in R depends on the error in I . Rather, we use $P = VI$; thus the fractional error in P is the same as the fractional error in R .

6.9 Errors and graphs

A problem which arises frequently in experimental work is to determine the relationship between two quantities, say y and x . By plotting y vs x on a linear, log or log-linear graph we can check for example if $y = ax - b$ or $y = ax^n$ or $y = ae^{bx}$ and obtain an estimate for the constants a and b by drawing a line of best fit through the experimental points. This line tends to average out the errors in each individual point. If we wish to determine the worst possible error in the slope of a graph, we can also draw lines of worst fit through the error bars associated with each point. Care should be taken, however, to ensure that the worst fit lines pass through all points (including $y = 0$, $x = 0$ if we have good reason to believe that they should). Plots of voltage vs current, for example, usually (but not always) pass through $(0, 0)$.

It should be noted that worst fit curves usually overestimate the actual error in the slope of a graph and that a so-called line of "best fit" drawn by eye may have significantly different slopes when drawn by different observers. Both of these limitations can be overcome by the regression techniques outlined below.

If random errors dominate, then the method of least squares can be used to estimate the line of best fit.

6.9.1 The method of least squares

Suppose y_1, y_2, \dots, y_n are the values of a measured quantity y corresponding to values x_1, x_2, \dots, x_n of another quantity x . For these data, we wish to determine the “best fit” line of the form

$$y = ax + b \quad (18)$$

The usual way is, of course, to estimate by eye the best line and draw it in with a ruler. If the data points lie very close to a straight line this method is as good as any, but if there are large errors it is difficult to estimate a “best” line, and different lines will be drawn by different experimenters.

The method of least squares provides an unambiguous way of choosing the best line. It is the method used by *Excel* and *Origin*. Suppose we fit the data by a line of the form of equation 18. Then for each data point there will be a small error

$$\Delta = y_i - (ax_i + b) \quad (19)$$

between the line and the actual measured values of y .

An estimate of how well the line fits the data is given by the sum of squares S :

$$S = \sum [y_i - (ax_i + b)]^2 \quad (20)$$

The line of best fit, by definition, is the one that minimises S . As S is a function of the slope, a , and intercept, b , we can find the minimum of S by differentiating with respect to a and b and setting the derivatives equal to 0:

$$\frac{\partial S}{\partial a} = 2 \sum x_i(ax_i + b - y_i) = 0 \quad (21)$$

and

$$\frac{\partial S}{\partial b} = 2 \sum (ax_i + b - y_i) = 0 \quad (22)$$

Solving for a and b gives:

$$a = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} \quad (23)$$

and

$$b = \frac{\sum y_i \sum x_i^2 - \sum x_i \sum x_i y_i}{n \sum x_i^2 - (\sum x_i)^2} \quad (24)$$

Exercises

1. The following voltage and current readings were obtained for a particular device. Obtain the best fit line (a) by eye and (b) by the least squares method, and hence estimate the resistance of the device.

Note that the least squares line does not pass through $(0, 0)$.

[Ans: $a = 1.02$, $b = -0.03$]

2. How can we change the above formulae to ensure that the line does pass through $(0, 0)$? Show that in that case,

$$a = \frac{\sum x_i y_i}{\sum x_i^2}$$

and hence find a .

[Ans: $a = 1.007$]

3. Show by the method of least squares that the best fit of a series of n readings y_1, y_2, \dots, y_n to the function $y = k$ (where k is a constant) is given by $k = \bar{y}$.
4. Show that the best fit line of the form $y = ax + b$ passes through the point (\bar{x}, \bar{y}) .

6.9.2 Confidence intervals for the slope and intercept

One advantage of the least squares method is that it can provide reasonable estimates for the uncertainty in the slope and intercept of the line of best fit. This is difficult to do by the “eyeball” method of best and worst fit lines. The $100(1 - \alpha)\%$ confidence intervals for the slope and intercept are given by the following formulae:

$$a \pm (t_{n-2; \alpha/2}) E_a \quad (25)$$

and

$$b \pm (t_{n-2; \alpha/2}) E_b \left(\frac{\sum x_i^2}{n} \right)^{1/2} \quad (26)$$

where

$$E_a^2 = \frac{(n \sum [y_i - ax_i + b])^2}{(n - 2)[n \sum x_i^2 - (\sum x_i)^2]} \quad (27)$$

Note that the t -distribution is taken with $n - 2$ degrees of freedom; this is because we know have two dependent or derived quantities (a and b).

The errors in a and b can usually be treated like any other errors. They are not independent, however, and if you need to calculate a function $f(a, b)$ which includes both quantities, care must be taken. When *Origin* is used for fitting a least squares line to data it automatically calculates errors for the slope and intercept.

6.9.3 A word of warning

The method of least squares is essentially a mathematical black box: put data in and it will produce the numbers a and b . The results may be complete nonsense, however, and it is always necessary to graph your data and draw in the least squares line. If the fit looks plausible by eye, then there is probably no difficulty. If the line is clearly wrong, then the method should not be used. The least squares technique can fail for several reasons:

- The data do not fit a straight line. A quadratic curve or some other shape may be more appropriate.

- The independent variable (x) may have significant errors. The least squares method assumes that all the error is associated with the dependent variable y .
- The scatter is not uniform along the line. The method is based on the assumption that y is distributed as a normal random variable about the line, with a constant standard deviation. This may not be true, especially if the fit is to a power law and the data have been plotted on log-log axes to give a linear relation.

Exercise

Suppose y ranges from 0.1 to 100, and has a standard deviation over this range of 0.05. Using log paper, draw in the $\pm s$ error bars for $y = 0.1, 1, 10$ and 100.

6.10 Slopes and intercepts

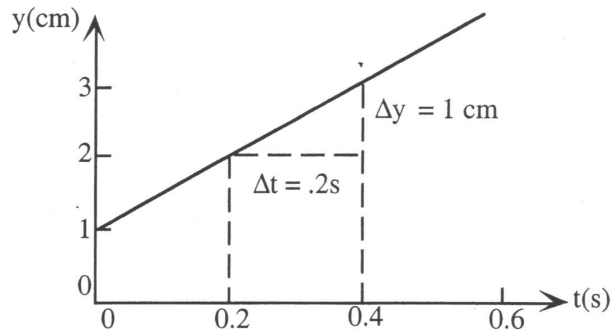
A straight line can be fitted to linear, power law or exponential curves provided the appropriate linear or logarithmic graph paper is used. Regardless of the type of curve plotted, the slope of the straight line $y = ax + b$ is given by $a = \Delta y / \Delta x$. The value of other constants can be found by substitution. Figure 3 shows several examples.

Example 1

$$y = (at + b) \text{ cm}$$

$$a = \frac{1 \text{ cm}}{0.2 \text{ sec}} = 5 \text{ cm/sec}$$

When $t = 0$, $y = b = 1 \text{ cm}$



Example 2

$$I = kV^a \text{ amps}$$

$$\text{So } \log_{10} I = \log_{10} k + a \log_{10} V$$

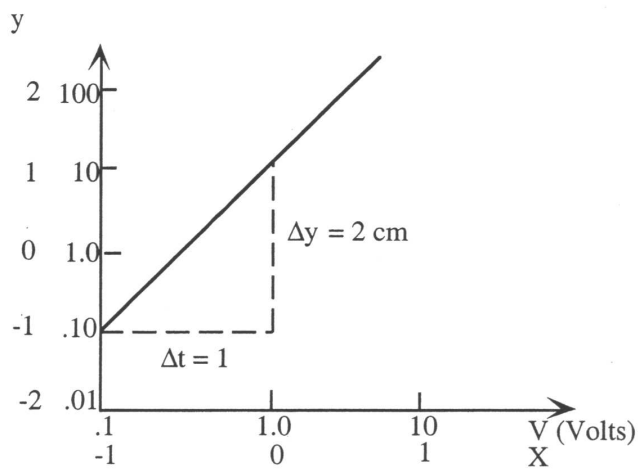
$$\text{or } y = b + ax$$

where $y = \log_{10} I$

and $x = \log_{10} V$

Slope $a = 2/1 = 2$ (no units for logs)

When $v = 1.0V$, $I = 10.0A$ so $k = 10.0/1^2 = 10 \text{ A/V}^2$



Example 3

$$V = V_0 e^{-at} \text{ volts}$$

$$\text{So } \log_{10} V = \log_{10} V_0 - at \log_{10} e$$

$$\text{or } y = b - 0.4343 at$$

where $y = \log_{10} V$

Slope $= 0.4343a = -1/(0.5\text{sec})$ so $a = 2/0.43 = 4.65 \text{ sec}^{-1}$

When $t = 0$ $V = V_0 = 10 \text{ volts}$

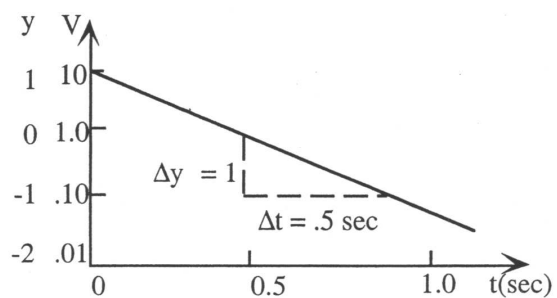


Figure 3: Calculating the slopes of straight lines

7 SOME USEFUL PHYSICAL CONSTANTS

Speed of light in vacuo	c	=	$2.99792458 \times 10^8 \text{ ms}^{-1}$
Planck's constant	h	=	$6.626176 \times 10^{-34} \text{ Js}$
		=	$4.1357 \times 10^{-15} \text{ eVs}$
electron charge	e	=	$1.6021892 \times 10^{-19} \text{ C}$
electron mass	m	=	$9.109534 \times 10^{-31} \text{ kg}$
		=	$0.510976 \text{ MeV}/c^2$
proton mass	M_p	=	$1.6726485 \times 10^{-27} \text{ kg}$
neutron mass	M_n	=	$1.67470 \times 10^{-27} \text{ kg}$
Boltzmann's constant	k	=	$1.380662 \times 10^{-23} \text{ J K}^{-1}$
		=	$8.6164 \times 10^{-5} \text{ eV K}^{-1}$
kT at $T = 300\text{K}$		=	0.02585 eV
Avogadro's constant		=	$6.02486 \times 10^{23} \text{ (mole)}^{-1}\text{K}^{-1}$
Fine structure constant	α	=	$e^2/(4\pi\epsilon_0\hbar c) = 1/137.0373$
g factor for electron		=	$2(1.001145) = 2.0023$
Permeability for vacuum	μ_0	=	$4\pi \times 10^{-7} \text{ H m}^{-1}$
Permittivity for vacuum	ϵ_0	=	$8.854187818 \times 10^{-12} \text{ F m}^{-1}$
		=	$1/(c^2\mu_0)$
Geographic position of		=	$33^\circ 53' 24'' (\pm 3'')$ South
School of Physics		=	$151^\circ 11' 00'' (\pm 3'')$ East
Gravitational constant	G	=	$6.6720 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}$

8 SI PREFIXES

Value	Prefix	EXAMPLES		
		DISTANCE	CAPACITANCE	ENERGY
10^{-18}	atto	am		
10^{-15}	femto	fm	fF	
10^{-12}	pico	pm	pF	
10^{-9}	nano	nm	nF	
10^{-6}	micro	μm	μF	
10^{-3}	milli	mm	mF	
10^0	unit	m	F	eV
10^3	kilo	km		KeV
10^6	Mega	Mm		MeV
10^9	Giga	Gm		GeV
10^{12}	Tera	Tm		TeV
10^{15}	Peta	Pm		PeV
10^{18}	Exa	Em		EeV

(1 Em = 32.4 parsec)

Revised on March 2, 2006