"Ruler physics:” Thirty-four demonstrations using a plastic ruler

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A collection of 34 demonstrations is described which use only a plastic ruler and other simple items. Many of the demonstrations have a quantitative component, making them suitable for student experiments or at-home exercises. Although most are appropriate to introductory physics courses, a few involve more advanced concepts.

I. INTRODUCTION

It is surprising how many interesting physics demonstrations can be done with a 0.304 m (12 in.) long grooved transparent plastic ruler. Such rulers may be obtained at most discount, toy, stationary, and business supply stores. The transparency of the ruler allows many of the demonstrations to be done on an overhead projector, but if this is not considered important an opaque ruler may be used in all cases. (Almost) all of the 34 demonstrations described here uses only very limited other “apparatus” besides a plastic ruler, such as paper clips, pennies, balls, clay, etc. In a majority of cases, the demonstrations could also be done as student lab experiments, or home experiments. Although many of them may be well known, some may not be. Moreover, some of them do not appear in any of the well-known collections of physics demonstrations.

1. Measuring someone’s reaction time by dropping a ruler

Hold a ruler by one end, and let it hang vertically so that the lower end lies between someone’s thumb and index finger. Challenge the other person to catch it as quickly as possible, when you drop the ruler at an unannounced time. You can easily calculate the person’s reaction time, \( t \), from the distance, \( d \), the ruler falls: \( t = \sqrt{2d/g} \). You will probably find a fairly reproducible value for a given person, even when she changes the separation between her fingers.

2. Period of a compound (physical) pendulum using a swinging ruler

Make a small hole very near the end of the ruler, and allow it to swing on a straightened paper clip. Measure the period, and compare with the value predicted for 1 foot long compound pendulum: 0.91 s. Another alternative would be to make the pivot hole on the ruler at a point where the predicted period is exactly 1 s. In general, the period of a compound pendulum of mass \( m \), and moment of inertia \( I \), can be expressed as \( T = 2\pi \sqrt{I/\rho g h} \), where \( h \) is the distance from the rotation axis to the center of gravity. Let \( d \) represent the distance from the top end of the ruler to that axis, \( A \), for which the period is one second, [so that \( l/2 - d \) is the distance from \( A \) to the center of mass]. For a uniform rod of mass \( m \) and length \( l \), the moment of inertia about the center of mass is \((1/12)ml^2 \) so that, according to the parallel axis theorem, we have the moment of inertia about axis \( A \) given by \( I = (1/12)ml^2 + m(l/2 - d)^2 \). The period can, therefore, be expressed as

\[
T = 2\pi \frac{I^{1/2}}{\rho g (l/2 - d)}
\]

with \( I \) given above. Requiring the period to be 1 s yields a value of \( d = 11.6 \) cm. An oscillation with a period of 1 s makes it trivial for students to check the prediction by counting off the seconds while watching the swings. Incidentally, it seems remarkable that the period of oscillations of a compound pendulum about an axis only 3.6 cm from the center of the ruler is almost the same as the period about an axis at one end, especially considering that the period when the axis is at the center of the ruler is infinite.

3. Period of oscillations of a ruler balanced on a cylinder

Another example of simple harmonic motion occurs when a ruler is balanced horizontally atop a cylinder, such as a soda can, and given a push away from its equilibrium horizontal orientation. It can easily be shown that, for small oscillations, the period is given by

\[
T = 2\pi \sqrt{l^2/12g(r-d)}
\]

where \( l \) is the length of the ruler, \( r \) is the radius of the cylinder, and \( d \) is the very small distance the ruler’s center of mass lies above the contact point of the ruler with the cylinder. You can test the validity of this equation by observing the oscillation periods using cylinders having various radii. However, if you try the experiment using cylinders having radii that are not much larger than \( d \), such as stick pens or pencils having a round cross section, you will probably find that achieving balance is difficult, and that only very small oscillations can be observed. For cylinders having radii less than \( d \) the equilibrium is unstable, and no oscillations can be observed.

4. Collisions of balls rolling in the groove of a ruler

Observe a collision between one smooth steel ball incident on a second stationary steel ball of equal mass placed in the groove of a level plastic ruler. If the collision were elastic, and we ignored rotational motion, an incident ball would be brought to rest after the collision. But rotational motion, of course, cannot be ignored for rolling objects. Because of spin effects, you will probably find that the speed of the incident ball after collision, expressed as a fraction of its initial speed, depends dramatically on how forcefully you project it. If the incident ball is given a forceful push, you will probably find that it does, in fact, lose nearly all its original velocity after collision. This is just like the case of a cue ball, which is nearly stationary after being projected forcefully into another ball in a head-on collision.

In such cases, we can explain the observations by noting that the incident ball's motion before impact tends to be
more sliding than rolling, as a result of its forceful push, so that the complications of rotational motion are minimized. On the other hand, if you slowly roll one steel ball towards another, you will find that the incident ball continues to roll forward after collision with an appreciable fraction of its initial speed—just as in the case of a collision between pool balls. In such cases, we may assume that the incident ball is rolling without slipping before the collision, and that it is momentarily brought to rest by the collision. If we further assume that little spin is transferred between the smooth balls during the collision, the incident ball maintains the spin it had just before the collision. This spin causes the ball to accelerate forward, until it is rolling without slipping.

In demonstrations to introductory classes you may wish to downplay the effects of spin, and emphasize the first case, where the incident ball is given a vigorous push—so that it is nearly sliding. In fact, the higher the speed of the incident ball, the less it spins before impact, and the more nearly it is at rest after collision. If, on the other hand, you wish to accentuate the effects of spin, you can try deliberately giving the incident ball backspin or topspin, when you launch it. (You can give it backspin, for example, by pressing your finger down on the back half of the ball, causing it to "squirt" forward, while spinning backwards.) A backspinning ball should be found to rebound backwards after hitting a stationary ball. It is also interesting to examine collisions between balls of unequal mass—especially, the difference found when you switch the roles of the incident and target balls. The transparency of the ruler allows the demonstration to be performed on an overhead projector. Even if the projector is not leveled, you can always find some orientation of the ruler which will be level.

5. Ball rolling in the groove of an inclined ruler

Using a ruler, you can verify Galileo’s observation that balls roll from rest down a flat incline a distance that is proportional to the square of the time, as required by a constant acceleration. Place the ruler on an overhead projector, and level it by placing enough strips of index card under one end, so that a metal ball does not roll when placed on it (or else just find the orientation of the ruler for which it is level). You may find that, due to ruler warping, a ball will not be in equilibrium at various points on the ruler. In this case, either find a flatter ruler, or else tape the ruler to a flat surface or back-to-back to a second ruler. If you are using a single ruler as the incline, be sure to use a light enough ball, so that the ruler does not flex appreciably when the ball is placed on it. The distance, \( x \), a solid ball rolls down an incline from rest is given by \( x = \frac{1}{2}at^2 = \frac{1}{2}(\frac{3}{2}g \sin \theta) t^2 \). For a ruler of length \( L \), if we use the small angle approximation, \( \sin \theta = \theta \), we find that the distance one end of the ruler needs to be raised for the ball to roll a distance \( x \) in time \( t \) is given by \( \Delta y = L \theta = 14Lx/(5g\theta^2) \). If you want the ball to roll 2.54 cm in 1 s, four times that in 2 s, and nine times that in 3 s, the required elevation of one end of the ruler \( \Delta y \) would be 1.85 mm, based on the preceding equation. Prop one end of the ruler up by the required amount, using the needed number of index cards (about eight). Make marks on the ruler at a distance one, four, and nine times 2.54 cm from some starting point, and observe the ball as it rolls down. You can count off the seconds as the ball rolls, and see if the ball passes each mark at 1.2, and 3 s.

6. Sliding two fingers under a ruler

Rest a horizontal ruler on your two extended index fingers. No matter what the original placement of the ruler, your fingers always meet in the center of the ruler if you move them together, as long as you do not suddenly accelerate one of them. Notice how your fingers alternate in their movement relative to the ruler, independent of your intentions. This effect is a result of the difference between the sliding and kinetic friction coefficients, whose ratio can be deduced by observing exactly at what point one finger begins to slide, and the other one stops. If the distances of each finger to the center of the ruler are \( x_1 \) and \( x_2 \) at this point, then it can easily be shown that the ratio of the friction coefficients equals \( x_1/x_2 \). Note, that since this ratio remains constant, the values of \( x_1 \) and \( x_2 \) get progressively smaller for each switch in finger motions. Theoretically, if the ruler were to remain exactly horizontal, there would be an infinite number of switches in which finger is moving relative to the ruler, before they met at the middle.

7. Center of mass of a weighted ruler

Tape a few pennies to one end of a ruler. If you have already measured the masses of the ruler and pennies, you can calculate the location of the combined center of mass, based on the placement of the pennies. See if the ruler balances on your finger at the predicted point. You can also mark the center of mass location with tape, and see that this point is the center of rotation when you fling the ruler in the air, causing it to rotate.

8. Faster-than-\( g \) acceleration of one end of a falling ruler

Rest one end of a horizontal ruler on the edge of a table, and support the other end with your finger. If you suddenly remove your finger support, that end of the ruler is predicted to have an initial downward acceleration of 1.5 g. In order to observe that the end of the ruler does, in fact, have an acceleration in excess of \( g \), place a ball or a penny on the end of the ruler supported by your finger. Suddenly remove your supporting finger, and observe that a gap can be seen between the ruler and the ball or penny, during the initial part of their descent. This gap shows that the ruler’s end must be descending with an acceleration in excess of the 1 g acceleration of the penny or ball. It can be shown that for no gap to be seen, the ball or penny would need to be placed within two-thirds the length of the ruler from the pivot point. If you are showing this demonstration to a large group, you may wish to use a meter (or longer) stick rather than a plastic ruler for better visibility. Sometimes the demonstration is given with a row of pennies arranged all along the meter stick. Only pennies up to the two-thirds point have accelerations less than \( g \) and only they can remain in contact with the stick as it falls. The break in the row of pennies at a point two-thirds of a meter from the pivot can usually be clearly seen.
9. Balancing weighted and unweighted rulers on your finger

Your ability to balance a stick upright on the end of your finger is greater, the longer the stick—a consequence of the fact that longer sticks have greater moments of inertia, and smaller angular accelerations as they tip over. Unless your reflexes are extremely fast, you probably will not be able to balance an object as short as 12 in. ruler vertically on the tip of your finger. On the other hand, if you place a clay ball on the top of the ruler, you probably will be able to balance it fairly easily. If the ruler is much lighter than the clay ball, it can easily be shown that the angular acceleration of a toppling ruler with a clay ball on top is two-thirds the value without a clay ball. It is likely that the 33% reduction in angular acceleration of the toppling ruler may be sufficient for you to keep it balanced, unless you have rather slow reflexes.

10. Catching a row of pennies in mid-air

Line a row of about a dozen pennies along the non-grooved side of a ruler. Hold your forearm horizontal, with your palm and elbow facing the ceiling—somewhat in the manner of a waiter holding a tray, but with your forearm horizontal not vertical—see Fig. 1. (The position of your forearm should be roughly on the same level as your head.) Place the ruler with its row of pennies on your horizontal forearm. Be sure that the ruler points towards your open palm. With one rapid motion, swing your arm, and sweep your hand forward. As you do so, visualize the row of pennies "hanging in air," and you should be able to scoop them all up as your hand moves forward pushing the ruler out from under them. A rapid swing of your arm is, of course, essential since the pennies will fall about 5 cm, if your arm swing takes a tenth of a second. This demonstration is the "poor man's" version of the trick where you pull a table cloth out from under a dinner setting. Aside from requiring much less preparation, it works more reliably (with some practice, and less potential disaster!), and can be explained more simply, because you do not have to worry about the complications of friction—since the pennies are weightless while in free-fall.

11. Maximum angle of a ruler leaning against a smooth wall

Hold a note pad vertically, and lean a ruler against it like a ladder resting against the side of a building. It can easily be shown that, if we ignore friction with the note pad, the maximum angle that the ruler can make with the vertical before slipping is given by \( \tan \theta_{\text{max}} = 2 \mu \), where \( \mu \) is the static friction coefficient between the bottom of the ruler and the surface on which it rests. Thus you can determine the static friction coefficient \( \mu \), by observing the maximum angle with the vertical before slipping occurs. Now tape some paper clips or pennies near one end of the ruler. (Let \( x \) represents the added mass expressed as a fraction of that of the ruler.) If the added mass is at the top end of the ruler, and we neglect friction between the top of the ruler and the wall, you can easily prove that the predicted maximum angle for no slipping is given by

\[
\tan \theta'_{\text{max}} = \left( \frac{x+1}{2x+1} \right) \tan \theta_{\text{max}}.
\]

To check this relation, you can compare the measured angles \( \theta_{\text{max}} \) and \( \theta'_{\text{max}} \) with and without the added mass. If you do not want to bother with a protractor, you can find the angle of the ruler with the vertical in terms of \( \sin^{-1}(d/I) \), where \( d \) is the distance of the base of the ruler to the note pad, and \( I \) is the length of the ruler. The reason for neglecting wall friction is that otherwise the problem is indeterminate. In any case, the neglect of wall friction should not be too bad an approximation if the angle \( \theta_{\text{max}} \) is small, and the wall is smooth.

12. Tangential speed at the top of a rolling wheel

Rest one end of a horizontal ruler on a sphere or cylinder that is free to roll, while holding the other end. Move the ruler forward, and observe that the sphere or cylinder rolls forward only half as far as the ruler moves. This result, which shows up quite well on an overhead projector, proves that the contact point with the ruler on the top of a rolling wheel has twice the linear velocity of the wheel's center, if the wheel rolls without slipping.

13. Constancy of the angular deceleration of a rotating ruler

Many plastic rulers have a small hole in their center, but you can easily make one if yours does not. Balance a horizontal ruler on the conical tip of a ball point pen placed in the ruler's center hole, thereby allowing the ruler to rotate freely about its center in a horizontal plane. Give the ruler a hard spin, and count the number of rotations, \( N \), before it comes to rest. Have someone else measure the time, \( T \), it takes to come to rest, measured from your initial push. Repeat the observations two or three additional times, giving it different amounts of spin each time. If the angular deceleration of the ruler due to friction is constant, you should find that the ratio \( N/T^2 \) is the same for all trials. Equivalently, you should find that a plot of \( N \) vs \( T^2 \) yields a straight line.
14. Dependence of centripetal force on $\omega$ and $r$

Tape a plastic ruler, grooved side down, onto a low-friction turntable. Ideally, it should be sufficiently friction-free so that it spins for a time of at least 3 s when given a gentle push. (You cannot use the previous arrangement of a ruler spinning on the tip of a ball point pen in this demonstration because it is not stable enough.) Place a row of pennies all along the ruler with the pennies in contact. Give the turntable a spin, and observe which pennies have slid outward. In general, you will find that up to a certain distance from the axis the pennies remain in contact, and beyond that point the pennies have slid outward. For several different trials, observe in each case the distance, $r$, to the axis of the last penny not to move, and also the angular velocity $\omega$. If the turntable has very little friction, you can determine $\omega$ by timing the first couple of rotations after you give it a push. Otherwise, you can instead measure the total time, $t$, it takes for the turntable to stop spinning, measured from your initial push. The initial angular velocity $\omega$ of the ruler is inversely proportional to $t$. To find the relation between $r$ and $\omega$ (or $r$ and $t$), we note that the maximum force of static friction: $f = \mu mg = ma_0^2 r$ = constant. Thus the ratio $a_0^2$ or $(r/t^2)$ should be the same for all spins. Specifically, if one spin lasts half as long as another, the last penny not to move should be four times further out from the axis for the briefer spin.

15. Zero net torque using pennies on a balanced ruler

Place pennies at various points along a ruler, such that the sum of the torques computed about the mid-point of the ruler is predicted zero for that placement. You may wish to tape the pennies in place so that they do not slide off. See if the ruler, in fact, balances at its center as predicted when placed horizontally on the edge of a pen or pencil of circular cross section. (If it does not you might want to check whether the ruler without any coins on it balances at its mid-point.) You can easily give the demonstration on an overhead projector, where the lack of balance can be best seen by looking at the ruler itself rather than its projection. By requiring that the sum of the torques equal zero at the ruler's mid-point, we need not consider the torque due to the weight of the ruler.

16. Conservation of linear momentum based on recoil speeds

In the absence of outside forces, two initially stationary objects that push off against each other must recoil with speeds inversely proportional to their masses. Find two smooth balls whose masses are in the ratio of two or three to one. Do not use rubber balls, but instead use balls made of various metals or glass, which have equal diameters of around 1 in. Incidentally, if you do use a glass ball (marble) as one of the two, be sure it is sufficiently round, and rolls smoothly. (Metal balls can be purchased at specialized companies, or more generally, companies selling educational scientific supplies.) Place the two balls in contact in the groove of a leveled ruler. Remember, if the demonstration is done on an overhead projector, you can always find one orientation for which the ruler is level. Place a folded index card sandwiched between the balls, and squeeze the card closed by finger pressure on the balls. When you suddenly remove your fingers, the unfolding card will gently drive the balls apart with equal and opposite momenta. The way to verify this without a stopwatch is to initially place the balls in contact at a point on the ruler where they should reach their respective ends of the ruler at the same moment. Thus if one ball is three times the mass of the other, its distance to the ruler end should be a third of the other. (You may wish to first show the case of two equal mass balls placed at the center of the ruler.) The advantage of using a folded index card, rather than, say a spring, to drive the balls apart is that the unfolding card applies a gentle force over some distance, and it is less sensitive to nonsimultaneous finger releases as a spring would be. Just be sure that you do not have sticky fingers.

17. Hooke's Law for a ruler clamped at one end

Clamp one end of a horizontal ruler to the edge of a table, or else just hold it firmly in place. Hang different masses from the end of the ruler, and observe its deflection in each case. Determine for what range of values the mass and deflection are linearly related (Hooke's Law), and find force per unit deflection, $k$, in Newtons per meter. Repeat the observations allowing the ruler to overhang the table by different amounts to see how $k$ depends on the amount of overhang.

18. Periods of oscillation of a vibrating ruler

Clamp one end of a horizontal ruler to a table, and pluck the free end causing it to vibrate. You can accurately measure the frequency of vibrations using a strobe. The predicted frequency of vibrations can be easily shown to be $f = (1/2\pi) \sqrt{3k/M}$, where $k$ is the force constant measured in the previous demonstration, and $M$ is the mass of the ruler. Try using different amounts of ruler overhang, and the appropriate value of $k$ in each case from the last demonstration. Also, try adding some mass, $m$, to the end of the ruler, and observe the frequency in this case. The predicted frequency can be found using the preceding equation with $M$ replaced by $M + 3m$.

19. Acceleration of the end of a vibrating ruler

An object undergoing simple harmonic motion of amplitude $A$ should have a maximum acceleration given by $a = \omega^2 A = (2\pi f)^2 A$. For a sufficiently large amplitude: $A > g/(2\pi f)^2$, the maximum acceleration, therefore, exceeds $g$. You can verify this relation using a vibrating ruler. Clamp one end of a horizontal ruler to a table with its grooved side down (or else just hold the end of the ruler on the table firmly in place by hand). You may find it convenient to decrease the ruler's vibration frequency by tapping about six pennies to its underside near its free end. (This will make the amplitude $A$ at which $a = g$ larger, and easier to measure.) Place one free penny on top of the ruler at its end. Pluck the ruler, and observe the penny as the ruler oscillates. You should find that for small oscillations the penny remains in contact with the ruler at all times, but for large oscillations it loses contact, causing an audible clatter. It may even jump off the ruler in some cases. Measure the largest vibration amplitude for which the penny remains in contact with the ruler at all times (no clatter). The penny should begin to lose contact with the ruler when the ruler's downward acceleration just exceeds $g$. See if the
20. Resonant vibrations of a ruler

Place a board having a thickness of about 2 cm perpendicular to a firm surface such as a table. Balance a horizontal ruler with its mid-point resting on the edge of the board. Press your thumb down very firmly on the center of the board, and pluck one end. Notice that the other ("free") end vibrates with large amplitude. These resonant oscillations occur because the vibrations of the plucked end cause small vibrations in the supporting board, which resonantly drive the free end. Now repeat the demonstration, but with the ruler held down firmly at a point other than its mid-point. The amplitude of oscillations at the free end will be smaller than before, because of the mismatch in the frequency of the "driving" oscillations (of the point of support), and the natural frequency of the free end. Obviously, the closer the point of support is to the mid-point of the ruler, the smaller the mismatch in frequencies, and the greater the amplitude in oscillations of the free end. A simple modification of the demonstration allows you to perform it on an overhead projector. Press the mid-point of the ruler against the vertical column that supports the projector lens, and the ruler vibrations will occur in a horizontal plane, making them readily visible on the screen.

21. Using a ruler "diving board" as a projectile launcher

This demonstration allows you to launch pennies vertically upward with a controllable velocity, so you can see if they rise to the predicted height. Clamp one end of a horizontal ruler to a table with its grooved side down (or else just hold it against the table firmly by hand). Tape one penny to the free end of the ruler, and observe its oscillation frequency with a strobe. The maximum velocity of the penny during its oscillation of amplitude, $A$, and frequency, $f$, can be expressed as $v = 2\pi f A$. Replace the taped penny by a free penny at the end of the ruler. Pull the end of the ruler downwards by a measured amplitude, $A$, and release it, causing the penny to fly upward. Have someone estimate its maximum height with a meter stick. For greater accuracy, you might want to average the results for five successive launches, using a fixed ruler deflection amplitude. The predicted maximum height is given by $y = v^2 / (2g) = 2\pi^2 f^2 A^2 / g$. You might want to try a range of deflection amplitudes. For larger amplitudes, you can make more precise measurements, but in these cases the penny is more likely not to fly directly upwards—a source of some error. This occurs because as the ruler swings upwards, the penny slides along the ruler, due to "centrifugal force." As a result, it leaves the ruler before the ruler reaches its equilibrium position. One way to avoid this source of error is to make a cardboard "ledge" at the end of the ruler, which prevents the penny from sliding off.

22. Wave source for a ripple tank

In a ripple tank, expanding circular waves can be created by a vibrating source that periodically dips into the water. When two nearby sources are driven in phase by a common vibrator, the resultant waves form an interference pattern that can be observed on an overhead projector, as shown in Fig. 2. You can make a ripple tank using a plastic box picture frame (the type used for photographs). A good size would be around 30 by 35 cm—which is not too large to fit on an overhead projector. Fill the frame with water to a depth of about a centimeter. Place several wooden blocks next to the tank, so that when one end of a ruler is held down on the blocks, the bulk of the ruler projects over the water like a spring board. Make a double wave source from the plastic ruler by taping two bent paper clips at its end. The straight ends of the clips should be at right angles to the ruler, and be about 2 cm apart. They should just reach the water surface when the ruler diving board is in place on the blocks. (If they do not try using different size blocks to rest the ruler on, or else vary the water level.) Put two 0.5 cm diameter clay balls at the ends of the paper clips, so that when the diving board is in place, the clay balls lie half below the water surface.

When the end of the ruler on the blocks is pressed down firmly, and the free end is plucked, circular waves emanate outward from each clay ball for a brief time while the oscillations last. You should observe a beautiful interference pattern on the overhead projector, with clearly defined directions for the minima in the pattern. For best results focus the projector on the ruler, rather than on the water surface, and allow the water to settle completely between trials. (Adding a shelving "beach" of folded card would minimize reflections of waves from the edges of the tank, but this is not really a problem here, since the reflected waves do not affect the interference pattern, which only lasts a second or two.) Try adding some mass to the end of the ruler, which slows the oscillations and makes them last longer. This last addition should also increase the wavelength, $\lambda$, from which you can predict the directions for minima: $\sin^{-1}\left(\frac{m + \frac{1}{2}}{d}\right)\lambda/d$, where $d$ is the source separation, and $m$ is an integer. By watching the waves below the ruler, you may be able to estimate their approximate
wavelength. Once you have estimated $\lambda$ for a given length of ruler overhang, you may wish to prepare a transparency on which you have drawn the directions predicted for minima, to place under the ripple tank.

23. Radius of gyration for different shapes rolled down an inclined ruler

Rest one end of a plastic ruler on a ledge about 3 mm high, and roll different shaped objects down the slight incline. Two shapes of particular interest would be the solid sphere, and the cylindrical shell (hoop). Both should be very smooth and round, and not so heavy as to deform the ruler as they roll down the incline. And, in addition, the cylindrical shell should not be too long, otherwise it will fall off the ruler as it rolls down. A good choice for the hoop might be a 5 cm length of 2 cm diameter tubing, but the dimensions are not especially important. When rolling the hoop you probably want to make the grooved side of the ruler facing down, but when you roll the ball, its grooved side should be up. For any round shaped object, the predicted time to roll down an incline of vertical descent $y$ is given by $t = \sqrt{\frac{2y}{g}}(1+k)$, where $k$ is the radius of gyration. Thus if you were to roll a sphere ($k = 2/5$), and a hoop ($k = 1$), down the same incline, the predicted ratio of times of sphere to hoop would be $\sqrt{0.7} = 0.84$. If you average the results from five or ten trials, you are likely to find a measured value no more than a few percent different from this value. (The advantage of working with ratios is that uncertainties in the height of the incline do not affect the results.) The results should be independent of the masses and radii of the bodies.

24. Oscillations of a ball in a potential well made from a flexed ruler

It can be shown that a solid ball rolling in a potential well whose radius of curvature is $r$ will oscillate about the bottom of the well with a small amplitude period of $T = 2\pi \sqrt{r/5g}$. To make a potential well, put one ruler on top of a second, and tape them together at the middle. (Both should have their grooved sides facing up.) Pry the ends of the rulers apart, and wedge a 2 cm thick piece of sponge or foam rubber in at each end, causing each ruler to flex into a concave-outward shape. Remove any tape that obstructs the groove on the top ruler. Place the two rulers on a horizontal surface, such as an overhead projector. When a smooth metal ball is rolled in the groove of the top ruler, it will roll back and forth between turning points equidistant from the center, with the oscillations persisting for a considerable time. Count a large number of oscillations, and see how well the period matches the small amplitude prediction for a solid ball: $T = 2\pi \sqrt{r/5g}$. If the ruler is bent into the approximate shape of the arc of a circle, the radius of curvature, $r_c$, can be found from: $r_c = l^2 / 2d$, where $l$ is the length of the ruler, and $d$ is the separation of the ends caused by the insertion of the sponge. (Even though the exact shape of a bent beam is noncircular, the approximation is not too bad in the present case, where the bending is small.) In addition to observing free oscillations, you can also demonstrate driven oscillations, and resonance, by very gently moving the rulers back and forth at the proper frequency and phase.

25. Oscillations of a ball in other types of potential wells

You can make an asymmetric potential well by inserting different thicknesses of sponge between the rulers at the two ends—using perhaps twice as much sponge at the left end as the right. The asymmetric potential well can be used to explain why most materials expand when heated. To show this, place a ball in the top ruler groove, and give it a push, causing the ball to oscillate. Observe the end points of each oscillation. Due to the asymmetry of the well, the right turning point (where the well rises less steeply), should be twice as far from the ruler center as the left one. Thus if you observe the oscillations over time as they gradually damp out, the mid-point of the oscillations should gradually move towards the center of the ruler. The idea is quite analogous to what happens as a material cools, and atomic oscillations become less vigorous. This causes the mid-point of the oscillations to shift towards smaller values, resulting in contraction of the material.

You could also make a double (symmetric) potential well. First tape a stack of two pennies to the underside of the middle of a ruler, and tape three pennies to the underside at each of its ends. Place the ruler on top of a second one, and press the rulers together at their 1/4 and 3/4 points, tapering them together there. Remove any tape that obstructs the groove. The result is a double well shape, whose middle hump is not as high as the rise at the two edges. A ball rolled from one end will make it over the middle hump if it has enough energy. Assuming it does not have too much energy, the ball will oscillate, and eventually settle down in one of the wells. You could also illustrate how, through the process of driven resonant oscillations, you can induce the ball, initially at the bottom of one well to make the transition to the other.

Using two such double wells placed side by side, you can give a demonstration of chaos on the overhead projector. Place a ball in the left well of each double well. With the balls initially stationary at the bottom of their respective wells, very gently start moving the double wells back and forth together. Observe how the balls oscillations about the bottom of the wells remain in phase. Now, slowly increase the amplitude of your shaking, and observe the change in the ball's behavior. As one or both of the balls nears or reaches the top of the hump separating the wells, the onset of chaos occurs—i.e., initially small differences in the position or speed of the balls (or the shapes of the wells), become greatly amplified, and the balls no longer oscillate in phase.

26. Oscillations of a ball in a rotating single well potential

Make a single well potential using two rulers, as described in Demonstration 24. Place the two rulers on a rotatable turntable, such as a "lazy Susan." (If you want to show this demonstration on an overhead projector, you can make a transparent low friction turntable using a ball bearing sandwiched between two lucite disks.) The center of the rulers should coincide with the center of the turntable. Place a smooth metal ball on the groove of the ruler, a distance $x$ from the axis, and rotate the turntable. If the turntable is rotated at a certain critical angular velocity, $\omega$, the centripetal force $ma\omega^2x$ just matches the component of gravity along the incline, $mg \tan \theta$, where $\theta$ is the angle of the ruler with the horizontal at a distance $x$ from the other.
If the curved ruler shape is parabolic (so that \( \tan \theta \) is proportional to \( x \)), the match between centripetal force and gravity along the incline wall will hold at all \( x \).

Similarly, if the centripetal force is greater than or less than the component of gravity along the incline, that imbalance will have the same sign at all points. As a consequence, a stationary ball at an arbitrary point in the groove should either move towards the center or away from it, depending on whether the ruler is rotated at less or more than a specific rotation rate. To find the critical rotation rate, we take the shape of the well to be given by \( y = 2d(x/l)^2 \), so that \( y' = \tan \theta = 4 \, dx/l^2 \), where \( d \) is the separation between rulers at their ends, and \( l \) is the length of the ruler. Requiring that \( mg \tan \theta = ma \) yields the critical angular velocity: \( \omega = \sqrt{4dg/l^2} \) rad/s. If you wish to find the critical angular speed experimentally, you need to rotate the turntable (by hand) at a slowly increasing speed, and see when a ball in the groove starts moving outward. When a ball does begin to fly outward you need to decrease the rotation speed slightly. In practice, continued speed adjustments may be necessary to keep the ball moving back and forth between the center and end of the ruler. You could then have someone measure the time for a specific number of rotations to find the angular speed, and see if it matches the predicted value. If your turntable is quite friction-free, you can observe that as you approach the critical angular velocity from below, the period of oscillations of the ball steadily increases. (Right at the critical angular velocity the ball can remain at rest at any point on the ruler.)

The reader may wonder whether a better way to do the demonstration would be to use a rotating turntable of known speed, such as a 33\( \frac{1}{2} \) rpm record turntable, and make the curvature of the ruler what it needs to be to match that particular speed. However, one problem with this alternative is that, in practice, the ruler curvature will either be slightly smaller or slightly larger than it needs to be for the particular speed. In the former case, the ball will stay at the center (or oscillate very slowly about the center), and in the latter case, it will move outward, so a controlled variable speed turntable is probably required.

27. Conservation of angular momentum in a rotating single well

Angular momentum conservation can be demonstrated if you have a turntable that has very low friction, is highly stable, yet not too massive—see the preceding description of one suitable for an overhead projector. You can demonstrate conservation of angular momentum by rolling a steel ball along a rotating single well potential made from two rulers (constructed as described in the preceding demonstration), which has been placed on a rotating turntable. You should be able to observe that as the ball passes through the mid-point of its oscillation in the well, the angular velocity of the turntable increases owing to the decrease in the moment of inertia of the system. How noticeable the effect is depends, of course, on the relative masses of ball and turntable, and on the amount of friction at the axle. It is also important that the turntable not tend to tip when the ball is at the extremes of its oscillations.

28. Stability of inverted pendulum

A compound pendulum in its inverted position is an example of a system in unstable equilibrium. The inverted pendulum can, however, be stabilized if the axis is oscillated in an appropriate manner. Make a light compound pendulum from a light piece of wood, such as a coffee stirrer or popsicle stick. Make a hole at one end of the stick, and place a thin nail through the hole to serve as an axle for the pendulum. When the nail is taped to the end of a horizontal plastic ruler the pendulum should be able to rotate freely in a vertical circle. Press one end of the ruler down at the edge of a table, so that the ruler projects forward like a diving board with the pendulum at its end. If you put the pendulum in the inverted position, and release it, it will, of course, topple over. However, if the end of the ruler is plucked, with the pendulum initially in the inverted position, it will not topple over for a few seconds, while the ruler is vibrating with sufficient amplitude—see Fig. 3. Why does this occur?

Consider the torque on the inverted pendulum when it is a small angle \( \theta \) away from its inverted orientation: \( \tau = -mgl\theta \), where the plus sign reminds us that this is not a restoring torque. In the absence of ruler vibrations, Newton's Second Law: \( I\ddot{\theta} = +\frac{1}{2}mgl\theta \), yields only divergent solutions of the form: \( \theta = A \cosh at + B \sinh at \). But, now suppose the ruler vibrates with a frequency \( \omega_0 \), so that the end of the ruler has an instantaneous acceleration \( a \sin \omega_0 t \). In the noninertial reference frame moving with the end of the ruler, according to the equivalence principle, we may take the acceleration of gravity to be \( g + a \sin \omega_0 t \), so that Newton's Second Law becomes

\[
I\ddot{\theta} = +\frac{1}{2}mgl(g + a \sin \omega_0 t).
\]  

(3)

Note, that when \( a > g \), then for part of each ruler vibration the torque on the pendulum is of the restoring type (negative sign). If this occurs for a sufficiently large fraction of each vibration (if \( a \) is large enough), then Eq. (3) admits an oscillatory solution about the vertical, and the inverted
ruler will not topple. However, when the ruler vibrations gradually damp out, the fraction of each oscillation over which the sign of the torque is negative decreases, and eventually the inverted ruler becomes unstable, and it topples.

There is still another reason that partly explains the stability of an inverted pendulum when the point of support oscillates. In practice, the end of the ruler moves in an arc of a circle, rather than straight up and down. Consequently, the pendulum rocks back and forth in a direction at right angles to the plane of its swings. This rocking motion causes the friction at the axle to increase, and helps stabilize the pendulum in the inverted position. A fuller explanation of the theory behind the stability of the inverted pendulum, and experimental studies using more sophisticated apparatus can be found elsewhere.9

29. Whirling a ruler on a string

Tie a string to the end of a ruler (through a hole), and rapidly whirl the ruler on the string in a vertical circle. Now, try it again, but this time, before you begin to whirl it, give the string enough twists so that the ruler starts to spin when hanging from the string. Begin whirling the ruler in a vertical circle once it has begun to spin rapidly. The difference observed in the two cases—whirling with and without spin—should be quite pronounced. You should find that when the ruler is whirled while spinning, a loud whirring sound is heard, and the resistance to rotation feels much greater—both phenomena being attributable to the greater turbulence that arises when the ruler is spinning.

30. Dependence of stopping distance on initial velocity

In this experiment you will observe a couple of pennies move across a table top, and observe how their stopping distances depend on their initial velocities. For this purpose, it is important that the two pennies have the same coefficient of kinetic friction. To check this, line up the two pennies against the edge of a plastic ruler. Rapidly push the ruler forward about 5 cm perpendicular to its length, so that when the ruler stops the pennies fly across the table together with a common speed. If the pennies travel the same distance before stopping, they must have the same coefficient of friction. Check pairs of pennies a few times, and select the best pair.

Now, we want to launch the pennies across the table at different speeds, for example, in the ratio 2:1. To accomplish this, you need to pivot the ruler, so that one penny before it flies away from the ruler, travels in a circular arc half the radius of the other. The technique is to use your left thumb as the pivot (against which the left end of the ruler rests). Push the right end of the ruler forward with your right hand, and use your left index finger to stop its rotation, thereby causing two pennies placed against its leading edge to fly off when the ruler stops. If one penny was initially twice as far from the pivot, it will fly off with twice the speed of the other when the ruler suddenly stops—see Fig. 4. According to the quadratic dependence of stopping distance on initial speed, the faster penny should travel four times further than the other. (You may want to draw a line showing where the rotating ruler is to be stopped by your finger, so as to facilitate measurement of stopping distances.)

31. Dropping and launching two coins with a ruler

The independence of horizontal and vertical motions requires that, ignoring air resistance, an object dropped from rest hits the ground the same moment as one simultaneously projected horizontally. You can easily use a ruler to launch a coin horizontally off a table at the same time you drop a second one from rest. To accomplish this hold the ruler horizontally, with one end resting on the edge of a table. Place a quarter on the table, so that if you sweep the ruler across the table, it will launch the quarter off the table in the horizontal direction. But, first place a second quarter on top of the ruler, so that during its sweep, the ruler moves out from under the coin, thereby causing its descent to begin at nearly the same moment as the other coin is launched horizontally. Both coins should hit the ground the same time, if we can neglect air resistance, which is reasonable for two pennies. However, if the same experiment is repeated using two very light objects, such as two pieces of styrofoam, the one that falls directly downwards hits the floor first. This is because it experiences an appreciably smaller vertical component of the force of air resistance (proportional to \( v^2 \)), since it travels more slowly.

32. Ruler diffraction grating

In a darkened room, place a laser on a table so that its beam shines on a wall several meters away. Align a ruler with the beam, and raise the far end of the ruler about a half inch higher than the end nearer the laser, so that the laser beam reflects off the ruler at near-grazing incidence. The placement of the ruler should be such that the beam
illuminates the finest markings near the edge of the ruler—either the millimeter markings on one edge, or the 16th in. markings on the other. If the alignment of the ruler is such that the laser beam hits the ruler very nearly at its edge (which is the only place a near-grazing beam can illuminate the spaces between the raised marks), you should see a diffraction pattern on the wall, from which the wavelength can readily be measured.\textsuperscript{10} (Although a plastic ruler can give good results, it cannot be denied that an appropriate steel ruler is better.)

33. Measuring the diameter of a fog droplet with a ruler

Let us not forget that we can also use a ruler to measure distance! If you view a distant white street light on a foggy night, you should observe a colored blob, or halo, centered on the light, with its outer edge being red. The blob is due to the combined interference patterns from many fog droplets; its angular diameter is inversely proportional to that of the average fog droplet. Hold up a plastic ruler at arms length, and measure the diameter of the outer edge of the blob, \(d\), in meters. Assuming your arm is roughly a meter long, the angular radius of the first minimum in radians is roughly \(\frac{1}{d}\). The diameter of the average fog droplet in nanometers can then be found from: \(D = 700 \div \sin \theta\), where we have taken the wavelength of red light to be 700 nm. Actually, you need not wait until the next foggy night to do the experiment. If you wear glasses, just fog them up with your breath, and view an unfrosted light bulb some distance away in a dark room.

34. Toppling ruler and the uncertainty principle

As this demonstration shows, the uncertainty principle, normally reserved for discussions of quantum systems has implications for macroscopic systems as well. Balance a ruler on end on a table, and allow it to topple over, while someone measures the toppling time with a stop watch. One way to make the ruler's initial orientation nearly vertical is to press it against a right angle, such as the vertical edge of a draftsman's triangle placed on a horizontal surface. A ruler placed on end topples over in accordance with the equation for a compound pendulum. Now, a simple pendulum of length \(l\), which is released from rest at an angle \(\theta\) with the positive \(y\) axis, reaches the bottom of its swing \((\theta = \pi)\), in a time: \(T = \frac{F(k)}{\sqrt{l/g}}\), where \(k = \sin \frac{\pi}{2} (\pi - \theta)\), and \(F(k)\) is the complete elliptic function of the first kind. The only difference for a compound pendulum, consisting of a uniform rod pivoting about one end, is that \(l\) is replaced by \(\overline{l}\). Now, consider a ruler standing on one end, which initially makes a small angle \(\theta\) with the vertical. The predicted time for the ruler to topple over \((\text{reach } \theta = \pi/2)\), is given by

\[
T = [F(k) - F(k')] \sqrt{\frac{2l}{3g}}.
\]

where \(k' = \sin \pi/4, \text{ and } F(k') = 2.08\). Now, for small \(\theta\) \((k \text{ close to one}), F(k) \text{ takes the asymptotic form}
\(F(k) = \frac{1}{2} [\ln(16/(1-k))]\). If we let \(\theta = 10^{-n}\) rad, so that \(k = \sin \frac{\pi}{2}(\pi - \theta) = 1 - 10^{-2n}/8 = 1 - 10^{-2n}/8\), we find that \(F(k) = 2.426 + 2.302n\). Substituting into Eq. (4) yields

\[
T = \sqrt{\frac{2l}{3g}}(0.346 + 2.302n).
\]

For \(\theta = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\) rad, Eq. (5) predicts toppling times of 0.382, 0.715, 1.046, and 1.388 s. Based on your measured time, you can use these results or Eq. (5) to estimate how closely the ruler was initially aligned with the vertical. Note, that increasing the ruler alignment with the vertical by a factor of a 1000 increases the toppling time only by 1 s. The infinite toppling time predicted for a perfectly aligned ruler is, of course, unattainable because of quantum fluctuations.

To find the uncertainty principle limit, start with

\[
\Delta L \Delta \theta > \frac{\hbar}{2},
\]

where we can express the uncertainty in angular momentum, \(\Delta L = l \Delta \omega = \hbar m \Delta \omega\), so that

\[
\Delta \omega \Delta \theta > \frac{3 \hbar}{2ml^2}.
\]

If either \(\Delta \omega\) or \(\Delta \theta\) is made too small, the resulting increased size of the other variable causes the toppling time to grow. The optimum relationship between the two variables can easily be shown to be

\[
\Delta \omega = \sqrt{\frac{3g}{2l}} \Delta \theta,
\]

from which Eq. (7) yields an optimum value of \(\Delta \theta\)

\[
\Delta \theta = \sqrt{\frac{3l^2}{2gm^2} (\hbar)^{1/4}}.
\]

A typical plastic ruler has a mass of about 11 g, for which Eq. (9) yields \(\Delta \theta = 1.5 \times 10^{-16}\) rad. Substituting this angle in Eq. (5) yields a maximum toppling time of 5.5 s.

II. CONCLUSION

This diverse collection of demonstrations with a plastic ruler was compiled to illustrate how much can be done with the simplest "equipment" to illuminate the principles of physics. In these days of tight budgets, the existence of such experiments and demonstrations have considerable value—though, of course, they should not serve as a rationale for reducing equipment budgets! But even apart from budgetary considerations, there is great value in experiments and demonstrations that use apparatus, which students are likely to find around the home. Could one come up with 34 demonstrations for some other simple objects? Perhaps there is another suitable object—but, that is for another paper.

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Thermodynamics of Crawford's energy equipartition journeys

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Energy equipartition "journeys" proposed originally in the context of relativistic mechanics [F. S. Crawford, Am. J. Phys. 61, 317–326 (1993)] are examined thermodynamically. Equipartition is between ideal gases separated by an insulating piston. Each journey alternates slow adiabatic volume changes and zero-work adiabatic piston resettings effected by a benevolent Maxwell's demon until temperature and pressure equalities exist. For the systems considered, reversible adiabatic thermodynamic processes are equivalent to slow mechanically adiabatic, constant-action processes. Because phase space mixing is assumed implicitly in thermodynamics, the piston jiggling step needed in Crawford's mechanics-based treatment is unnecessary. Zero-work journeys maximize the gas system's entropy change, and maximum work output journeys leave that entropy unchanged. Thermodynamics confirms Crawford's result for the piston's effective spring constant, and statistical mechanics enables rigorous justification of his interesting recursion relation for the moments of $q=momentum \times velocity$.

I. INTRODUCTION

Frank Crawford has given a creative and enlightening treatment of relativistic energy equipartition in ideal gases.¹ He used a massive, rigid, damped, insulating piston to couple distinct gases in two chambers of a cylinder (see Fig. 1). Based primarily on relativistic classical mechanics, his treatment illustrates novel ways to approach thermodynamic equilibrium. It shows that pressure and temperature equality can be achieved using constant-action (i.e., mechanically adiabatically invariant) work-generating processes, alternated with variable-action piston resettings (reminiscent of manipulations by James Clerk Maxwell's fictitious demon)² that do not alter the gas energies. "First-moment" equilibrium, $\langle pv \rangle_{gas} = \langle pv \rangle_{sol}$, where $p =$ momentum and $v =$ velocity, can always be attained in this way. Crawford shows that first-moment equilibrium is necessary but not sufficient for the achievement of complete thermodynamic equilibrium, and that the latter requires an additional "pot-stirring" process that randomizes gas particle velocities. This is because the purely mechanical model lacks a mechanism for changing the shape of an existing momentum distribution.

Crawford's work is important because it illuminates the critical interface between mechanics and thermodynamics. It shows that although purely mechanical processes can lead to both mechanical equilibrium and equal average energies per particle in the two chambers, they do not necessarily lead to complete equilibrium for which each gas has a proper equilibrium momentum distribution. The objective here is to give a thermodynamic analysis of Crawford's equilibration algorithms, providing a macroscopic alternative to his mechanics-based work.

The treatment here differs from Crawford's in three primary ways. First, our use of traditional classical thermodynamics accepts the implicit assumption that each gas is in complete thermodynamic equilibrium throughout reversible, adiabatic, nonzero work processes and after piston resettings. Without this assumption, one cannot speak meaningfully of temperature and cannot use common expressions for the internal energy and entropy. With this assumption, Crawford's extra pot-stirring process is not needed.

Second, while Crawford concentrates on equilibration and energy equipartition for generally relativistic particles, the emphasis here is on thermodynamic considerations of equilibration and equipartition for ideal gases of distinguishable particles that are not necessarily relativistic. For mathematical simplicity, we focus on systems that are either nonrelativistic (a classical ideal gas) or totally relativistic (a gas of photons).

Third, in Ref. 1, piston damping helps effect equilibra-