PHYS 1901: OSCILLATIONS, WAVES & CHAOS
Lecture Notes (Part 2): A Brief Introduction to Chaos Theory

1 Chaos

Chaos is the term used to describe the *apparently* complex behaviour of what we consider to be simple, well-behaved systems. Chaotic behaviour, when looked at casually, looks erratic and almost random. Before the development of chaos theory, such behaviour would have been attributed to one of two reasons:

1. Complexity. For example, a system with many moving parts (a box of gas, a galaxy of stars) has many "degrees of freedom" (variables needed to specify the state of the system at any given time). Such a system obviously has the possibility of showing very complicated behaviour.
2. Noise. If the behaviour of the system is strongly influenced by outside, random effects (e.g., temperature fluctuations, mechanical vibrations, etc.) then it might show complicated and unpredictable behaviour.

Chaos provides a third explanation for complicated behaviour. We know that a third explanation is necessary because we sometimes see complicated, apparently random behaviour is systems with only a few degrees of freedom that are not significantly affected by noise.

2 Sensitive dependence on initial conditions

The defining feature of chaotic behaviour is the *sensitive dependence on initial conditions*. Consider the following quotation:

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective positions of the beings which compose it, if moreover this intelligence were vast enough to submit these data to analysis, it would embrace in the same formula both the movements of the largest bodies in the universe and those of the lightest atom; to it nothing would be uncertain, and the future as the past would be present to its eyes.
>
> Pierre Simon De Laplace (1820)

In other words, provided we know the initial conditions of a system with sufficient accuracy, we can predict its future behaviour. Clearly this would be impractical in most cases, but it sounds plausible for simple systems (only a few degrees of freedom) that are not affected by noise. The first statement that such predictability might not always be possible, even for simple systems, is attributed to Poincaré:

> If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon.
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> Jules Henri Poincaré (1913)

A trivial example of the sensitive dependence on initial conditions is a ball at the peak of a hill. The ball could roll either way, and this clearly has a big influence on its future. However, the sensitivity to initial conditions only arises for one particular case. If the ball starts on one side of the hill, the sensitivity disappears. To describe motion as chaotic, we require sensitivity to initial conditions for a wide range of initial conditions.

To be even more precise, it turns out that chaos involves an *exponential divergence* of trajectories. Here, a *trajectory* refers to a particular way in which the system might evolve. If a system is chaotic then two trajectories that differ by Δx_0 in their initial conditions will, after time t, differ by

$$\Delta x(t) = \Delta x_0 e^{\lambda t}. \tag{1}$$

Here, λ is called the Lyapunov exponent and must be positive for chaotic behaviour.

Note that the future behaviour of a chaotic system *is* predictable, but only if the state of the system is known to infinite precision. In practice, this is never possible, and so any real prediction is useless after a certain time (how long?).

3 The ingredients for chaos

It turns out that for a system to show chaotic behaviour, two things are necessary:

1. The system must have at least three degrees of freedom. For example, a simple pendulum and a mass on a spring both have only two degrees of freedom. In each case, the state of the system (i.e., what the system is doing at any given moment) can be completely described by one position and one velocity. Adding a periodic driving force adds one degree of freedom (to describe the system at time t, we need to specify the phase of the driving force at that time). Making a single pendulum into a double pendulum adds two degrees of freedom (the position and velocity of the second pendulum).
2. There must be nonlinearity in the system. In a linear system, all the variables (position, velocity, pressure, etc.) appear in the equations to the first power (or not at all). If we "kick" a linear system twice as hard then the response will always be twice as large. In reality, all physical system are nonlinear at some level and linearity is only an approximation.

4 The Duffing nonlinear oscillator

The equation for a simple harmonic oscillator is linear:

$$m\frac{d^2x}{dt^2} + kx = 0.$$

This remains true, even when we introduce damping that is proportional to speed:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

To make the equation non-linear, we can modify the restoring force:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx + \beta x^3 = 0.$$

Note that we added a term with x^3 rather than x^2 because the latter does not change sign as x passes through zero. This equation is now nonlinear but the system will not exhibit chaos because it has only two degrees of freedom. But now we can add a periodic driving force

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx + \beta x^3 = F_{\mathrm{max}}\cos(\omega_{\mathrm{d}} t)$$

Now the system has one extra degree of freedom, since at any given time we need to specify the position and speed of the mass (x and $\frac{dx}{dt}$), and also the phase of the driving force ($\omega_{\mathrm{d}} t$). This system is called the Duffing oscillator and it exhibits chaotic behaviour for some (but not all) values of the parameters (m, k, b, β and ω_{d}).

5 The pendulum

The equation for a pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0,$$

which is nonlinear (except at small amplitudes, when $\sin\theta \approx \theta$). The physical pendulum (for example, if the string is replaced by a solid rod) has a similar equation:

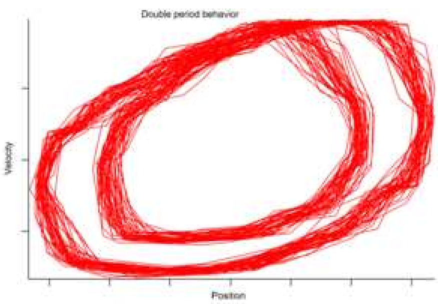
$$\frac{d^2\theta}{dt^2} + \frac{mgL}{I}\sin\theta = 0,$$

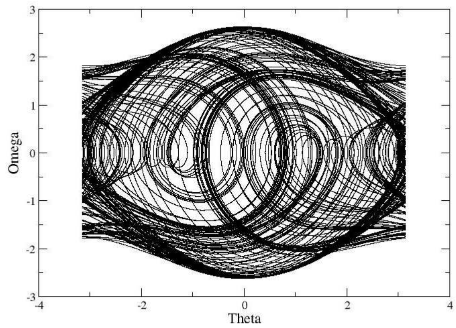
Both these systems have only two degrees of freedom and so will not exhibit chaos. Again, we can add a periodic driving force. We could do this by placing a charge on the bob and apply an alternating electric field, as simulated in the laboratory. A physical pendulum can also be driven by applying a periodic torque at the pivot point. A driven pendulum has three degrees of freedom and nonlinearity, and it exhibits chaos for some (but not all) values of the parameters.

We can also produce chaos with no driving at all, in a double pendulum. This is seen in the system in the ground-floor corridor of the Physics Building.

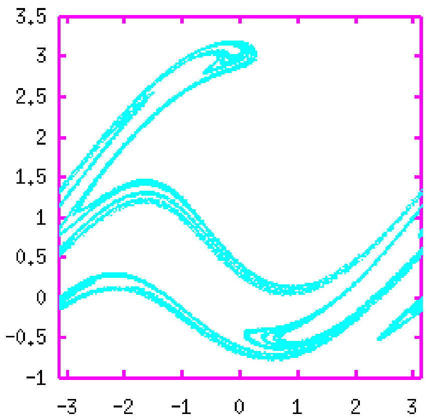
6 Visualizing chaotic behaviour

We are used to plotting position or velocity versus time. It is also helpful to plot velocity versus position (this is called a phase diagram). For an undamped harmonic oscillator, this traces a circular trajectory. For a damped harmonic oscillator, the trajectory spirals in towards the origin as the system loses energy. For nonlinear systems, we see much more complicated patterns. For example, we sometimes see a period doubling in which the system goes through two cycles before repeating itself, as shown in the left-hand figure. This is not chaos, but is often shown by a system that is approaching chaotic behaviour. For a chaotic system (right-hand figure), the trajectory does not repeat and eventually covers a large part of the phase diagram.





For a chaotic system, the phase diagram is very complicated. It is useful to instead plot a point at regular time intervals. Such a diagram is called a Poincaré section. For example, in a periodically driven system we can plot the position and velocity at exactly the same point in the drive cycle. The Poincaré section for a periodic oscillator is just a single point, and for a system that has undergone period doubling it is a pair of points. For a chaotic system it is a complicated pattern, such as the one shown below, which is for a periodically forced pendulum. Note that this has fine structure and looks very similar when you zoom in. This property of invariance under a change of scale is called self-similarity, and the structure is called a fractal.



7 Differential equations and maps

Differential equations (DEs) are central to physics. Most laws in physics, such as Newton's Second Law, are differential equations. (So are Maxwell's equations in electromagnetism, Schrödinger's equation in quantum mechanics and Einstein's field equations in general relativity.)

Let us look at the motion of a particle and examine the role played by the DE. Newton's Second Law says that the acceleration of an object at any instant is determined by the forces acting on it:

$$\frac{d^2x}{dt^2} = F/m. \tag{2}$$

The forces usually depend on the position and velocity of the object (x and $\frac{dx}{dt}$). Sometimes we can solve the DE and write down an explicit solution for position at all future times. For example, an object undergoing simple harmonic motion has the DE

$$m\frac{d^2x}{dt^2} + kx = 0.$$

which has a sinusoidal solution:

$$x(t) = A\cos(\omega t + \phi).$$

Another simple example is an object undergoing constant acceleration (e.g., in a uniform gravitational field), which has the DE

$$\frac{d^2y}{dt^2} = a; \quad \frac{d^2x}{dt^2} = 0.$$

The solution is motion in a parabola:

$$x(t) = x_0 + u_x t; \quad y(t) = y_0 + u_y t + \tfrac{1}{2}at^2.$$

It is important to realise that these are special cases: in most situations, we cannot write down the solution to the DE explicitly.

What does the object itself do? It does not solve the DE and "decide" to follow a sinusoid or a parabola. All it can do is react to the forces acting on it at each instant. If we want to calculate the trajectory of the object, we must do the same, as follows. The position of the object a short time from now, at time $t + \Delta t$, is given by its *current* position plus the *change* in its position during that small interval ($v\Delta t$):

$$x(t + \Delta t) = x(t) + \Delta t\frac{dx}{dt}.$$

Note that this follows directly from the definition of the derivative:

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$

So that tells us where the object will be a short time from now. But what will its velocity be? The same type of reasoning tells us that the velocity of the object at time $t + \Delta t$ is given by its *current* velocity plus the *change* in its velocity during that small interval ($a\Delta t$):

$$\left(\frac{dx}{dt}\right)_{t+\Delta t} = \left(\frac{dx}{dt}\right)_t + \Delta t \frac{d^2x}{dt^2}.$$

Again, this follows from the definition of the second derivative.

So now we know how to calculate both the position and velocity of the object a short time from now, provided we know the acceleration. Do we have to keep going with higher and higher derivatives? Well, no, because we know that the acceleration is determined by the forces via Newton's Second Law (Equation 2). The difficulty arises because the forces themselves also depend on position (and sometimes velocity). To calculate the trajectory of a particle using a computer is therefore an iterative process that is done in a series of small steps.[1]

Calculating the trajectory of a system by iteration, as described above, allows us to study quite complicated systems. For example, we can observe the chaotic behaviour that arises when the DE is nonlinear. This iterative process also reminds us of *maps* in mathematics, and it is not surprising that these maps can also show chaotic behaviour. Indeed, studies of the so-called Logistic Map:

$$x_{n+1} = \mu x_n (1 - x_n)$$

by Robert May (now Lord May, who was a student and then a lecturer and professor in this School of Physics), were of great importance in the development of chaos theory.

[1] Solving DEs with computers involves some clever tricks and is one of the topics covered in the Unit of Study COSC 1903 (Intro. to Computational Science), which is offered in Semester 2.