# Grip-slip behavior of a bouncing ball

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Measurements of the normal reaction force and the friction force acting on an obliquely bouncing ball were made to determine whether the friction force acting on the ball is due to sliding, rolling, or static friction. At low angles of incidence to the horizontal, a ball incident without spin will slide throughout the bounce. At higher angles of incidence, elementary bounce models predict that the ball will start to slide, but will then commence to roll if the point of contact on the circumference of the ball momentarily comes to rest on the surface. Measurements of the friction force and ball spin show that real balls do not roll when they bounce. Instead, the deformation of the contact region allows a ball to grip the surface when the bottom of the ball comes to rest on the surface. As a result the ball vibrates in the horizontal direction causing the friction force to reverse direction during the bounce. The spin of the ball was found to be larger than that due to the friction force alone, a result that can be explained if the normal reaction force acts vertically through a point behind the center of the ball. © *2002 American Association of Physics Teachers.*

## I. INTRODUCTION

In ball sports such as tennis, baseball, and golf, a fundamental problem for the player is to get the ball to bounce at the right speed, spin, and angle off the hitting implement or playing surface. Players approach the problem by trial and error and years of practice. Even then, most players have difficulties obtaining consistent and accurate results. Physicists have not done much better because the existing models are somewhat oversimplified. Approximate solutions of a ball bouncing at an oblique angle on a rigid surface are described by Brody[1] and Garwin.[2] Brody analyzed the bounce of a tennis ball and Garwin analyzed the bounce of a superball. The bounce models adopted by these workers are quite different. Garwin assumed that the collision is perfectly elastic in both the vertical and horizontal directions, implying that the vertical and horizontal components of the ball velocity at the contact point are both reversed by the bounce. In Brody's model, the collision is inelastic in the vertical direction and may be completely inelastic in the horizontal direction, in which case the contact point comes to rest and the ball then commences to roll during the impact.

Another difference between the Brody and the Garwin bounce models involves the friction force acting at the bottom of the ball. Garwin assumed that the friction coefficient was large enough for the bottom of the ball to grip the surface, allowing the ball to stretch in a horizontal direction while it is compressed in the vertical direction. Tennis players and commentators use the word ''bite'' rather than ''grip'' to describe a ball that kicks up off the court at a steep angle. Neither term provides an accurate description of conditions at the bottom of a ball when it bounces, but the term ''grip'' will be used to describe conditions where a significant fraction of the bottom of the ball is at rest on the surface. A better term is ''grip-slip'' because some annular sections of the ball in contact with the surface can slide or vibrate in a horizontal direction while other sections remain at rest.

Recovery of the horizontal component of the stored elastic energy results in enhanced ball spin, which is consistent with the fact that a superball spins faster than other balls of similar mass and diameter. Brody assumed that a ball incident without spin would commence to slide along the surface. Because sliding friction acts to reduce $v_x$, the horizontal component of the velocity, and to increase the angular speed $\omega$, Brody assumed that the ball would commence rolling if at some point $v_x = R\omega$, where $R$ is the radius of the ball. Such a result would be expected for a rigid ball impacting a rigid surface, in which case there would be no deformation of the ball or the surface. A ball that starts rolling before it bounces undergoes no further change in $v_x$ or $\omega$ because the coefficient of rolling friction is essentially zero. A ball that enters a rolling mode will therefore bounce with a larger horizontal velocity and smaller angular speed than one that grips the surface, other things being equal.

Experimental data indicate that a dry superball indeed grips the surface on which it bounces with the result that the ball spins faster after the bounce than one would expect from the rolling condition $v_x = R\omega$.[3,4] A superball spins so fast that it slides backward on the surface as it lifts off the surface. It was observed that a tennis ball can also spin slightly faster than allowed by the rolling condition, a result that can be attributed to partial recovery of elastic energy stored in a direction parallel to the surface.[4]

In this paper additional data are presented regarding the nature of the bounce of five ball types: a tennis ball, a superball, a baseball, a basketball, and a golf ball. The new data concern measurements of the friction force acting on the bottom of a ball. If a ball slides along a surface during the entire bounce period, the friction force is proportional to the normal reaction force and does not reverse direction during the bounce. If a ball enters a rolling mode, the friction force drops instantaneously to zero. The new experimental data show that all ball types grip the surface under conditions where they were previously thought to roll. Instead of dropping instantaneously to zero, the friction force decreases gradually to zero and then reverses direction during the bounce. It can reverse direction several times for some ball types. For example, the friction force on a basketball reverses direction six times when it bounces at an oblique angle on a surface.

Reversal of the friction force was predicted theoretically by Maw, Barber, and Fawcett.[5] These authors subsequently
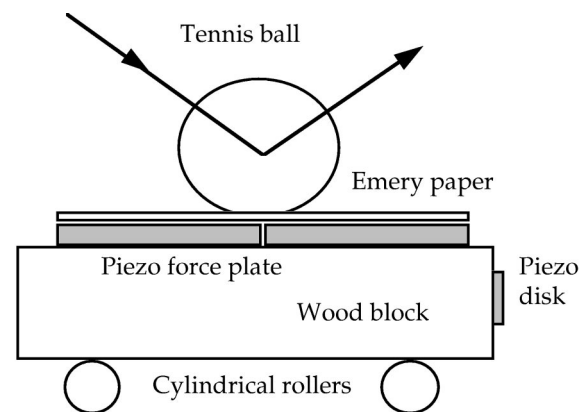
Fig. 1. Experimental arrangement used to measure the friction force and normal reaction force.
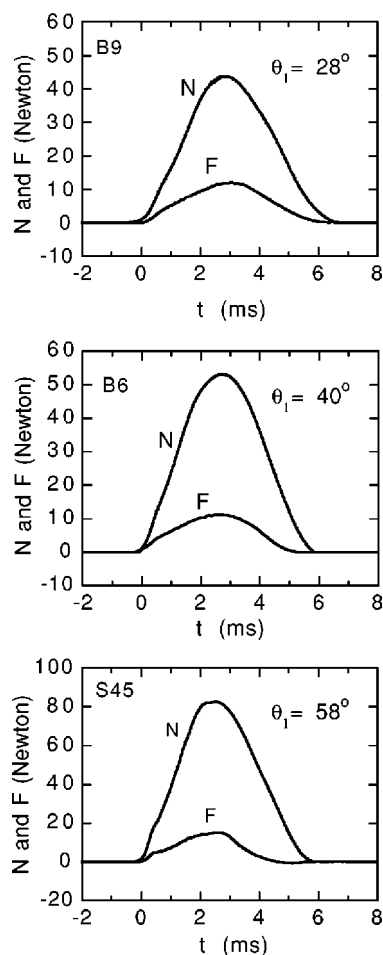


Fig. 2. Measurements of $N$ and $F$ for a tennis ball incident obliquely on a smooth surface; $\theta_1$ is the angle between the incident ball and the horizontal.

described measurements of the rebound angles for the oblique bounce of circular steel and rubber disks supported on an air table and colliding with steel and rubber blocks, respectively.[6] Their results indicate that steel balls also grip when they bounce. Their hardened steel disk was constructed by slicing up a 4-in.-diam ball bearing in order to retain a spherical surface. Measurements of the normal and friction forces for a steel ball impacting obliquely on a steel plate were described by Lewis and Rogers[7] and were analyzed by Stronge.[8] Instrumental problems and the short impact duration prevented Lewis and Rogers from observing a reversal in the direction of the friction force.

The reversal in the direction of the friction force can be attributed to two possible causes. The duration of the bounce is determined by the mass of the ball and by the stiffness of the ball in the vertical direction. If the bottom of the ball grips the surface, it will allow the ball to vibrate back and forth in the horizontal direction at a frequency that is determined by the tangential stiffness of the ball in the contact region. The static friction force will then vary in proportion to the horizontal stretch of the ball in the contact region. Alternatively, the ball may acquire sufficient spin during the bounce to slide backwards on the surface. Maw *et al.*[5,6] provide a numerical solution of the bounce problem indicating that both effects occur simultaneously as a grip-slip phenomenon, with some parts of the ball slipping and other parts gripping.

## II. EXPERIMENTAL TECHNIQUES

The arrangement used to measure the friction force on a bouncing ball is shown in Fig. 1. A wood block was supported on two cylindrical rollers so that the block could move freely in the horizontal direction with almost no frictional resistance. A ball incident obliquely on the block bounces off the block at an oblique angle, exerting a time-varying force on the block only during the brief period of the impact. For a tennis ball, the duration of the impact is typically about 5 ms. The force can be resolved into components perpendicular and parallel to the wood surface. The perpendicular component is equal in magnitude to the normal reaction force on the ball, and the horizontal component is equal in magnitude to the friction force on the ball. The horizontal component causes the block to accelerate in the horizontal direction. A 19-mm-diam, 0.3-mm-thick ceramic piezo disk was fastened to one end of the block as a simple and inex-

pensive ($2) accelerometer. Piezo disks of this type can be extracted from piezo buzzers or musical greeting cards. The output voltage from the piezo disk is directly proportional to the acceleration of the block, thus providing a direct measurement of the time-varying friction force acting on the bottom of the ball. The only signal processing required was to connect an external 20-nF capacitor in parallel with the piezo disk to extend the time constant well beyond the duration of the impact. If the output signal is monitored with a high impedance probe (for example, a standard 10-MΩ voltage probe), then the output signal will provide a reliable measure of the friction force for times up to 20 ms or more.

Ideally, we would like to measure the friction force acting on a block of effectively infinite mass, but such a measurement would be more difficult. In any case, there is no fundamental difference between the bounce off a moving or accelerating surface and the bounce off a surface that remains at rest. The horizontal speed and spin of the ball may be altered by the motion of the block as described in Sec. VI, but these effects can be minimized by using a block that is much heavier than the ball. Also, there is no effect at all if the ball slides on the block throughout the bounce. For the low speed bounces studied in this experiment, a block mass of 340 g (about six times the mass of a tennis ball) was used to obtain an adequate output signal from the accelerometer. A
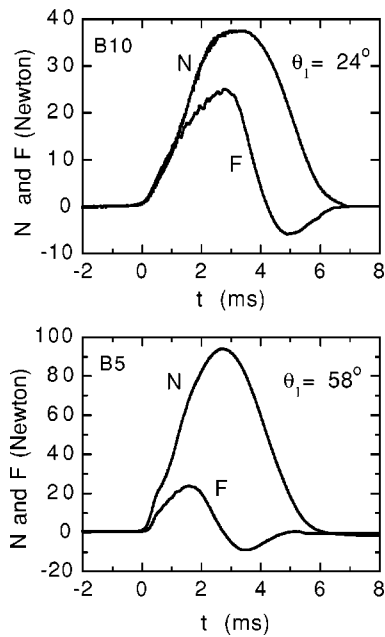
Fig. 3. Measurements of $N$ and $F$ for a tennis ball incident obliquely on P800 emery paper.



Fig. 5. Measurements of $N$ and $F$ for a superball incident obliquely on a smooth surface.

study the bounce off a surface with a higher coefficient of friction. It is referred to in the following as the high friction surface.

Bounces at several different angles on both surfaces were filmed at 100 frames/s with a digital video camera to determine the ball speed, spin, and angle before and after each bounce. Video clips were transferred in real time via a firewire connection to an iBook computer for analysis. The horizontal and vertical distance scales were calibrated simply by measuring the actual and image diameter of each ball because any other object or scale would necessarily lie in a different plane and introduce parallax errors.

## III. FRICTION FORCE RESULTS

Simultaneous measurements of the normal reaction force, $N$, and the horizontal friction force, $F$, are shown in Figs. 2–6 to illustrate and characterize the bounce of a ball under a variety of impact conditions. Figure 2 shows the bounce of a new tennis ball on the low friction surface, and Fig. 3 shows the bounce on the high friction surface. Figure 4 illustrates the case of a tennis ball incident with topspin on the high friction surface. Figure 5 shows the bounce of a superball and Fig. 6 shows the bounce of a baseball and a basket-

block with a larger mass could be used if a more sensitive accelerometer were available or if one wished to study high speed collisions.

A ball dropped vertically onto the wood block gave zero output from the piezo disk, demonstrating that the piezo disk responded only to acceleration of the block in the horizontal direction. However, to achieve this result it was necessary to provide some vibration isolation by fastening the piezo disk to a block of rubber attached to one end of the wood block. Two large area ceramic piezo blocks were mounted on top of the wood block so that both the vertical and horizontal components of the force on the wood block could be determined simultaneously. Each of the large piezo blocks was 51 mm square and 4 mm thick. They were connected electrically in parallel to act as a single, large surface area (51 mm×102 mm) force plate. The upper surface of the plate was mechanically protected using a 0.3-mm-thick circuit board attached directly to the plate with double-sided adhesive tape. This surface is quite smooth and is referred to below as the low friction surface. For some experiments, fine grade (P800) emery paper was taped firmly to the circuit board to
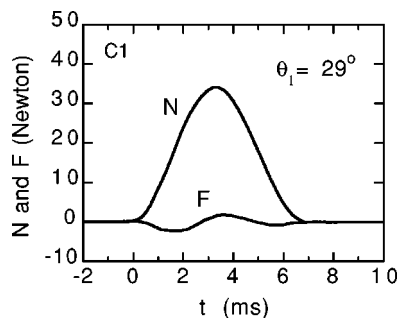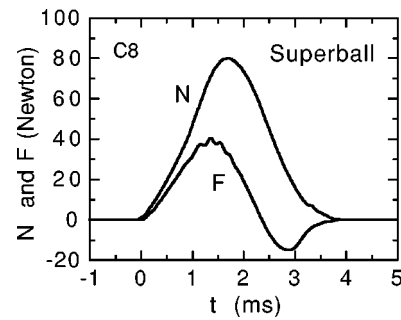


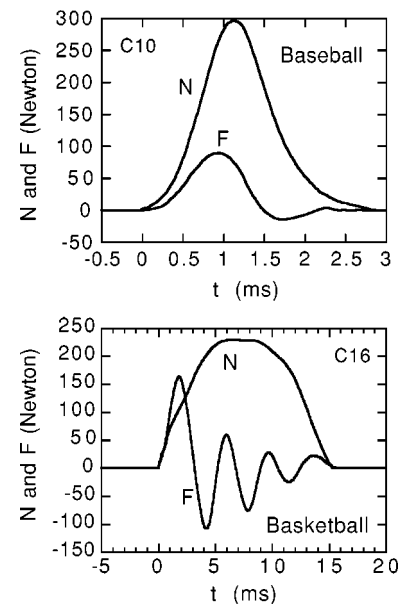Fig. 4. Measurements of $N$ and $F$ for a tennis ball incident obliquely with heavy topspin on P800 emery paper.





Fig. 6. Measurements of $N$ and $F$ for (a) a baseball and (b) a basketball incident obliquely on a smooth surface.

| Ball | Mass $m$ (g) | Radius $R$ (mm) | $\alpha$ |
|---|---|---|---|
| Tennis | 57.4 | 33.0 | 0.55 |
| Superball | 46.4 | 23.0 | 0.40 |
| Golf ball | 45.5 | 21.3 | 0.40 |
| Baseball | 149.4 | 36.5 | 0.40 |
| Basketball | 589 | 120 | 0.66 |

ball, all three balls being incident on the low friction surface. In all cases, the ball was incident at low speed, typically about 3 m/s. The properties of each ball are listed in Table I and a summary of each bounce is given in Table II.

The results in Fig. 2 demonstrate that a tennis ball incident on a low friction surface bounces without significant reversal of the friction force during the bounce. At low angles of incidence the ball slides throughout the bounce, and the ratio F/N remains approximately constant during the bounce. In Fig. 2(a), F/N=0.27±0.03. At higher angles of incidence, the friction force drops to zero during the bounce, and it drops to zero earlier as the angle of incidence was increased. These results are qualitatively consistent with the bounce model described by Brody, who predicted that the ball would start to roll at a progressively earlier stage as the angle of incidence was increased. However, there is no sudden transition from sliding to rolling during the bounce, and the ball was observed to spin faster than allowed by the rolling condition as indicated in Table II. The fact that the friction force dropped to zero during the bounce can be interpreted to mean either that the ball rolled or that the friction force was positive in some areas of contact and negative in others. For reasons described in more detail below, it can be inferred that the ball did not roll during the bounce, but it first gripped the surface and then commenced to slide backward in the manner predicted in Ref. 5.

In practice, the coefficient of sliding friction between a tennis ball and a court surface is usually about 0.5 or larger. On such a surface the ball is predicted by Brody[1] and by Maw et al.[5] to slide throughout the bounce if the angle of incidence is less than about 15°. At higher angles of incidence, Brody predicted that the ball would enter a rolling mode and hence the friction force would drop to zero. Figure 3 shows that at these higher angles of incidence, the bounce of a tennis ball on a high friction surface is characterized by a significant reversal of F during the bounce. The reversal in the direction of F occurs earlier in time as the angle of incidence is increased, allowing two reversals to occur in Fig.

3(b). In Brody's model, the ball rolls at an earlier time as the angle of incidence is increased, but F does not reverse direction.

Figure 4 shows the effect of significant topspin on the incident ball. To generate topspin, a tennis ball was allowed to roll down an inclined ramp so that it was incident at an angle of 29° to the horizontal and spinning at 75 rad/s. In this case, the initial contact point slides backwards on the surface at impact, generating a small negative friction force at the beginning of the impact. A rigid ball would quickly start to roll after the initial sliding stage. The experimental result is qualitatively consistent with Brody's rigid ball model in that F remains quite small throughout the bounce. F reverses direction twice during the bounce with the result that the time integral of F is almost zero. Consequently, the changes in horizontal velocity and spin of the ball were also very small.

The bounce of a superball incident without spin at 36° on a low friction surface is shown in Fig. 5. Replacing the low friction surface with emery paper made a negligible difference to the bounce. The behavior of the friction force is qualitatively similar to that for a tennis ball bouncing on a high friction surface in that F is almost equal to N during the early stage of the bounce and F reverses direction toward the end of the bounce period. However, the effect on the ball is different in that a superball spins much faster than a tennis ball after the bounce. The peripheral velocity of the superball after it bounced, $R\omega_2$, was 1.75 times larger than its horizontal velocity $v_{x2}$. For the tennis ball in Fig. 3, $R\omega_2$ is about 1.1 times larger than $v_{x2}$. As described below, the ratio $R\omega_2/v_{x2}$ increases slightly if $v_{x2}$ is measured with respect to the speed of the block on which the ball bounces. In this experiment, the block translated at a speed typically about $0.07v_{x2}$.

The bounce of a baseball incident without spin at 41° on the low friction surface is shown in Fig. 6(a). The impact duration is shorter than that for the tennis ball or the superball, but it is longer than the usual 1.0 ms impact duration of a high speed ball impacting on a bat.[9] The result is almost consistent with rolling during the bounce, but a small negative F arises because the ball spins backward on the surface as it bounces, with $R\omega_2 = 1.13v_{x2}$.

Figure 6(b) shows the bounce of a basketball incident with almost zero spin at 66° on the low friction surface. It is especially obvious in this case that the ball vibrates horizontally during the bounce, causing the friction force to reverse direction six times. The half period of oscillation is 15 ms in

Table II. Ball bounce data ($v_1$ in m/s, $\omega$ in rad/s, $\Delta(mv)$ and $\Delta(I\omega)/R$ in gm m/s). Experimental errors are typically 2 or 3%.

| Ball | Surface | $\theta_1$ | $v_1$ | $\omega_1$ | $\omega_2$ | $v_{x2}/v_{x1}$ | $R\omega_2/v_{xR}$ | $\Delta(mv)$ | $\Delta(I\omega)/R$ | $e_y$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Tennis | Smooth | 28° | 3.26 | 0 | 48.0 | 0.76 | 0.76 | 39.2 | 50.1 | 0.72 |
| Tennis | Smooth | 40° | 2.68 | 0 | 49.1 | 0.71 | 1.20 | 34.7 | 51.2 | 0.77 |
| Tennis | Smooth | 58° | 3.08 | 0 | 37.1 | 0.63 | 1.30 | 33.8 | 38.7 | 0.77 |
| Tennis | Rough | 24° | 3.31 | 0 | 67.6 | 0.68 | 1.19 | 55.5 | 70.6 | 0.77 |
| Tennis | Rough | 58° | 3.50 | 0 | 44.7 | 0.74 | 1.14 | 27.5 | 46.7 | 0.74 |
| Tennis | Rough | 29° | 2.55 | 74.9 | 73.5 | 0.98 | 1.11 | 2.8 | −1.5 | 0.66 |
| Superball | Smooth | 36° | 2.69 | 0 | 98.4 | 0.59 | 1.90 | 40.8 | 42.0 | 0.91 |
| Baseball | Smooth | 41° | 2.25 | 0 | 39.7 | 0.76 | 1.32 | 60.4 | 86.6 | 0.39 |
| Basketball | Smooth | 66° | 2.39 | 0.44 | 4.22 | 0.70 | 1.27 | 154 | 176 | 0.86 |

the vertical direction, and the period of oscillation in the horizontal direction is about 4 ms. There is no simultaneous 250 Hz oscillation in the $N$ waveform, indicating that this mode of oscillation involves purely tangential displacements in the wall of the ball rather than transverse displacements. Any vertical motion in the wall at 250 Hz would show up in the $N$ waveform as a result of changes in the pressure acting on the upper force plate. High frequency transverse oscillations were observed at the top of the ball by attaching a small piezo disk to the top of the ball, but they were not at 250 Hz.

## IV. BALL SPEED AND SPIN MEASUREMENTS

Each of the bounces described in Sec. III was filmed with a video camera to obtain simultaneous measurements of the ball speed and spin and the speed of the block. In all cases the change in the horizontal momentum of the ball was equal to the momentum of the block after the collision (within experimental error) as expected. The change in horizontal momentum of the ball is equal to the time integral of $F$. This information was used to calibrate the response of the piezo disk used to record the $F$ waveform. Similarly, the piezo blocks used to record the $N$ waveform were calibrated by using the fact that the time integral of $N$ is equal to the change in vertical momentum of the ball.

A summary of the information obtained for each bounce is given in Table II. The angle $\theta_1$ is the angle between the incident ball and the horizontal, $v_1$ is the incident ball speed, $\omega_1$ is the angular velocity of the incident ball, and $\omega_2$ is the angular velocity of the ball after bouncing. The horizontal velocity $v_{xR} = v_{x2} - V_2$ is defined with respect to the horizontal velocity $V_2$ of the block after the bounce. The ratio $R\omega_2/v_{xR}$ equals 1.0 if the ball commences to roll during the bounce and is less than 1.0 if it slides throughout the bounce. A value $R\omega_2/v_{xR} > 1$ implies that the ball slides backwards on the surface at the end of the bounce period.

In Table II, the quantity $\Delta(mv) = m(v_{x1} - v_{x2})$ is the change in horizontal momentum of the ball and $\Delta(I\omega) = I(\omega_2 - \omega_1)$ is the change in angular momentum of the ball. In theory, the change in angular momentum of the ball is given by the time integral of the torque $FR$. For the low speed impacts studied in this paper, $R$ is essentially constant. A video of the ball in contact with the block indicated that $R$ decreased by no more than 3 or 4 mm during the bounce, which is consistent with the relatively small values of the normal reaction force and the known stiffness of each ball type. However, the quantity $R\int F\,dt$ was consistently less than the change in angular momentum of the ball, typically by 30 or 40% (except for the superball where it was only 3% smaller). Allowing $R$ to be smaller than the actual ball radius makes the difference even greater. This result indicates that the torque acting on the ball is significantly larger than $RF$. Because the only other force acting on the ball is $N$, the normal reaction force, there must be a time interval during the impact when $N$ acts vertically through a line passing a distance $D$ behind the center of mass. The additional torque $ND \approx 0.3FR$, where $F$ is typically about $N/4$ so $D$ is typically about $R/10$. Such a result indicates that the ball tends to lean forward during the bounce.

A tennis ball served with heavy topspin is called a "kick" serve because the ball bounces at a steep angle off the court and bounces typically around head height. Players and commentators often remark that the ball "really bites" in this situation, but the results in Fig. 4 and in Table II are not consistent with this interpretation. The ball did not bounce steeply and it did not grip strongly. In a kick serve the ball is incident at much higher speed than in Fig. 4 and spins much faster. The ball will kick up at a steep angle if there is large decrease in the horizontal velocity or if the vertical coefficient of restitution is enhanced. Our evidence shows that the horizontal velocity is not decreased substantially when the ball is incident with significant topspin. The fact that the ball bounces to around head height indicates that the vertical coefficient of restitution is enhanced under conditions where the ball rotates by a significant fraction of a revolution during the bounce.[4] In Fig. 4 the ball rotated by only 25° during the bounce.

## V. QUALITATIVE EXPERIMENTS

Several experiments were conducted to assist in the interpretation of the above measurements. The first was to squash four large superballs in a shallow wood box in such a way that the balls could be dragged across a surface without rolling, and lead bricks could be placed on the upturned box to increase the normal reaction force. When a small horizontal force is applied to the box, the bottom of the balls remain stuck to the surface by static friction. However, the box itself was observed to move horizontally through a small distance due to the fact that the balls stretch slightly in the horizontal direction. At a sufficiently large value of the horizontal force, the balls release their grip and start to slide in the direction of the force. A ball can therefore store elastic energy due to the fact that it stretches horizontally when subject to a horizontal force. A central feature of the bounce model of Ref. 5 is that when a ball releases its grip and starts to slide, it does so progressively. The edge of the contact area slips first, while the central part of the contact area remains stuck because the normal reaction force acting in the central region is larger than that at the edge of the contact area. As the horizontal force increases, the area that is stuck shrinks until the whole contact area slides.

A dynamic version of the above experiment was performed by gluing a tennis ball to a 260-g wood block and attaching a small piezo disk to the side of the ball. A 12-kg load was placed on top of the ball while the glue dried so that a large circular area of the ball adhered to the block. After drying, the block was rotated by 90° and dropped on a horizontal surface to excite tangential oscillations in the ball. In this orientation the piezo disk was at the bottom of the ball and the ball vibrated in a vertical direction. The excitation of high frequency transverse modes was minimized because the ball did not compress in a direction perpendicular to the vertical surface of the wood block. The ball was observed to undergo five damped oscillations with a period of 5.2 ms. Because the impact duration of a low speed tennis ball is about 6 ms, the ball can undergo a fraction more than one complete cycle of tangential oscillation during the bounce. The data shown in Fig. 3(b) are consistent with this result. In Fig. 3(a), the friction force reverses direction only once, indicating that the bottom of the ball gripped the surface at a later stage of the impact than in Fig. 3(b).

It is well known that each contact point on the circumference of a rigid ball comes to rest momentarily on a surface when the ball rolls. When the ball is flexible and subject to a vertical force, the ball squashes and the contact point enlarges to a flat, circular area. To investigate whether some or
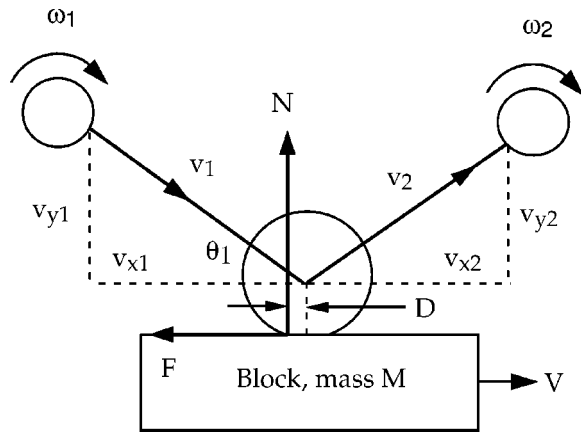
Fig. 7. Geometry of a ball of mass $m$ incident obliquely on a block of mass $M$.

all contact points remain at rest when the ball rolls, a large and relatively soft rubber ball was rolled on a table under a thick plate of glass to observe the effects by eye. The ball was marked by a series of dots around a circumference and a straight line was drawn with a felt pen across the width of the glass plate. The dots and the line on the plate were made to coincide by pushing down on the glass plate, and then the plate was pushed sideways by hand to allow the ball to roll. Each dot remained attached to the line without slipping until it reached the edge of the contact area and rotated away from the glass. Consequently, a squashed ball can roll on a surface in such a way that all points in contact with the surface remain at rest on the surface. When the same experiment was repeated with a tennis ball, the dots on the ball gradually slipped behind the line as the ball rolled forwards, indicating that the low coefficient of friction between a tennis ball and the glass plate allowed both grip and slip to occur. In the absence of friction the plate would simply slide across the top of the ball and the ball would not roll forwards.

## VI. BOUNCE MODELS

We now consider the bounce models developed by Brody and Garwin, modified to include motion of the block and an off-center normal reaction force. In Sec. VII we will compare the results of these simplified models with the more complete analysis of Maw et al.[5]

Consider a ball of mass $m$ and radius $R$ incident at speed $v_1$, angular velocity $\omega_1$, and at an angle $\theta_1$ on a block of mass $M$, as shown in Fig. 7. We can ignore the gravitational force because it is much smaller than the impact force even for a low speed bounce. The equations of motion for the ball are $N=m dv_y/dt$ and $F=-m dv_x/dt$, where $N$ is the normal reaction force, $F$ is the friction force acting parallel to the surface, and $v_x$, $v_y$ are the velocity components of the center of mass of the ball parallel and perpendicular to the surface, respectively. If $N$ acts through a point a distance $D$ behind the center of mass, then $FR+ND=I d\omega/dt$, where $I$ is the moment of inertia about an axis through the center of the ball. The moment of inertia of a spherical ball is given by $I=\alpha m R^2$, where $\alpha=2/5$ for a uniform solid sphere and $\alpha=2/3$ for a thin spherical shell. A tennis ball can be approximated as a spherical shell with $I=2m R_1^2/3$, where $R_1$ is the average radius of the shell. The wall is typically about 6 mm thick, including a 3-mm-thick outer cloth cover. For the later

calculations, we will take $R=33$ mm, $R_1=30$ mm, and $\alpha=0.55$. In a high speed impact a tennis ball may squash in half but for the low speed impacts studied in this paper, the ball radius remains approximately constant during the bounce. For simplicity, we also assume that $D$ remains constant throughout the bounce or that the effect of a time-varying $D$ can be represented by a constant value of $D$.

The motion of the block is described by the relation $F=M dV/dt$. The rebound speed $v_2$, spin $\omega_2$, angle $\theta_2$, and the final block speed $V_2$, can be determined by taking the time integrals of $N$ and $F$ over the impact interval $\tau$, so that

$$\int_0^\tau F \ dt=m(v_{x1}-v_{x2})=MV_2 \tag{1}$$

$$\int_0^\tau N \ dt=m(v_{y2}-v_{y1})=-m(1+e_y)v_{y1}, \tag{2}$$

and

$$R\int_0^\tau F \ dt+D\int_0^\tau N \ dt=\alpha m R^2(\omega_2-\omega_1), \tag{3}$$

where $e_y=-v_{y2}/v_{y1}$ is the coefficient of restitution in the vertical direction, and where $v_{y1}$ is negative because the ball is incident in the negative $y$ direction. We require a relation between $F$ and $N$ or a statement regarding energy conservation to determine the final state of the ball. We consider three possibilities, as follows.

(a) Pure sliding. If the ball slides throughout the bounce, then $F=\mu N$, where $\mu$ is the coefficient of sliding friction. In this case it can be shown from Eqs. (1)–(3) that

$$v_{x2}=v_{x1}+\mu(1+e_y)v_{y1}, \tag{4}$$

and

$$\omega_2=\omega_1-(R\mu+D)(1+e_y)v_{y1}/(\alpha R^2). \tag{5}$$

These relations are independent of the mass of the block because the friction force on the ball does not depend on the mass or speed of the block. An interesting consequence is that the total energy loss is independent of the mass of the block even though the kinetic energy transferred to the block does depend on the mass of the block.

(b) Slide then roll. The bottom of the ball will come to rest on the block just at the end of the impact period if $v_{x2}-R\omega_2=V_2$, in which case we find from Eqs. (1), (4), and (5) that

$$v_{x2}=\frac{R\omega_1+[m/M+1/\alpha+D/(\mu\alpha R)]v_{x1}}{[A+D/(\mu\alpha R)]}, \tag{6}$$

and

$$\mu=\frac{R\omega_1-v_{x1}}{(1+e_y)Av_{y1}}-\frac{D}{\alpha RA}, \tag{7}$$

where $A=1+1/\alpha+m/M$. If $\mu$ is smaller than the value given by Eq. (7), then the ball will slide throughout the bounce. If $\mu$ is larger, then the bottom of the ball will come to rest on the surface before the end of the impact period. A rigid ball would start rolling if the bottom of the ball comes to rest in which case the friction force would drop rapidly to a negligible value. Because there is no further change in spin or horizontal speed once a ball starts rolling, the final speed

and spin of the ball is independent of the time at which the ball starts to roll.

For the tennis ball results described above, $e_y = 0.75 \pm 0.02$. If we take $D=0$ and $\omega_1 = 0$, then Eq. (7) indicates that the ball will slide throughout the bounce if $\tan \theta_1 < 0.19/\mu$. On the smooth surface, $\mu = F/N = 0.27$, and hence the ball will slide if $\theta_1 < 35°$. This prediction is consistent with the results shown in Fig. 2. On the rough (emery) surface, $\mu$ varied from 0.7 to 1.0 depending on the incident angle. If we take $\mu = 0.7$ as a lower limit, then the ball will slide throughout the bounce only if $\theta_1 < 15°$. This prediction is consistent with the results shown in Fig. 3 because in both cases the ball did *not* slide throughout the bounce. Rather, the ball gripped the surface during the bounce, causing $F$ to reverse direction.

(c) *Slide then grip*. Two approaches can be used to describe a ball that grips the surface when it bounces. One is to analyze its dynamical behavior numerically. The other is to ignore the dynamics and characterize the bounce in terms of the measured coefficients of restitution. The vertical bounce velocity of a ball is rarely calculated from first principles. It is more commonly specified by the measured vertical coefficient of restitution, $e_y$. For example, a tennis ball bounces on a rigid surface with $e_y$ typically about 0.75. In the present context, the horizontal coefficient of restitution, $e_x$, can be defined by the relation

$$e_x = -\frac{(v_{x2} - R\omega_2 - V_2)}{(v_{x1} - R\omega_1)}, \qquad (8)$$

where $v_x - R\omega - V$ is the horizontal speed of a point at the bottom of the ball with respect to the block. This definition yields the result that $e_x = 1$ for a perfectly elastic ball with no energy losses. If the ball rolls along the block before bouncing, then $e_x = 0$. Garwin[2] provided an elegant description of a superball simply by assuming that $e_x = e_y = 1$, but this approach does not provide any insights as to what actually happens during the bounce.

Unlike $e_y$, $e_x$ can be positive or negative. If a ball is incident at sufficiently small $\theta_1$ and without spin, then it can slide throughout the impact and will bounce with $R\omega_2 < (v_{x2} - V_2)$, in which case $e_x < 0$. A value $e_x = -1$ corresponds to a bounce on a frictionless surface where $v_{x2} = v_{x1}$ and $\omega_2 = \omega_1$. Alternatively, $e_x = -1$ if a ball starts rolling at the beginning of the bounce and continues rolling throughout the bounce. If a ball grips the surface, then $e_x > 0$, but if the elastic energy stored in the horizontal direction is not completely recovered, then $e_x < 1$.

The torque acting on the ball is given by

$$FR + ND = -Rm(dv_x/dt) + Dm(dv_y/dt) = I\,d\omega/dt. \qquad (9)$$

Conservation of angular momentum about a point at the bottom of the ball is therefore described by the relation

$$I\omega_1 + mRv_{x1} - mDv_{y1} = I\omega_2 + mRv_{x2} - mDv_{y2}. \qquad (10)$$

Equations (8) and (10) can be solved to show that

$$v_{x2} = v_{x1} - \frac{(1+e_x)(v_{x1} - R\omega_1)}{A} - \frac{D(1+e_y)v_{y1}}{\alpha RA}, \qquad (11)$$

and

Table III. Bounce parameters for each of the bounces in Table II.

| Ball | Surface | $\theta_1$ | $e_x$ | $e_y$ | $D$ (mm) |
|---|---|---|---|---|---|
| Tennis | Smooth | 28° | −0.17 | 0.72 | 2.3 |
| Tennis | Smooth | 40° | 0.13 | 0.72 | 3.1 |
| Tennis | Smooth | 58° | 0.17 | 0.77 | 0.5 |
| Tennis | Rough | 24° | 0.12 | 0.77 | 3.8 |
| Tennis | Rough | 58° | 0.10 | 0.74 | 2.1 |
| Tennis | Rough | 29° | −1.0 | 0.66 | −1.2 |
| Superball | Smooth | 36° | 0.49 | 0.91 | 0.10 |
| Baseball | Smooth | 41° | 0.21 | 0.39 | 3.2 |
| Basketball | Smooth | 66° | 0.09 | 0.86 | 1.1 |

$$\omega_2 = \omega_1 + \frac{(v_{x1} - v_{x2})}{\alpha R} - \frac{D(1+e_y)v_{y1}}{\alpha R^2}. \qquad (12)$$

The bounce is completely determined if the initial conditions are specified together with appropriate values of $e_x$, $e_y$, and $D$. It is not appropriate to do so for a ball that slides throughout the bounce because then $e_x$ (and possibly $D$) is a function of the incident angle. However, if a ball grips the surface, and if $e_x$, $e_y$, and $D$ are all independent of the incident angle, then a description of the bounce in terms of $e_x$, $e_y$, and $D$ would be very useful.

Suppose that $\omega_1 = 0$, $e_x = 0$, $D = 0$, and $\alpha = 0.55$. Then $v_{x2}/v_{x1} = 0.645$ if $m/M = 0$, and $v_{x2}/v_{x1} = 0.665$ if $m/M = 0.17$ (the tennis ball on wood block value). If $D/R$ is increased to 0.1 and $e_y = 0.75$, then $v_{x2}/v_{x1} = 0.645 + 0.113 \tan \theta_1$ when $m/M = 0$. The effect of finite $D$ is to increase both $v_{x2}$ and $\omega_2$ compared with the case where $D = 0$ (given that $v_{y1}$ is negative). The effect of finite positive $e_x$ is to decrease $v_{x2}$ and to increase $\omega_2$.

The bounce parameters listed in Table II can be used to determine values of both $e_x$ and $D$. These are listed in Table III. When a tennis ball grips the surface, $e_x$ is typically between 0.1 and 0.2, and $D$ is typically about 2 mm. Two exceptions in Table III are the first entry, where the ball slides throughout the impact with $e_x < 0$, and the last tennis ball entry where the ball was incident with heavy topspin. In the latter case the friction force remained small throughout the bounce, there was almost no change in the horizontal speed or the spin, and hence the data are almost consistent with pure rolling throughout the bounce. Furthermore, $D$ was slightly negative, a result that has previously been reported for balls that roll.[10,11] The superball had a significantly higher value of $e_x$ than the other balls, but it was only half as large as the ideal $e_x = 1$ superball analyzed by Garwin.

## VII. BOUNCE OF A SOLID ELASTIC SPHERE

A brief summary of the model by Maw, Barber, and Fawcett[5] (MBF) for the oblique bounce of a solid elastic sphere is given here because the theoretical predictions are qualitatively consistent with the observed friction force measurements given in Sec. III. The MBF model is numerical, and the approach is to divide the contact circle into small annuli, some of which grip the surface and some of which slip. Because the component of the normal reaction force acting on the outermost annulus is zero, this and several adjacent annuli usually slip when the ball is subject to a
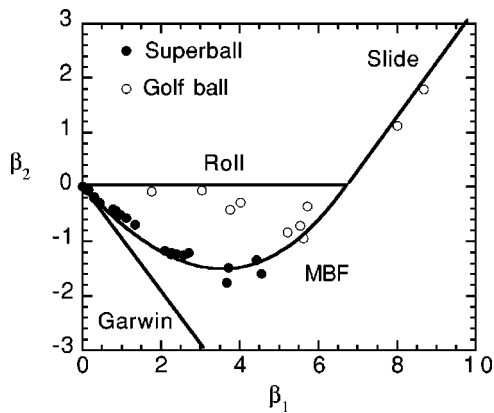
Fig. 8. Comparison of bounce models described by Brody, Garwin and Maw *et al.* (Refs. 5 and 6). Experimental results obtained for the superball and a golf ball impacting on a polished granite slab are also shown.
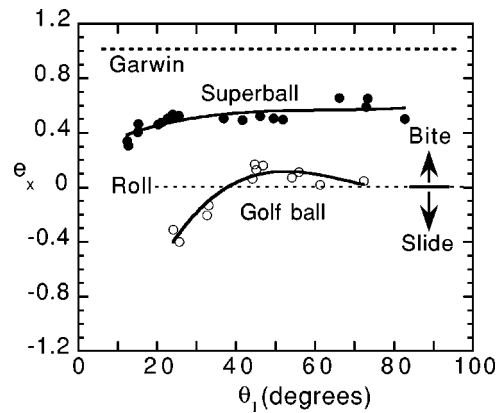


Fig. 9. Horizontal coefficient of restitution, $e_x$, as a function of incident angle, $\theta_1$. A ball that enters a rolling mode bounces with $e_x = 0$. Otherwise, it slides throughout the bounce with $e_x < 0$, or it bites the surface and then $e_x > 0$. The solid curves are best fit curves to the experimental data points.

horizontal force, while the inner annuli grip the surface if the coefficient of static friction is sufficiently large. Under conditions where Brody's model predicts that the ball will slide throughout the bounce period, all annuli slip and the two models are equivalent. The primary difference between Brody's model and the MBF model is that Brody assumed that the ball would commence rolling when $v_x = R\omega$, whereas MBF assume that the whole contact area sticks to the surface when $v_x = R\omega$ because all points within the contact area come to rest at that instant. In Brody's model, the friction force drops instantaneously to zero when the ball rolls. In the MBF model the friction force does not change instantaneously because MBF assume that the coefficients of static and sliding friction are equal.

The ball deforms elastically in the horizontal direction while it is sliding, and it continues to deform and vibrate in the horizontal direction while the contact area is stuck. However, annuli near the outer edge of the contact area quickly become unstuck and begin to slip because the torque on the ball acts to increase the angular velocity of the ball in those annuli. These annuli slide backwards on the surface, reducing the total friction force on the ball. As time progresses, the annuli in slip spread radially inward, reducing the friction force to zero and then reversing it. Near the end of the bounce period the whole contact area slides backwards on the surface.

The differences between the Brody, Garwin, and MBF models for a bounce on an infinitely massive surface are summarized in Fig. 8, which is a plot of the dimensionless quantity $\beta_2$ versus the dimensionless quantity $\beta_1$, where

$$\beta_1 = -\frac{(v_{x1} - R\omega_1)}{\mu v_{y1}} \tag{13a}$$

and

$$\beta_2 = -\frac{(v_{x2} - R\omega_2)}{\mu v_{y1}}. \tag{13b}$$

If $D = 0$, then Eqs. (4) and (5) give

$$\beta_2 = \beta_1 - (1 + e_y)(1 + 1/\alpha), \tag{14}$$

if the ball slides throughout the impact, while $\beta_2 = 0$ if the ball enters a rolling mode.

Garwin assumed that $e_x = 1$ and hence $\beta_2 = -\beta_1$. All theoretical results in Fig. 8 are given for a solid sphere with $\alpha$

$= 0.4$, $D = 0$, and $e_y = 1$ so that all three models can be compared using the same parameters. Garwin and MBF considered only a solid sphere with $e_y = 1$. Also shown in Fig. 8 are experimental data for the superball and for a golf ball filmed when bouncing at low speed (about 4 m/s) onto a heavy, polished granite slab. The ball was thrown by hand and was incident with negligible spin. Both balls had similar mass and diameter but very different bounce characteristics. A similar set of measurements for a superball is given by Johnson.[3]

The experimental data in Fig. 8 were plotted using the measured value $\mu = 0.18$ for the golf ball and an assumed value $\mu = 1.0$ for the superball. The value of $\mu$ for the golf ball represents an average obtained from several low angle bounces where the ball was sliding throughout the bounce. The superball did not slide under any conditions, even at angles of incidence as low as 12° to the horizontal. Consequently, it was not possible to obtain a reliable estimate of $\mu$ for the superball. A lower limit of 0.9 can be deduced using Eq. (4) and a maximum value of 2.4 can reasonably be deduced from the data obtained in Ref. 6 for a rubber disk incident on rubber. Experimentally it was found that $e_y = 0.97 \pm 0.03$ for the superball and $e_y = 0.90 \pm 0.02$ for the golf ball, for all angles of incidence. The small departures from the ideal value $e_y = 1$ are not significant. Similarly, experimental values for $D$ were typically less than 0.5 mm for both the superball and the golf ball and this effect is also not very significant.

An alternative view of each theoretical model is given in Fig. 9 which shows the horizontal coefficient of restitution, $e_x$, as a function of the angle of incidence, $\theta_1$. Because $e_x = -\beta_2/\beta_1$ and $\beta_1 = 1/(\mu \tan \theta_1)$ when $\omega_1 = 0$, there is no new information in Fig. 9, but the significance of the angle of incidence and the coefficient of friction is more apparent. The golf ball has a much lower coefficient of friction than the superball, and it slides throughout the bounce at angles of incidence up to about 40°. Equation (7) with $D = 0$ and $m/M = 0$ indicates that the golf ball should slide at angles of incidence up to 39.9°, and the superball should slide only when $\theta_1 < 8.1°$ if $\mu = 1$. It is difficult to obtain accurate data at such low angles of incidence. A more reliable value for $\mu$ in this case could be obtained if the ball was incident with
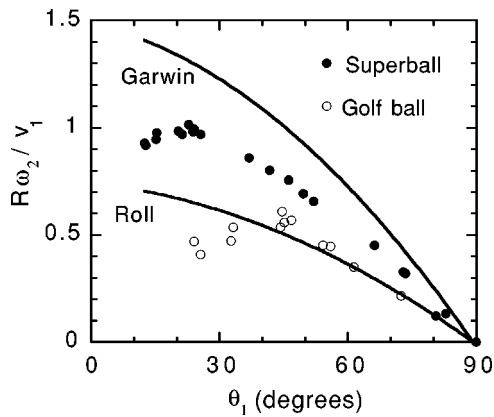
Fig. 10. Measured spin parameter $R\omega_2/v_1$ vs the incident angle $\theta_1$ for the superball and golf ball bounces shown in Figs. 8 and 9. The solid curves are simplified theoretical estimates assuming that the ball enters a rolling mode or that it bounces with $e_x = e_y = 1$ as assumed by Garwin (Ref. 2).

significant backspin to allow for measurements of sliding at higher angles of incidence, but this was not attempted.

The data on ball spin used to generate Figs. 8 and 9 are shown separately in Fig. 10, together with two simple theoretical estimates. If one assumes that the ball enters a rolling mode, then the spin for a solid sphere is given by $R\omega_2/v_1 = (5/7)\cos\theta_1$, where $v_1$ is the incident speed. Garwin's model with $e_x = 1$ indicates that $R\omega_2/v_1 = (10/7)\cos\theta_1$, twice the rolling value. At large angles of incidence, it can be seen from Fig. 10 that the golf ball spins at a rate that is almost the same as that for a rolling ball. At low angles of incidence, the spin is reduced, consistent with sliding. If the ball slides throughout the bounce, then $R\omega_2/v_1 = 2.5\mu(1 + e_y)\tan\theta_1$, and hence $\omega_2$ approaches zero at glancing incidence. The maximum spin of the golf ball, for a given incident speed, occurs at about $\theta_1 = 40°$, which is the angle at which the ball switches from a pure sliding to a biting mode. The superball spins faster than the golf ball, but not as fast as predicted by Garwin.

We can conclude that the MBF model provides a better qualitative description of the bounce of a ball than the more elementary models. However, Brody's model provides a better quantitative description in the case of a golf ball, presumably because the storage and recovery of elastic energy due to tangential compliance is less efficient for the golf ball, giving a value for $e_x$ of only about 0.1 when the ball grips. The low value of $e_x$ is not simply due to the low coefficient of friction. Maw et al.[6] obtained good agreement with their model using a steel disk with $\mu = 0.115$.

Maw et al.[5] considered the situation where a solid sphere is compressed against a half space of the same material. A similar result would be expected for a solid, elastic ball compressed against a rigid surface. However, a tennis ball that is compressed on a rigid surface behaves differently because the normal reaction force is zero at the center of the contact circle as well as at the edge of the circle. If a tennis ball is pushed onto a surface, the ball buckles in such a way that a central section of the contact region lifts off the surface and protrudes inside the ball. The same effect has been observed during a high speed vertical bounce of a tennis ball, in which case the initial contact area bounces up inside the ball while the rest of the ball continues its initial motion downwards.[12] As a result, the contact area is an annulus rather than a com-

plete circular region. The effect is not as dramatic in a low speed bounce, but the distribution of the normal reaction force for a hollow ball is likely to be quite different from the case in Ref. 5 at least when the diameter of the contact area exceeds the wall thickness of the ball.

An additional effect that was not considered by MBF is that $N$ does not necessarily act through the center of mass. In this paper it was found that $N$ acts a small distance behind the center of mass. Experiments currently being undertaken by the author and by colleagues at the University of Sheffield using a tennis ball projected at ball speeds greater than 20 m/s indicate that $N$ can act through a point up to about 12 mm ahead of the center of mass during a high speed impact. This work has not yet been published. The effect is analogous to the shift in weight toward the front of a vehicle when the brakes are suddenly applied. The torque due to the friction force causes the vehicle to rotate about its center of mass. The vehicle would roll over if it were not for the fact that the normal reaction force on the front wheels is then larger than the force on the rear wheels.

## VIII. CONCLUSIONS

Measurements of the friction force on a bouncing ball demonstrate that balls can slide on a surface or they can grip the surface but they do not roll. As a result, balls can spin faster than allowed by the rolling condition $\omega_2 = v_{x2}/R$ and they can bounce with a horizontal coefficient of restitution greater than zero. Even greater spin is imparted to the ball if the normal reaction force acts through a line passing a distance $D$ behind the center of mass. Values of $D$ around 2 or 3 mm were observed for a tennis ball and a baseball incident at low speeds.

When a ball grips the surface, the friction force reverses direction during the bounce and it may reverse direction several times. As a result, the average friction force during the bounce is not dramatically different from the value that would be obtained by assuming that the ball will roll. Simplified bounce models that allow the ball to roll rather than grip can therefore be used to make approximate predictions of the bounce parameters, but the predictions may differ from observations by a factor of 2 or more, especially for a superball. The behavior of any particular type of ball is best determined experimentally, in which case one can characterize the bounce properties in terms of the vertical and horizontal coefficients of restitution and a typical value of $D$. However, at small angles of incidence where the ball slides throughout the bounce, the horizontal coefficient of restitution is a function of the angle of incidence. If the ball slides, then the percentage reduction in horizontal speed depends on the coefficient of sliding friction, the vertical coefficient of restitution, and the angle of incidence. For a sliding ball the spin depends on all three of these parameters, and it also depends on $D$.

The qualitative behavior of a ball when it grips is consistent with the MBF model, but the MBF model was developed to a study a perfectly elastic, solid ball impacting on an elastic surface. Further refinements of the MBF model will be required to obtain quantitative results relevant to inelastic or hollow balls impacting on a rigid surface.

## ACKNOWLEDGMENTS

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