# The coefficient of restitution for collisions of happy balls, unhappy balls, and tennis balls

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A perfectly happy ball is one that bounces to its original height when dropped on a massive, rigid surface. A completely unhappy ball does not bounce at all. In the former case, the coefficient of restitution (COR) is unity. In the latter case, the COR is zero. It is shown that when an unhappy ball collides with a happy ball, the COR increases from zero to unity as the stiffness of the happy ball decreases from infinity to zero. The COR is independent of the mass of each ball. The implication of reducing the COR of a tennis ball, as a possible means of slowing the serve in tennis, is also considered. It is shown that (a) the COR for a collision with a racket varies with the impact point and is a maximum at the vibration node near the center of the strings, and (b) the serve speed is reduced by only about 20% if the COR for a bounce on the court is reduced to zero. 

## I. INTRODUCTION

Most students and physics professors are fascinated when they first see a rubber ball that refuses to bounce when it is dropped on the floor, especially when it is compared with an apparently identical ball that bounces like a superball. Such a ball is available for about $3 from Arbor Scientific[1] as an ''unhappy'' ball of mass 10.3 g, and is sold together with a happy ball of identical feel and appearance. When the unhappy ball is dropped vertically onto the floor, it stops dead without bouncing. If it is thrown at an angle onto the floor, it rolls forwards without bouncing off the floor. The happy ball bounces to a height almost equal to the drop height. The unhappy ball is made from a rubber compound sold under the trade name Norsorex, while the happy ball is made from neoprene rubber. The two balls have similar mass and static stiffness but have dramatically different dynamic properties.

The collision of any two balls or any other massive objects is always accompanied by a loss of energy. Newton was the first to recognize that such collisions can be conveniently described in terms of the coefficient of restitution, $e$, defined in the case of a head-on collision as the ratio of the relative speed of the objects after the collision to the relative speed before the collision.[2] In a perfectly elastic collision, $e=1$. In a completely inelastic collision, $e=0$. In general, the COR depends on the elastic properties of both objects, but under some conditions the COR may depend almost entirely on the elastic properties of only one of the objects. For example, if a relatively soft ball is dropped on a rigid surface such as a hard floor, the resulting value of $e$ provides a measure of the elastic properties of the ball, provided that there is no significant deformation of the surface on which it bounces. Under these circumstances, it is appropriate to refer to the COR as an inherent property of the ball. In addition, if the rigid surface has a much larger mass than the ball, the COR is easily determined from the ratio of the rebound height to the drop height. In the remainder of this paper, any reference to the COR of a particular ball is based on the assumption that it impacts with a perfectly rigid surface. In practice, it is easy to find a suitably hard surface to measure the COR for balls used in sport, but it would clearly be inappropriate to measure the COR of a steel ball by dropping it on a hard, wooden floor. In this case, the COR would be determined more by the elastic properties of the floor rather than those of the ball.

An interesting experiment is to collide a happy ball with an unhappy ball. Apart from its purely academic interest, the physics of this type of collision is relevant to problems in the physics of sport where a ball is struck by a bat or club or racket. The coefficient of restitution for a baseball or a golf ball or a tennis ball impacting with a relevant surface is more or less well established by the rules of the game, but large sums of money are now being spent by manufacturers to increase the COR using technically advanced baseball bats, golf clubs, and tennis rackets in order to drive the ball further or faster. Other attempts by sporting organizations are being made to prevent this happening, or to limit it.[3] The COR can be increased by making a bat or club more ''springy,'' in the same way that the strings in a tennis racket help to absorb most of the impact energy and then give this energy back to the ball. Rule 4-1e of golf specifies that ''the club face shall not have the effect at impact of a spring'' but this rule is impossible to enforce since nothing is infinitely stiff. In baseball, various illegal and unsuccessful methods have have been used in the past to give the bat some extra power, by inserting rubber or cork inside the bat. More recently, aluminum bats with two, closely spaced thin walls have been introduced in an attempt to reduce the stiffness of the outer wall while maintaining the mass and strength of a single, thick wall.

In tennis, big serves tend to dominate the game, leading to some boring matches, especially on the fast grass courts at Wimbledon. One of the suggestions to slow the serve is to reduce the COR of the ball, as is done in top levels of competitive squash. The COR of a tennis ball dropped onto a concrete slab is about 0.75. Suppose that the COR of a tennis ball is reduced, for academic purposes, to zero. If we ignore the fact that the ball will not bounce when it lands on the court, and a match played with such a ball would be incredibly boring, an interesting question arises as to whether the serve speed would be reduced drastically or only slightly. It is also of interest to calculate the COR for a conventional tennis ball impacting on the strings of a racket, given that the COR has never been measured directly under conditions where the racket is free to translate, rotate, and vibrate. Mea-

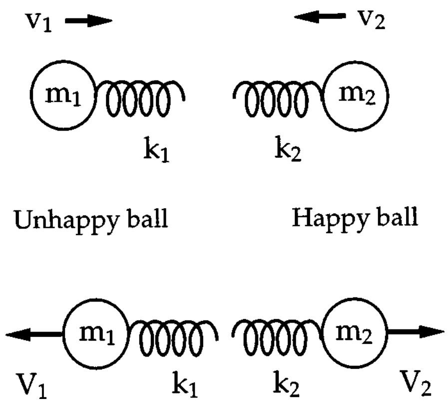


Fig. 1. Collision of a ball of mass $m_1$ and spring constant $k_1$ with another ball of mass $m_2$ and spring constant $k_2$, showing the velocity of each ball before (lower case $v$) and after (upper case $V$) the collision.

surements of the COR in the past have always been made on a head-clamped racket or for an impact of the ball on a rigid surface. Measurements of the incident and rebound ball speeds have been made on free-standing rackets, but the COR cannot be determined without a simultaneous measurement or calculation of the rebound speed of the racket.

## II. CONSERVATION EQUATIONS

Suppose that a ball of mass $m_1$ collides head-on at speed $v_1$ with another ball of mass $m_2$ approaching the first ball at speed $v_2$, as shown in Fig. 1. In order to account for the dynamics during the collision, one can model the elastic properties of each ball by assuming that $m_1$ is connected to a spring of spring constant $k_1$ and $m_2$ is connected to a spring of spring constant $k_2$. The springs may be linear or nonlinear and the spring constants may differ during the compression and expansion phases as a result of hysteresis losses in each ball.

After the collision, let $m_1$ recoil at speed $V_1$ and let $m_2$ recoil at speed $V_2$. It is convenient to analyze the collision in a reference frame where the center of mass remains at rest, in which case the total momentum remains zero at all times, so $m_1 v_1 = m_2 v_2$ and $m_1 V_1 = m_2 V_2$. The coefficient of restitution for the collision is then given by

$$e = \frac{V_1 + V_2}{v_1 + v_2} = \frac{V_1}{v_1}. \tag{1}$$

The total initial kinetic energy is

$$E_i = \frac{1}{2} m_1 v_1^2 \left(1 + \frac{m_1}{m_2}\right) \tag{2}$$

and the total final kinetic energy is

$$E_f = \frac{1}{2} m_1 v_1^2 e^2 \left(1 + \frac{m_1}{m_2}\right). \tag{3}$$

The fractional energy loss, $f$, can be defined as the loss in kinetic energy divided by the initial kinetic energy. From Eqs. (2) and (3), $f$ is given by

$$f = 1 - e^2. \tag{4}$$

For a perfectly elastic collision, $e=1$ and there is no energy loss. In a completely inelastic collision, $e=0$ and all of the initial kinetic energy is dissipated, at least in the center of mass frame. In any other reference frame, a collision with $e=0$ is characterized by the fact that both balls recoil with a common speed and with finite kinetic energy. If $m_2$ is effectively infinite, such as in the collision of a ball with the floor, then the center of mass frame is the same as the laboratory frame. A perfectly happy ball will then rebound with $V_1 = v_1$ and with $e=1$. A completely unhappy ball does not rebound at all since $V_1 = 0$ when $e=0$.

## III. MODELING THE ELASTIC PROPERTIES OF BALLS

The COR for a collision between any two balls depends on the energy loss resulting from deformation of each ball, which in turn depends on the relative stiffness of the two balls. The stiffness of each ball depends on its relevant material properties, as well as the size and shape of each ball. In general, there is no direct relation between the stiffness modulus and the loss modulus, although metal balls with a high stiffness modulus also tend to have a low loss modulus and vice-versa. For example, the COR for an impact between two steel balls is higher than that between two lead balls,[4] but it is similar to that between two superballs of much lower stiffness. The COR also depends on the shape of the colliding objects. For example, a hollow rubber ball has a lower COR than a solid rubber ball, even though the material properties might be the same, since a hollow ball is more easily deformed. The physical dimensions of the colliding objects may also affect the COR, particularly if the impulse duration time is comparable to the period of vibration of one or both of the colliding objects.[4,5] The effect of vibrations on the COR is considered below for the case of an impact between a tennis ball and racket.

At one point in time during the collision, both balls in Fig. 1 will momentarily come to rest in the center of mass frame, at which point any remaining energy is stored as elastic energy due to compression of the balls. A useful approximation is to assume that $k_1$ and $k_2$ remain constant and that there is no energy loss up to this point, in which case $E_i = k_1 x_1^2/2 + k_2 x_2^2/2$, where $x_1$ is the maximum compression of $m_1$, $x_2$ is the maximum compression of $m_2$, and $k_1 x_1 = k_2 x_2$. Hence,

$$E_i = \frac{1}{2} k_1 x_1^2 \left(1 + \frac{k_1}{k_2}\right). \tag{5}$$

A fraction $k_2/(k_1+k_2)$ of the initial energy is therefore stored in the ball with spring constant $k_1$ and a fraction $k_1/(k_1+k_2)$ is stored in the other ball. During the subsequent expansion phase, the total energy dissipated in the two balls is given by $(1-e^2)E_i$. In many cases of interest, almost all of the energy dissipation occurs in only one of the balls. For example, if $m_1$ impacts on a hard surface where $k_2 \gg k_1$ and if the COR for the impact is $e_1$, then the energy dissipated in $m_1$ is $(1-e_1^2) k_1 x_1^2/2$ since $k_2 x_2^2 \ll k_1 x_1^2$. Provided that the energy dissipation is not a function of the impact duration, then the same energy will be dissipated in the ball whenever it compresses by an amount $x_1$, regardless of the surface with which it impacts. If $m_1$ impacts on another surface or another ball of spring constant $k_2$, and if there is no dissipation in the second object, then the COR for such an impact will be given by

$$1-e^2=\frac{(1-e_1^2)k_1x_1^2/2}{E_i}=\frac{k_2}{k_1+k_2}(1-e_1^2). \qquad (6)$$

For example, the COR for an impact of a tennis ball on concrete is $e_1=0.75$. The ball has a spring constant $k_1\sim4\times10^4\ \mathrm{Nm^{-1}}$ during the compression phase. When it impacts on the strings of a head-clamped racket, the strings absorb a significant fraction of the impact energy but almost all of this energy is then returned to the ball. The spring constant of the strings is typically about $k_2\sim4\times10^4\ \mathrm{Nm^{-1}}$, giving $e\sim0.88$, as observed.[6] The impact duration on concrete is about 4 ms, and on the strings is about 5 ms, but the change in impact duration is not a strong factor in determining the loss in the ball. Nevertheless, it is known that there is a slight decrease in both $e_1$ and the impact duration as the ball speed increases.[6] The decrease in impact duration is due to a nonlinear increase in the ball stiffness as the magnitude of the deformation increases. The decrease in $e_1$ is due to a nonlinear increase in the hysteresis loss. A similar result is observed with colliding metal balls, which have been subject to more extensive analysis[4,7] than the relatively soft balls described in this paper.

The above equations can also be used to model the behavior of an unhappy ball when it impacts with a happy ball, given that energy dissipation in the happy ball is negligible compared with the dissipation in the unhappy ball. When the unhappy ball impacts on a rigid surface, $e_1=0$. If the unhappy ball has a spring constant $k_1$ and the happy ball has a spring constant $k_2$, then $1-e^2=k_2/(k_1+k_2)$, so $e^2=k_1/(k_1+k_2)$ will vary from 0 to 1 as $k_2$ decreases from infinity to zero. However, this result is based on the assumption made above that $k_1$ remains constant during the impact. In fact, the spring constant of an unhappy ball is strongly nonlinear and drops to zero at maximum ball compression. This effect is examined in more detail in the following section.

## IV. COLLISION DYNAMICS OF UNHAPPY BALLS

A measurement of the hysteresis curve, of force versus displacement of the center of mass, for an unhappy ball was made by dropping it onto a hard piezo disk attached to a heavy brass cylinder as described by Cross.[8] It was found that the hysteresis curve can be represented to a good approximation by the force law

$$F=F_o\sin\left(\frac{\pi X}{X_o}\right) \qquad (7)$$

where $F_o$ is the maximum force, $X$ is the ball compression, and $X_o$ is the maximum ball compression. This force law resembles Hooke's law for small $X$ but it has the property that $F=0$ when $X=0$ and also when $X=X_o$. For an impact on a rigid surface, $F$ increases to a maximum value $F_o$ when $X$ is half its maximum value, then $F$ decreases to zero as $X$ increases to $X_o$. When an unhappy ball is dropped on a rigid surface, the force rises to a maximum at about $t=0.36$ ms, depending slightly on the ball speed, indicating from the analysis given below that $k_0=F_o/X_o$ is essentially constant and equal to $2\times10^4\ \mathrm{Nm^{-1}}$ and that $F_o$ and $X_o$ are both proportional to the impact speed of the ball. The energy loss in the ball is given by the area under the $F$ versus $X$ curve, and is equal to $2F_oX_o/\pi$.

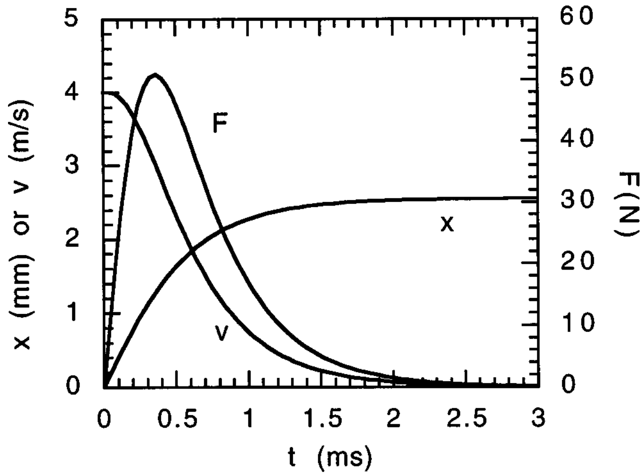


Fig. 2. Solutions of Eqs. (7) and (8) for an unhappy ball of mass $m_1=10.3$ g and stiffness $k_0=2\times10^4\ \mathrm{Nm^{-1}}$ incident at speed $v_1=4\ \mathrm{ms^{-1}}$ on a rigid surface.

The equation of motion of the unhappy ball, when it impacts on a rigid surface, has the form

$$\frac{d^2x}{dt^2}=-\frac{F}{m_1}, \qquad (8)$$

where $x$ is the displacement of the center of mass. If the compression of the ball is much smaller than its diameter, then a reasonable approximation is that $X=x$. Equations (7) and (8) can be solved numerically for initial and final conditions $dx/dt=v_1$ and $dx/dt=0$, respectively. The result for a ball of mass $m_1=10.3$ g is shown in Fig. 2. Since the ball does not bounce, $v=dx/dt$ decreases monotonically to zero and $x$ remains finite at the end of the impact. The ball is not permanently deformed by the impact, like a plasticene ball, but it returns to its original shape with a time constant of order 0.2 s. During the latter period, $F$ is very small and it plays no significant role in the collision dynamics. The force waveform shown in Fig. 2 is an excellent fit to the observed force waveform.

The collision in Fig. 1 can be analyzed using Eqs. (7) and (8) to describe the unhappy ball. The corresponding equation of motion for the happy ball is

$$\frac{d^2y}{dt^2}=\frac{F}{m_2}, \qquad (9)$$

where $y$ is the displacement of the happy ball and $m_2$ is its mass. If $Y$ is the compression of the happy ball, then

$$x-y-D=X+Y \qquad (10)$$

and

$$F=k_2Y=F_o\sin\left(\frac{\pi X}{X_o}\right), \qquad (11)$$

where $D$ is the initial distance between the ball centers. The value of $D$ is irrelevant in determining the dynamics. It is assumed that the spring constant of the happy ball, $k_2$, remains constant as it compresses and expands so that there is no energy loss in this ball. Equations (8)–(11) can be solved numerically for any given initial conditions, using an iterative procedure to solve (10) and (11) to determine $X$ and $Y$ at each time step. The nature of the solutions depends on the ratio of the stiffness of the happy ball to the stiffness of the unhappy ball. If the happy ball is relatively soft, then the impact is similar to that between any other balls and the COR increases towards unity as the happy ball is made softer. Conversely, if the stiffness of the happy ball is increased, the COR decreases towards zero. When the COR is zero, both balls move with the same speed as a single body after the collision.

## V. ANALYTICAL SOLUTION

An approximate analytical solution of the above equations is available when the happy ball is softer than the unhappy ball. The solution, which agrees very closely with numerical results when $k_2 < 3k_0$, is given by

$$\pi X/X_o = \omega t \tag{12}$$

where

$$\omega^2 = k_2\left(\frac{1}{m_1} + \frac{1}{m_2}\right). \tag{13}$$

Equation (12) can be substituted in (7)–(9) to show that

$$x = \left(v_1 - \frac{F_o}{\omega m_1}\right)t + \frac{F_o}{\omega^2 m_1}\sin(\omega t) \tag{14}$$

and

$$y = \frac{F_o}{\omega^2 m_2}[\omega t - \sin(\omega t)]. \tag{15}$$

The ball velocities are obtained by differentiating (14) and (15). Using $F_o = k_0 X_o$, and $\omega t = \pi$ at the end of the collision, it is easy to show that the COR is given by

$$e = \frac{\pi k_0 - k_2}{\pi k_0 + k_2}. \tag{16}$$

The COR is independent of the masses of the two balls and depends only on the relative stiffness of the balls. If $k_2$ is much smaller than $k_0$, then $e$ is close to unity. According to (16), $e=0$ when $k_2 = \pi k_0$. However, the approximate analytical solution is not valid if $k_2$ is larger than about $3k_0$ since it does not agree well with the numerical solutions at the end of the impact period and since $e$ cannot be negative. Nevertheless, the numerical solutions show that the COR is independent of ball mass even when $k_2$ is much larger than $k_0$.

## VI. NUMERICAL SOLUTIONS

Numerical solutions of Eqs. (7)–(11) are shown in Figs. 3 and 4 for a case where an unhappy ball of mass $m_1 = 10.3$ g and initial speed $dx/dt = 4\ \mathrm{ms}^{-1}$ collides head-on with a happy ball of mass $m_2 = 10.3$ g initially at rest. In Fig. 3, $k_2 = 2\times10^4\ \mathrm{Nm}^{-1}$, while in Fig. 4, $k_2 = 1\times10^5\ \mathrm{Nm}^{-1}$. For the softer ball, the happy ball moves at speed 3.034 $\mathrm{ms}^{-1}$ and the unhappy ball moves at speed 0.966 $\mathrm{ms}^{-1}$ after the collision, so $e = 0.517$. For the harder ball, $e = 0.001$ and both balls move at a speed close to 2.00 $\mathrm{ms}^{-1}$ after the collision. When the happy ball is relatively soft, the compression of the happy ball is relatively large, the compression of the unhappy ball remains relatively small, and the energy loss is small.

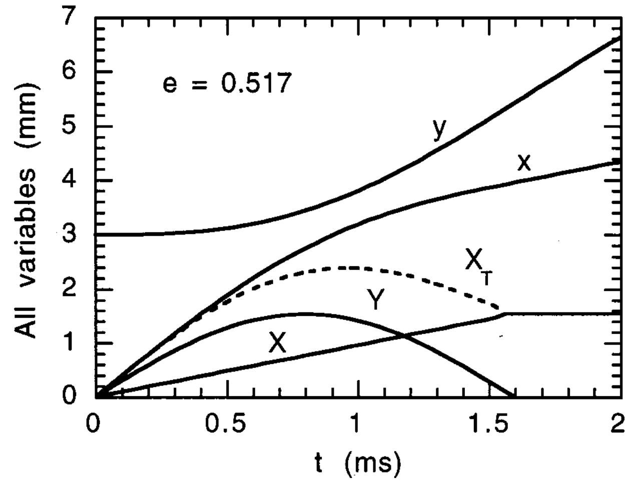


Fig. 3. Numerical solutions of Eqs. (7)–(11) for a case where an unhappy ball of mass $m_1 = 10.3$ g and initial speed $dx/dt = 4\ \mathrm{ms}^{-1}$ collides head-on with a happy ball of mass $m_2 = 10.3$ g, and spring constant $k_2 = 2\times10^4\ \mathrm{Nm}^{-1}$, initially at rest. After the collision, the happy ball moves at speed 3.034 $\mathrm{ms}^{-1}$ and the unhappy ball moves at speed 0.966 $\mathrm{ms}^{-1}$, so $e = 0.517$. $X_T = X + Y$ is the total compression of the two balls.

It is interesting to consider these collisions in different reference frames. For example, in a reference frame where the unhappy ball is incident at speed $dx/dt = 2\ \mathrm{ms}^{-1}$, the happy ball is incident at speed $dy/dt = -2\ \mathrm{ms}^{-1}$. In this case, the total initial momentum is zero, so the total momentum after the collision is also zero. Both balls therefore recoil with an equal and opposite speed, equal to $v = 1.034\ \mathrm{ms}^{-1}$ after the collision, for the case shown in Fig. 3. Alternatively, if the happy ball is incident at 4 $\mathrm{ms}^{-1}$ on an unhappy ball initially at rest, then the unhappy ball moves off at 3.034 $\mathrm{ms}^{-1}$ after the collision. For the example shown in Fig. 4, both balls come almost to rest if they collide head-on with equal and opposite speeds.

## VII. COLLISION OF A TENNIS BALL AND A TENNIS RACKET

In order to evaluate the effects of reducing the COR of a tennis ball, it is first necessary to determine the COR for an

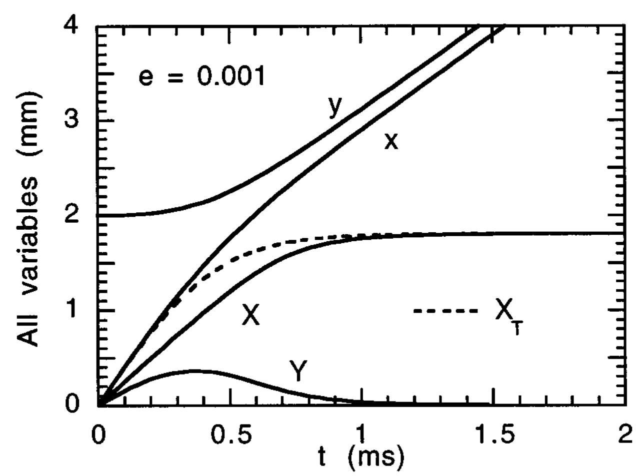


Fig. 4. As for Fig. 3, but $k_2 = 1\times10^5\ \mathrm{Nm}^{-1}$. In this case, $e = 0.001$ and both balls move at a speed close to 2.00 $\mathrm{ms}^{-1}$ after the collision.

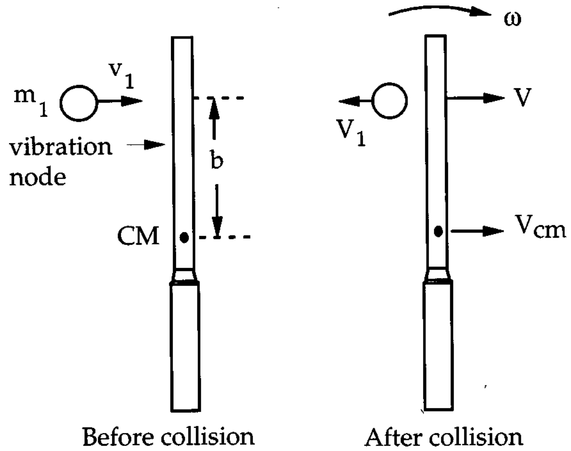


Fig. 5. Impact of a ball with an initially stationary, free-standing racket. The impact point on the racket recoils at speed $V = V_{\mathrm{cm}} + b\omega$.

impact of a standard tennis ball on the strings of a racket. In practice, it is difficult to measure the COR for such an impact, since it is not a trivial task to measure the speed of a swung racket at the actual impact point, as well as the speed of the ball, both before and after the collision. Consequently, the COR is almost always measured by projecting a ball onto the strings of a stationary racket clamped around the head to a heavy support. It is usually assumed that the COR will remain unaltered when the head is unclamped.[9] This assumption is not generally valid, since the COR can be strongly affected by energy loss due to vibrations induced in the racket. In addition, some of the impact energy is absorbed by recoil and rotation of the racket, thereby reducing the deformation of the ball and the resulting energy loss in the ball. At first sight, it might appear that the latter effect should make a significant difference to the COR, but it has no effect on the COR, for the following reason.

Suppose that a ball of mass $m_1$ is incident on a stationary, free-standing racket of mass $M$ as shown in Fig. 5. If the ball exerts a force $F$ at a distance $b$ from the center of mass (cm) of the racket, then the cm will recoil at speed $V_{\mathrm{cm}}$ given by

$$F = M\, dV_{\mathrm{cm}}/dt, \tag{17}$$

and the racket will rotate at angular frequency $\omega$ given by

$$F_b = I_{\mathrm{cm}}\, d\omega/dt, \tag{18}$$

where $I_{\mathrm{cm}}$ is the moment of inertia of the racket about an axis through its cm. One can define the effective mass, $m_2$, of the racket by the relation

$$F = m_2\, dV/dt = m_2\left(\frac{dV_{\mathrm{cm}}}{dt} + b\,\frac{d\omega}{dt}\right), \tag{19}$$

where $V = V_{\mathrm{cm}} + b\omega$ is the recoil speed of the racket at the point of impact. From Eqs. (17)–(19) it is easy to show that

$$\frac{1}{m_2} = \frac{1}{M} + \frac{b^2}{I_{\mathrm{cm}}}. \tag{20}$$

The dynamics of the impact can therefore be studied by replacing the extended racket with an equivalent point mass $m_2$, without having to consider separately the translational and rotational motion of the racket. The effective mass varies with position along the racket and is typically about one-half to one-third of its actual mass for an impact near the center of the strings.

Suppose that a ball of mass $m_1$ and spring constant $k_1$ is incident in the laboratory frame on a stationary racket of effective mass $m_2$ and spring constant $k_2$. Suppose also that the ball strikes the fundamental vibration node near the center of the strings so that there is no energy loss due to vibrations of the racket frame. In practice, this is a good approximation since the impact duration is too long to excite higher order modes efficiently. If the impact is analyzed in the center of mass frame, then the fraction of the initial impact energy stored in the ball as a result of its compression is given, from Eq. (5), by $k_2/(k_1 + k_2)$. If a fraction $f$ of this energy is subsequently dissipated in the ball, and no energy is dissipated in the strings, then the COR for the collision is given, from Eq. (4), by $e^2 = 1 - f$. If the racket head is now clamped, so that the effective mass of the racket is infinite, and the ball is projected onto the strings at the same speed (in the laboratory frame) as before, then the initial impact energy is the same in the laboratory frame but it is increased in the center of mass frame. The ball compression will be somewhat larger since none of the impact energy is absorbed by recoil of the racket. However, the fraction of the initial energy stored in the ball as a result of its compression remains the same, since it depends only on $k_1$ and $k_2$, and not on $m_2$, as shown by Eq. (5). The fraction $f$ is independent of the energy stored in the ball, so $e$ is unaltered. Even if the strings dissipate a significant fraction of the initial energy, it is easy to show by the same reasoning that $e$ would be unaltered if the racket head is clamped.

For an impact at any point other than the vibration node, it is possible to calculate the energy coupled to vibrational modes, and to calculate the apparent coefficient of restitution, $e_A = V_1/v_1$, using a flexible beam model for the racket. Numerical calculations of this type have previously been reported by Van Zandt[10] for a baseball bat and by Cross[11] for a tennis racket. If the racket vibrates, then the velocity of the racket at any point, well after the collision is over, can be decomposed into a time-independent or dc component and a time-dependent or ac component. The ac component arises from energy stored in racket vibrations, this energy representing part of the total energy lost or dissipated during the collision. The time-independent component must satisfy the same conservation equations for linear and angular momentum that one would obtain using a rigid body model of the racket. This situation is the same as that involved in the collision of any two masses, one of which vibrates as a result of the collision. The momentum of the vibrating mass, after the collision is over, is unaffected by its vibrational motion.

If a ball of mass $m_1$ is incident at speed $v_1$ in the laboratory frame on an initially stationary racket of effective mass $m_2$, the ball rebounds at speed $V_1$, and the impact point on the racket recoils at speed $V_2$, then conservation of momentum can be used to show that

$$e = \frac{V_1 + V_2}{v_1} = e_A + (1 + e_A)m_1/m_2. \tag{21}$$

Van Zandt[10] used a similar expression for the COR but he incorrectly included the vibrational component of $V_2$ in the numerator of Eq. (21). Graphs of $e_A$ versus impact parameter for a tennis racket are given by Cross.[11] From these data, one can show that the COR has a maximum value of about 0.86 at the vibration node near the center of the strings, and falls monotonically to about 0.54 at the tip of the racket and about 0.71 at the throat. The simple estimate of the COR given in Sec. III is therefore consistent with these more detailed calculations, at least for an impact at the vibration node. Despite the fact that $e$ is a maximum at the vibration node, since the vibration losses are a minimum at this point, the rebound speed of the ball is not necessarily a maximum at this point. For an initially stationary racket, $e_A$ is a maximum near the throat of the racket since the effective mass is increased in this region and hence the energy coupled to translation and rotation of the racket is reduced.

## VIII. IMPLICATIONS OF REDUCING THE COR OF A TENNIS BALL

The effect of serving a zero COR tennis ball can be estimated by considering a head-on impact of a racket of effective mass $m_2$ incident at speed $v_2$ in the laboratory frame on a stationary ball of mass $m_1$. Treating the mass $m_2$ as a point particle, it is easy to show that the ball speed, $V_1$, after the collision, is given by

$$V_1=\frac{(1+e)m_2v_2}{m_1+m_2}. \tag{22}$$

The effective mass of a racket is typically about three times the mass of the ball for an impact near the center of the strings. If $m_2=3m_1$ in Eq. (22), then $V_1=0.75(1+e)v_2$. The maximum possible ball speed is then $1.5v_2$, assuming that $e=1$. The maximum speed of a conventional ball with $e=0.86$ is $1.39v_2$. This can be as high as 60 or 65 $\mathrm{ms}^{-1}$ (130 to 140 mph), which many people feel is too fast for the good of the game.

If a tennis ball is constructed with zero COR for an impact on a rigid surface, and if the ball and the strings have about the same stiffness, then Eq. (16) shows that $e$ will be about 0.5 for an impact with the strings. This was confirmed by dropping an unhappy ball on the strings of a head-clamped racket. The racket was strung at a tension of about 55 lb (245 N), which is typical of most rackets, giving a string plane stiffness of about $3\times10^4\ \mathrm{Nm}^{-1}$. An even higher bounce was observed when the ball was dropped on a stretched rubber membrane constructed from a rubber glove. If $e=0.5$ for the tennis racket, then $V_1=1.12v_2$, corresponding to a speed reduction of 19% compared with balls currently in use. If the racket is strung at a higher tension, the speed reduction would be larger. Conversely, a racket strung at lower tension would have a smaller effect on the ball speed.

The speed reduction for a ball with zero COR is surprisingly small, given that none of the elastic energy stored in the ball is recovered during the collision. The only elastic energy available is that stored in the strings and the racket frame. The explanation lies in the fact that, in the laboratory frame, not all of the initial kinetic energy of the racket is converted to elastic energy. In the above example where $m_2=3m_1$, it is easy to show that only $\frac{1}{4}$ of the initial kinetic energy of the racket is converted to elastic energy. Even if the racket and the ball both dissipated all of their stored elastic energy, so that the COR for the collision was zero, Eq. (22) shows that the ball could be served at a speed $V_1=0.75v_2$. This is consistent with the fact that $\frac{3}{4}$ of the initial energy is retained by the racket and the ball. The ball would stick to the strings under these conditions, at least until the server reduced the racket speed.

It is also of interest to consider more generally the case of a bat or racket incident at speed $v_2$ on a ball traveling at speed $v_1$, as in Fig. 1. The collision was analyzed in the center of mass frame in Sec. II. If $v_1$ and $v_2$ are measured in the laboratory frame, then it can be shown that the fractional energy loss is given by

$$f=\frac{(1-e^2)m_1m_2(v_1+v_2)^2}{(m_1+m_2)(m_1v_1^2+m_2v_2^2)}, \tag{23}$$

which reduces to Eq. (4) when $m_1v_1=m_2v_2$. The fraction of the initial kinetic energy converted to elastic energy is easily determined by setting $e=0$ in Eq. (23). For example, if $m_2=3m_1$ as in the above case, and if $v_1=v_2$, then $\frac{3}{4}$ of the initial kinetic energy is converted to elastic energy during the collision. These conditions are typical of a forehand or backhand in tennis, or of a vigorous hit in baseball. The rebound speed of the ball is then much more sensitive to the ball COR than for a collision where the ball is initially at rest.

## IX. CONCLUSIONS

A rubber ball that sticks to the floor when it is dropped is a curious sight. Equally curious, since it is counter-intuitive, is the sight of the same unhappy ball bouncing to a considerable height when it is dropped on a soft, elastic surface such as a stretched rubber membrane. This provides a dramatic classroom demonstration of the nature of elastic and inelastic collisions, highlighting the significance of relative stiffness in determining the deformation and energy loss in each of the colliding objects. If one were to test the elastic properties of a surface such as a hard floor or a soft membrane by bouncing a happy ball on the surface, then both surfaces would appear to have identical properties, since the ball would bounce equally well. The differences are much more obvious when one also tests the surfaces using an unhappy ball, but this test alone does not indicate whether the energy loss occurs mainly in the floor or mainly in the ball. The happy ball test shows that there is no energy dissipation in the happy ball or the floor or the membrane, but it provides no information on the relative stiffness of these objects unless the impact duration is also measured. The unhappy ball test shows that the floor is much stiffer than the membrane and that none of the elastic energy stored in the unhappy ball is released during a collision of the ball.

The implications in the physics of sport are equally interesting. For example, tennis players can choose a wide variety of different strings for their racket and can string the racket to any desired tension, but all strings appear to have the same elastic properties, regardless of the string tension, when tested by bouncing a solid wood or metal ball on the strings. This test shows that there is essentially no energy dissipation in the ball or in any of the strings, even if the strings have lost tension over many years of abuse. At first sight, this test seems to indicate that all strings are essentially the same and that players should not waste their money on expensive strings. However, differences do become apparent when a tennis ball impacts with the strings, since rackets strung at low tension lead to smaller energy losses in the ball,[6] as do strings such as natural gut which have a relatively low dynamic stiffness. Any attempt to decrease the serve speed in tennis by using balls with a low coefficient of restitution is likely to be unsuccessful since the reduction in speed is surprisingly small. Even if the ball COR is reduced to zero, the ball can still be served at about 80% of the speed of a normal ball since the strings absorb some of the impact energy, and only a small fraction is dissipated in the ball. For similar reasons, the ball speed in baseball and golf can be increased by making the bat or the club softer or more springy in order to decrease the energy loss in the ball. Of course, this will only work if the fraction of the energy dissipated in the bat or the club remains significantly smaller than the fraction dissipated in the ball.

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[1]Arbor Scientific, www.arborsci.com or P.O. Box 2750, Ann Arbor, MI 48106-2750.

[2]Newton was the first to make detailed measurements on the impact of imperfectly elastic bodies and to describe such collisions in terms of what is now known as the coefficient of restitution. In doing so, Newton extended the laws of impact previously established by Wren, Wallis, Huygens, and Mariotte for perfectly hard or perfectly elastic bodies. A fascinating historical account is given by R. Dugas, *Mechanics in the Seventeenth Century* (Central Book, New York, 1958).

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## INTUITION

Our intuitive fear of heights would be ridiculous for an albatross; our intuitive appreciation of the flight of a ball is silly if we want to trace a quark. Intuition gives us plausible nonsense like astrology, homeopathy, or quantum-mechanics-turned-into-Zen. Intuition does not help us much in doing physics, be it quantum theory or classical mechanics (ever tried to understand the motions of a spinning top intuitively?)

Vincent Icke, *The Force of Symmetry* (Cambridge University Press, Cambridge, 1995), pp. xiii–xiv.