# Comment on "The sweet spot of a baseball bat," by Rod Cross [Am. J. Phys. 66 (9), 772–779 (1998)]

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Rod Cross[1] has published conclusions concerning the interaction of baseball bats and baseballs that appear to differ in some respects from my own results.[2] Here I attempt to correct, clarify, and reconcile our conclusions.

In particular, I address three points, (1) the position and meaning of the ''sweet spot'' of the bat, (2) the time required for the ball–bat impact signal to travel to the hands and back to the impact point, and (3) the duration of the ball–bat impact. The last two factors are important inasmuch as they bear on the possible influence of the grip of the hands on the bat on the ball–bat impact.

*The sweet spot*: The ''sweet spot'' of a baseball bat is a subjective baseball term, not a physics term, and is to be determined by batters, not defined by physicists. But our positions on the ''sweet spot'' do not differ significantly. I say[2] (page 64), ''... the vibrational node or 'sweet-spot' of the bat ... .'' Cross says[1] (page 778, Sec. VIII), ''An impact at the fundamental node is about optimum ... .''

However its basic irrelevance to the physics of the bat, the ''sweet spot'' has an importance in the language of the sport and we point out a simple experiment that will define it operationally to most players and fans. Hold a wooden bat as if you were batting in a game, and strike a relatively unyielding narrow vertical structure firmly with the barrel of the bat at various points in the hitting zone. (While a small tree, a door frame, or even a telephone pole will do as a target, the 4-in.-diam steel—Lally—columns filled with concrete found in the basements of most American homes are particularly good.) At one impact point it will feel *just right*. You will have found your sweet spot, which you can mark.

Then, following Brody,[3,4] you can determine the node of the fundamental vibration by hanging the bat from a short string, striking it lightly with a hammer, and listening for the fundamental vibration of about 170 Hz. You should hear the sound plainly when you strike the bat at the end or at the trademark. But, at some point between you will hear no hum. You can mark that as a node of the vibration and we expect that you will find it to be the same as your sweet spot within the uncertainties of measurements (which, with reasonable care in the measurements, will be as little as 1 cm).

For most wooden bats held at the end, the center of percussion is very near the vibrational node. But if you choke up on the bat, changing the position of the center of percussion (which is conjugate to the grip point), you will find that your sweet-spot position is unchanged.

*Neurobiology of the ''sweet spot''*: The basis of the identification of the ''sweet spot'' with the node of the fundamental vibration can be found in the neurophysiology of sensation. Different modalities of somatic pressure sensation are detected by different receptors which send signals through different dedicated $A\beta$-type afferent nerve systems through the spinal cord to the thalamus and then to the cerebral cortex.[5,6] In the hairless (glabrous) skin of primates such as that on the batter's hands, Merkel receptors in the derma detect steady pressure or skin indentation; Meisner's corpuscles in the derma detect low frequency oscillations or ''flutter'' near or below 50 Hz. Vibrations at frequencies above 50 Hz vibration are sensed by subcutaneous Pacinian corpuscles. Measured in terms of the skin indentation near sensitive points, the Pacinian corpuscle systems have a peak sensitivity from 250 to 300 Hz where they send out an action signal output pulse upon each vibration amplitude excursion that is greater than 1 $\mu$m. The amplitude sensitivity falls off by more than a factor of 10 at 50 Hz and the system is quite insensitive at lower frequencies. Similarly, the sensitivity falls off strongly for frequencies above 300 Hz. Indeed, above about 500 Hz, the signals are degraded in that the channel capacity of the myelinated afferent nerve fibers that serve the corpuscles is limited to about 500 Hz.

Hence the fundamental vibrations of a baseball bat are detected sensitively by the Pacinian corpuscles in the batter's hands, which send a signal to the central nervous system that is qualitatively different than the sensations from other pressure modalities. Insensitive to low frequency impulses on the hands from the bat and insensitive to the higher vibrational harmonics, that sensory system detects the fundamental vibration of the bat exclusively. Then the sweet spot, as defined by baseball batters, is determined by the absence of the unique sensation derived from such signals and that absence occurs when the ball is hit at the node of the fundamental bat vibration. This node is near the point at which the ball comes off of the bat fastest and travels the furthest and is thus identified in the batter's experience with good hitting.

*The signal velocity*: In his Fig. 2, Cross[1] shows the variation of the acceleration with time of a spot on the handle of the bat after the bat is struck sharply by a steel ball near the handle and interprets the time between acceleration maxima as the transmission time of a signal from that point to the end of the bat and back. Taking the distance from the measurement point to the end of the bat as 0.78 m and the elapsed time between acceleration peaks as about 1.3 ms, the signal velocity is about 1200 m/s. However, Cross notes dispersion effects that show that the lower frequency components generated by the impact travel more slowly than the high frequency components. The accelerometer, with a response proportional to $F^2A$, where $F$ is the frequency and $A$ is the displacement, strongly emphasizes high frequency components.

  

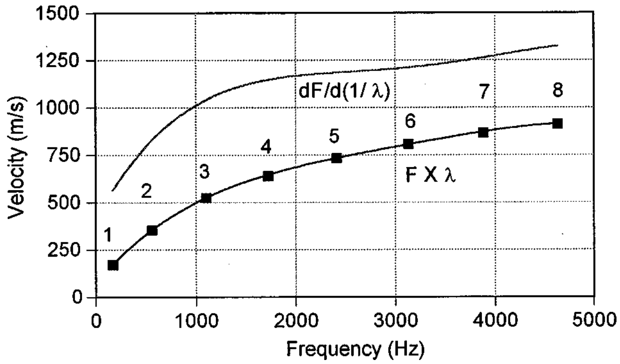


Fig. 1. The square points show phase velocities derived from Van Zandt and Hansen while the curve is a polynomial best fit to the points. The upper curve represents group-like velocities derived from that functional expression of the phase velocities.

Figure 1 shows bat vibration velocities taken from the work of Van Zandt and Hansen.[7] The lower points give the values of a phase velocity, $v_\phi = F \times \lambda$, where $F$ is the frequency and $\lambda$ the wavelengths determined for the first eight vibrational modes of the bat. The upper curve shows a velocity $v_{\rm grp} = dF/d(1/\lambda)$ for the same frequencies where $F(1/\lambda)$ was fitted to the discrete frequencies and wavelengths.

At high frequencies where the density of modes is sufficiently high so that a meaningful wave packet can be constructed, $v_{\rm grp}$ corresponds to the group velocity of the wave packet. Considering the approximations implicit in the limited set of frequencies available and that they are measuring different bats, $v_{\rm grp}$ is in excellent agreement with the velocity of 1200 m/s estimated from Cross's measurements, which emphasize high frequency vibrations. At such a velocity, certainly relevant for high frequencies, the signal from a collision with a ball near the fundamental node about 70 cm from handle end of the bat will take about 0.56 ms to travel to the mean hand position about 10 cm from the end of the bat. Plausibly, any effect of the grip will then be transmitted to the hitting region after, approximately, another 0.56 ms, reaching that point about 1.1 ms after the original impact signal.

For the lower frequency components, $v_{\rm grp}$ is appreciably smaller, as Cross notes from the dispersion evident in his data. If we take an effective velocity of 700 m/s from the value shown in Fig. 1 at the first harmonic, the transit time will be about 0.94 ms.

Either time differs significantly from the 2-ms time that I postulated (Ref. 2, page 136) for the time required for the signal to travel from the impact to the hands. Elsewhere,[8] I gave a value that is effectively about 4 ms. This is clearly excessive.

While a discussion in terms of ''signal velocity'' is especially useful—perhaps essential—in describing baseball for laymen, for physicists a complementary description of the motion of the bat based on an analysis of the excitations upon bat–ball impact of the orthonormal vibrational modes of the bat as described by Van Zandt[7] may be more illuminating. Van Zandt's Fig. 4 describes the configuration of a bat after a bat–ball collision that takes place about 4 cm closer to the handle than the sweet-spot node of the fundamental. The handle of the bat moves but little in the first 0.8 ms after impact but has moved considerably by the time of 1.2 ms. This action is dominated by the 170-Hz fundamental and the 560-Hz first-harmonic. The two amplitudes are out of phase and cancel at times less than $1000/(4\times560) = 0.4$ ms but are in phase and add at a time of 1.2 ms. Higher frequency modes seem to modify this result by adding some further damping of the fundamental.

In summary, we consider that an impact in the hitting zone at a time $t=0$ will not affect the bat significantly where it is held near the end for about 0.9 ms. Plausibly, the effect of the hands will take another 0.9 ms to reach the point of the collision and affect that action. These times are a little longer than that 0.6 ms that Cross states and significantly shorter than my estimate[2] of 2 ms.

*The ball–bat collision time*: Cross[1] measured the bat–ball collision time as about 1.5 ms for balls with the velocity (of about 4 mph) obtained from the ball's falling about 1 m. Assuming approximate linearity of the ball elastic parameters, he then used this collision time in his consideration of ball–bat impacts at baseball velocities of the order of 100 mph. With so long an impact time, the grip of the bat could influence the collision.

However, from general considerations as well as from measurements of static stress–strain curves by Paul Kirkpatrick,[9] the value of the effective ball ''spring constant'' increases as the ball is compressed. This leads to a momentum transfer time that decreases with increasing collision velocity. Using a reasonable model of the nonlinear spring after Kirkpatrick, I showed (Ref. 2, Figs. 5.5 and 5.6) that for typical ball–bat collisions in baseball, the momentum transfer is completed in about 0.6 ms. Since this is much shorter than the signal time from impact zone to the grip on the bat and back, the conclusion that the ball–bat collision can be considered as equivalent to the collision of the ball with a free bat can be retained.

Cross's use of his unrealistically long impact time of 1.5 ms in considering the forces on the hands from the ball–bat impact (Ref. 1, Sec. VII) leads to no significant error since those forces are dominated by low frequency vibrations that are not different for a shorter impact time.

## ACKNOWLEDGMENTS

This note reflects contributions from valuable discussions with Rod Cross and Allan Nathan.

[1]R. Cross, ''The sweet spot of a baseball bat,'' Am. J. Phys. **66**, 772–779 (1998).
[2]R. K. Adair, *The Physics of Baseball* (HarperCollins, New York, 1994), 2nd ed. This book, written for the layman, is an expansion of a report written for A. Bartlett Giamatti, then President of the National League, when I served as Physicist to the National League. Hence the book has no equations in the body of the text and but a few in the end notes.
[3]H. Brody, ''The sweet spot of a baseball bat,'' Am. J. Phys. **54**, 640–643 (1986).
[4]H. Brody, ''Models of a baseball bat,'' Am. J. Phys. **58**, 756–758 (1990).
[5]G. M. Shepherd, *Neurobiology* (Oxford U.P., New York, 1994), 3rd ed.
[6]E. R. Kandel, J. H. Schwartz, and T. M. Jessel, *Principles of Neural Science* (Appleton & Lange, East Norwalk, CT, 1991), 3rd ed.
[7]L. L. Van Zandt, ''The dynamical theory of a baseball bat,'' Am. J. Phys. **60**, 172–181 (1990). This paper contains the results of measurements of bat vibrational frequencies by Professor Uwe Hansen in an appendix. As a consequence of the renormalization of the calculations to fit Hansen's, the times in Figs. 5 and 6 should be reduced by a factor of about 0.8.
[8]R. K. Adair, ''The physics of baseball,'' Phys. Today **48**, 26–31 (May 1995).
[9]P. Kirkpatrick, ''Batting the ball,'' Am. J. Phys. **31**, 606–613 (1963).

# Response to ''Comment on 'The sweet spot of a baseball bat'' [Am. J. Phys. 69 (2), 229–230 (2001)]

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Baseball, tennis, golf, and cricket players all identify the sweet spot of their striking implement in terms of the absence of shock and/or vibration coming from the handle. Players often report that they feel no force at all when they strike the ball at the sweet spot. My measurements on bats[1] and rackets[2] indicate that this is a slight exaggeration. The forces on the hands and arms are minimized for an impact at the sweet spot, but they are not zero. When a batter hits a ball, the handle changes speed suddenly, resulting in an impulsive force on the hands and arms. The bat does not behave as a rigid body on the time scale of the collision, since there is a measurable delay between the time at which the bat strikes the ball and the time at which the handle starts to deflect suddenly from its previous path. In fact, the ball usually loses contact with the bat at about the same time that the first, high frequency, high phase velocity ripples in the impulse arrive at the hand. The batter therefore experiences an impulsive force starting at about the same time as the ball departs, and persisting for a period of about 10 or 20 ms afterwards. The impulsive force is typically much larger than the force applied to swing the implement prior to the impact. The impulsive force waveform is quite complex and contains components due to translation, rotation, and vibration of the bat. The waveform itself is modified by the hands, since the hands act to dampen the vibrations strongly and since motion of the handle is resisted by the force exerted by the hands.

My initial measurements on a bat, and subsequent measurements and calculations for a tennis racket,[2,3] indicate that the sweet spot (as located by players) is likely to coincide with a narrow impact zone that leads to a minimum in the total (translation+rotation+vibration) energy in the handle. Given that the energy coupled to the second vibration mode of a bat is almost as large as the energy in the fundamental mode,[4] and that the peak force on the hands due to this mode is as large if not larger than that due to the fundamental mode,[1] then the vibrational energy of the handle should be minimized for an impact at a point roughly half way between the nodes of the fundamental and second modes. The translational plus rotational energy at any given point in the handle is minimized for an impact on the barrel at the center of percussion (COP), defined in terms of the conjugate point in the handle. The COP is not a unique point on the bat since each point in the handle has a different COP in the barrel. However, all COP points in the barrel, with corresponding conjugate points under the hands, lie in a narrow band between the nodes of the fundamental and the second mode. The total handle energy is therefore minimized in this zone.

For my wood bat, the fundamental node is 17 cm from the end of the barrel and the node of the second mode is 13 cm from the end of the barrel. The region from 13 to 17 cm is therefore a ''sweet vibrations'' zone. I located the sweet spot zone, as described in Ref. 1, by striking a ball and feeling the effects on my hands and arms. This zone extended from 15 to 18 cm. Adair[5] disagrees with my sweet-spot location and claims that his sweet spot is the fundamental vibration node, the locations of the COP and second vibration node being irrelevant. As evidence, he cites the fact that his sweet spot remains fixed when he holds the bat further along the handle and closer to the impact point. This evidence is not convincing. A change of say 10 cm in the position of the conjugate point in the handle, toward the barrel, leads to a shift of only 1 or 2 cm in the location of the COP toward the end of the barrel, as indicated by Fig. 8(b) in Ref. 1. This would not shift the center of the COP zone more than 1 cm, which is about the stated uncertainty in Adair's measurement. If the bat is choked a lot further up the handle, then the COP eventually moves out beyond the end of the bat and plays no role. In that case, I would expect that the sweet spot would lie in the sweet vibrations zone, not necessarily coincident with the fundamental node, and that the feel of the bat would be quite different. Nevertheless, it is clear that a sample of two amateur batters (assuming an Aussie cricket player actually qualifies) is not representative of the broader community of batters, and a larger sample is needed to locate the sweet spot and to determine whether it is in the same location for all players.

Adair argues that the hands are most sensitive to vibrations in the frequency range from about 100 to 300 Hz, and a batter is therefore more likely to identify the sweet spot with the fundamental mode vibration at about 170 Hz. If one measures the handle and arm motion using piezo sensors, as described in Refs. 1 and 2, then one can distinguish the higher frequency vibrational motion from the lower frequency translational and rotational motion of the handle. However, sudden rotation of the handle of a bat and the simultaneous impact with the hands appear to generate a narrow force spike of width about 2 ms which is difficult to distinguish from the heavily damped second mode.[1] A spike of duration 2 ms contains a broad spectrum of frequency components and will be detected by the batter, regardless of the frequency dependence of the receptors in the hand. For a similar reason, a player catching a fast ball may feel a sting in the hands, even though the ball does not vibrate.

There is another reason to believe that the node and the COP both conspire to generate a sweet zone. If the fundamental node alone was responsible for the sweet spot, then there would be two sweet spots along the bat, one at each of the two fundamental nodes. These nodes are located about 17 cm from each end of the bat. If one strikes a ball or a heavy object at each of the nodes in turn, it is immediately obvious that one node is a lot sweeter than the other. The sweeter one is right beside the COP, for the reasons described above.

Adair's comments on the effect of the hands during the collision are valid. The impulsive reaction force exerted by the hands on the bat has no effect on the exit speed of the ball, at least for impacts along the fat part of the bat. This is entirely consistent with detailed measurements and calculations of the ball speed.[3,4] The main effect of the hands occurs during the 20-ms period after the initial impact. The hands cause the vibration modes to be strongly damped, and they also shift the axis of rotation of the bat.


[1]R. C. Cross, ''The sweet spot of a baseball bat,'' Am. J. Phys. **66**, 772–779 (1998).

[2]R. C. Cross, ''The sweet spots of a tennis racket,'' Sports Eng. **1**, 63–78 (1999).

[3]R. C. Cross, ''Impact of a ball with a bat or racket,'' Am. J. Phys. **67**, 692–702 (1999).

[4]A. M. Nathan, ''Dynamics of the baseball–bat collision,'' Am. J. Phys. **68**, 979–990 (2000).

[5]R. K. Adair, ''Comment on 'The sweet spot of a baseball bat,' '' Am. J. Phys. **69**, 229–230 (2001).


# The zeta function method and the harmonic oscillator propagator


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We show how the pre-exponential factor of the Feynman propagator for the harmonic oscillator can be computed by the generalized $\zeta$-function method. Also, we establish a direct equivalence between this method and Schwinger's proper time method.



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In a recent paper in this journal[1] the harmonic oscillator propagator was evaluated in a variety of ways, all of them based on path integrals. In fact, some of them did not involve any explicit computation of the Feynman path integral, but their common starting point was actually an expression for the harmonic oscillator propagator which was explicitly derived by path integral means, namely (we are using the notation of Ref. 1 as much as possible):

$$D_F(z_f,t_f;z_i,t_i)=\left(\frac{\det \mathcal{O}}{\det \mathcal{O}^{(o)}}\right)^{-1/2} \times \sqrt{\frac{m}{2\pi i\hbar(t_f-t_i)}}\exp\left\{\frac{i}{\hbar}S[z_{\rm cl}]\right\}, \tag{1}$$

where

$$\mathcal{O}=\omega^2+\frac{d^2}{dt^2},\quad \mathcal{O}^{(o)}=\frac{d^2}{dt^2}, \tag{2}$$

and the determinants must be computed with Dirichlet boundary conditions. In Eq. (1), $S[z_{\rm cl}]$ means the classical action, that is, the functional action evaluated at the classical solution satisfying the Feynman conditions $z(t_i)=z_i$ and $z(t_f)=z_f$ and the factor before the exponential is the so-called pre-exponential factor, which we shall denote by $F(t_f-t_i)$. In Ref. 1 three distinct methods were presented for the computation of $F(t_f-t_i)$: (i) It was computed directly by the products of the corresponding eigenvalues of $\mathcal{O}$ and $\mathcal{O}^{(o)}$ (some care must be taken here since both products are infinite, but their ratio is finite); (ii) it was computed with the aid of Schwinger's proper time method[2] (an introductory presentation of this method with simple applications can be found in Ref. 3); (iii) it was computed by the Green function approach (a variety of simple examples worked out with this approach can be found in Refs. 4 and 5).

In this note we just add to the previous list one more method for computing the pre-exponential factor of the harmonic oscillator propagator, namely, the generalized $\zeta$-function method, so that this note can be considered as a small complement to Holstein's paper.[1] In fact, every time we make a semiclassical approximation, no matter whether in the context of quantum mechanics or quantum field theory, we will get involved with the computation of a determinant of a differential operator with some boundary conditions. If we try naively to compute these determinants as the products of the corresponding eigenvalues we will get ill-defined expressions. Hence, it is imperative to give a finite prescription for computing determinants for these cases. The generalized $\zeta$-function method is precisely one possible way of doing that. It was introduced in physics in the mid-1970s[6] and it is in fact a very powerful regularization prescription which has applications in several branches of physics (a detailed discussion can be found in Ref. 7). This method, as we will see, is based on an analytical extension in the complex plane. We think that the harmonic oscillator propagator is the perfect scenario for introducing such an important method, because undergraduate students are all familiar with the quantum harmonic oscillator and besides, it is the first nontrivial example after the free particle. In what follows, we shall first introduce briefly the $\zeta$-function method, then we shall apply it to compute $F(t_f-t_i)$ for the harmonic oscillator propagator, and, finally, we shall establish a direct equivalence between this method and Schwinger's proper time method.

Consider an operator $A$ and let us assume, without loss of generality, that it has a discrete set of nondegenerate eigenvalues $\{\lambda_n\}$. When there is only a finite number of eigenvalues, $\det A$ is just given by the product of these eigenvalues and we can write:

$$\det A = \prod_n \lambda_n$$
$$= \prod_n \exp\{\log \lambda_n\}$$
$$= \exp\left\{ \sum_n \log \lambda_n \right\}$$
$$= \exp\left\{ -\sum_n \left( \frac{\partial \lambda_n^{-s}}{\partial s} \right)_{s=0} \right\}$$
$$= \exp\left\{ -\frac{\partial \zeta}{\partial s}(0;A) \right\}, \tag{3}$$

where we define the generalized zeta function associated with the operator $A$ as

$$\zeta(s;A) = \operatorname{Tr} A^{-s}. \tag{4}$$

However, when there is an infinite number of eigenvalues (and these are the cases of interest in physics), as occurs when $A$ is a differential operator, the product of the eigenvalues will be an ill-defined quantity and will no longer serve as a good prescription for $\det A$. In other words, expression (3) with $\zeta(s;A)$ given by (4), as it stands, is meaningless because it is not valid anymore to write:

$$\sum_n \left( \frac{\partial \lambda_n^{-s}}{\partial s} \right)_{s=0} = \left\{ \frac{\partial}{\partial s} \left( \sum_n \lambda_n^{-s} \right) \right\}_{s=0}. \tag{5}$$

Hence, for these cases we need to define a finite prescription for $\det A$. The generalized zeta function prescription consists basically of the following three steps: (i) We first compute the eigenvalues of $A$ subject to the appropriate boundary conditions and then write down the corresponding $\zeta$ function $\zeta(s;A) = \operatorname{Tr} A^{-s} = \Sigma_n \lambda_n^{-s}$. (ii) Since the last sum does not converge at $s=0$, we make an analytical extension of this function to the whole complex plane of $s$ (or at least to a domain that contains the origin). (iii) After the analytical extension is made, we just write $\det A = \exp\{-\zeta'(s=0;A)\}$.

In order to apply the $\zeta$-function method described above in the computation of $F(t_f - t_i)$, we first need to find the eigenvalues of $\mathcal{O}$ with Dirichlet boundary conditions. For convenience, we shall make the rotation in the complex plane $t = e^{-i\pi/2}T = -iT$. Let us also define the corresponding finite interval in $T$ by $t_f - t_i = -i(T_f - T_i) = -i\tau$. We then have that

$$\mathcal{O} = \omega^2 + \frac{d^2}{dt^2} \rightarrow \mathcal{O}_T = \omega^2 - \frac{d^2}{dT^2}. \tag{6}$$

This analytical extension guarantees that all the eigenvalues (now of the operator $\mathcal{O}_T$) are positive. Of course, after the calculations are finished, we must undo this transformation, that is, we must substitute $\tau = i(t_f - t_i)$. Solving the eigenvalue equation $\mathcal{O}_T f_n(T) = \lambda_n f_n(T)$ with Dirichlet boundary conditions $f_n(0) = 0 = f_n(\tau)$, we get

$$f_n(T) = \left\{ \sin\left( \frac{n\pi}{\tau} T \right); n = 1,2,... \right\},$$
$$\lambda_n = \omega^2 + \frac{n^2\pi^2}{\tau^2}, \quad n = 1,2,\ldots. \tag{7}$$

Consequently, the associated generalized $\zeta$ function is given by

$$\zeta(s;\mathcal{O}_T) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \left( \frac{\tau}{\pi} \right)^{2s} \sum_{n=1}^{\infty} \frac{1}{(n^2 + \nu^2)^s}, \tag{8}$$

where we defined $\nu = \omega\tau/\pi$. Since the above series converges only for $\operatorname{Re} s > 1/2$, we need to make an analytical extension in the complex plane of $s$ to include the origin. However, this can be done with no effort at all, for this series is precisely the so-called nonhomogeneous Epstein function, which we shall denote simply by $E_1(s;\nu^2)$ and whose analytical extension to the whole complex plane is well known and is given by[7,8] (see the Appendix for a brief deduction)

$$E^{\nu^2}(s;1) = -\frac{1}{2\nu^{2s}} + \frac{\sqrt{\pi}}{2\nu^{2s-1}} \frac{\Gamma(s-1/2)}{\Gamma(s)} + \frac{2\sqrt{\pi}}{\Gamma(s)} \sum_{n=1}^{\infty} \left( \frac{n\pi}{\nu} \right)^{s-1/2} K_{s-1/2}(2n\pi\nu), \tag{9}$$

where $K_\mu(z)$ is the modified Bessel function of second kind. This is a meromorphic function in the whole complex plane with simple poles at $s = 1/2, -1/2, -3/2, \ldots$, so that we can take its derivative at $s=0$ without any problem. Substituting the sum appearing on the right-hand side of (8) by the analytical extension given by (9), we may cast $\zeta(s;\mathcal{O}_T)$ into the following form:

$$\zeta(s;\mathcal{O}_T) = -\frac{1}{2} \left( \frac{\tau}{\pi\nu} \right)^{2s} + \frac{F(s)}{\Gamma(s)}, \tag{10}$$

where

$$F(s) = \left( \frac{\tau}{\pi} \right)^{2s} \left\{ \frac{\sqrt{\pi}}{2\nu^{2s-1}} \Gamma(s-1/2) + 2\sqrt{\pi} \sum_{n=1}^{\infty} \left( \frac{n\pi}{\nu} \right)^{s-1/2} K_{s-1/2}(2n\pi\nu) \right\}$$

is analytic at $s=0$. Taking, then, the derivative with respect to $s$ at $s=0$ and using that $\Gamma(s) \approx 1/s$ for $s \rightarrow 0$, we get

$$\zeta'(s=0;\mathcal{O}_T) = -\log(\tau/\pi\nu) + \lim_{s\rightarrow 0} \left\{ -\frac{\Gamma'(s)}{\Gamma^2(s)} F(s) + \frac{F'(s)}{\Gamma(s)} \right\}$$
$$= -\log(\tau/\pi\nu) + F(0). \tag{11}$$

From the above expression for $F(s)$ we readily compute $F(0)$, so that

$$\zeta'(s=0;\mathcal{O}_T) = -\log(\tau/\pi\nu) + \left[ \frac{\sqrt{\pi}\nu\Gamma(-1/2)}{2} + 2\sqrt{\pi} \sum_{n=1}^{\infty} \sqrt{\frac{\nu}{n\pi}} K_{-1/2}(2n\pi\nu) \right]. \tag{12}$$

Using that $\Gamma(-1/2) = -2\sqrt{\pi}$ and $K_{-1/2}(z) = \sqrt{\pi/2z}\, e^{-z}$, we obtain

$$\zeta'(s=0;\mathcal{O}_T)=\log(\pi\nu/\tau)-\pi\nu+\sum_{n=1}^{\infty}\frac{1}{n}e^{-n2\pi\nu}. \qquad (13)$$

It is not a difficult task to show that the above sum is given by (take its derivative with respect to $\nu$, sum the resultant geometric series, and then integrate in $\nu$; in order to eliminate the arbitrary integration constant, just use the fact that this sum must vanish for $\nu\rightarrow\infty$)

$$\sum_{n=1}^{\infty}\frac{1}{n}e^{-n2\pi\nu}=\pi\nu-\log[2\sinh(\pi\nu)]. \qquad (14)$$

From Eqs. (13) and (14) we then have

$$\zeta'(s=0;\mathcal{O}_T)=\log\left[\frac{\omega}{2\sinh(\omega\tau)}\right], \qquad (15)$$

where we used that $\nu=\omega\tau/\pi$. For the operator $\mathcal{O}_T^{(o)}$ we immediately get [it suffices to make $\omega\rightarrow 0$ in Eq. (15)]

$$\zeta'(s=0;\mathcal{O}_T^{(o)})=\log\left[\frac{1}{2\tau}\right]. \qquad (16)$$

Collecting all the previous results and rotating back to the real time $[\tau=i(t_f-t_i)]$, we finally obtain

$$\begin{aligned} F(t_f-t_i)&=\sqrt{\frac{\exp[-\zeta'(0,\mathcal{O})]}{\exp[-\zeta'(0,\mathcal{O}^{(o)})]}}\times\sqrt{\frac{m}{2\pi i\hbar(t_f-t_i)}}\\ &=\sqrt{\frac{m\omega}{2\pi i\hbar\sin[\omega(t_f-t_i)]}}, \end{aligned} \qquad (17)$$

where we used that $\sinh(i\theta)=-i\sin\theta$, in perfect agreement with Ref. 1.

Before we finish this note, we think it is interesting to establish a general equivalence between the $\zeta$-function method and Schwinger's proper time method. From the $\zeta$-function method just presented, we can write

$$\log\det\mathcal{O}=-\zeta'(s=0,\mathcal{O}), \qquad (18)$$

On the other hand, with the aid of the Mellin transform[9] applied here to an operator $\mathcal{O}$ with positive eigenvalues we can write

$$\zeta(s;\mathcal{O})=\mathrm{Tr}\,\mathcal{O}^{-s}=\mathrm{Tr}\frac{1}{\Gamma(s)}\int_0^{\infty}d\tau\,\tau^{s-1}e^{-\mathcal{O}\tau}. \qquad (19)$$

However, the last expression is not analytic at $s=0$ (though the presence of the exponential guarantees good behavior for large $s$, the limit $s\rightarrow 0$ is a divergent one), so that as it stands it is not valid to take the $s$ derivative at $s=0$. In order to circumvent this problem, we make the modification (regularization)

$$\zeta(s;\mathcal{O})\rightarrow\zeta(s,\alpha;\mathcal{O})=\frac{1}{\Gamma(s)}\int_0^{\infty}d\tau\,\tau^{s+\alpha-1}e^{-\mathcal{O}\tau}, \qquad (20)$$

where $\alpha$ is big enough to ensure that Eq. (19) is well behaved at $s=0$. Hence, first taking the $s$ derivative at $s=0$ and then taking the limit $\alpha\rightarrow 0$, we obtain

$$\begin{aligned} -\zeta'(s=0;\mathcal{O})&=-\lim_{\alpha\rightarrow 0}\lim_{s\rightarrow 0}\frac{\partial\zeta}{\partial s}(s,\alpha;\mathcal{O})\\ &=-\lim_{\alpha\rightarrow 0}\lim_{s\rightarrow 0}\mathrm{Tr}\left\{-\frac{\Gamma'(s)}{\Gamma^2(s)}\int_0^{\infty}d\tau\,\tau^{s+\alpha-1}e^{-\mathcal{O}\tau}\right.\\ &\qquad\left.+\frac{1}{\Gamma(s)}\int_0^{\infty}d\tau\log\tau\,\tau^{s+\alpha-1}e^{-\mathcal{O}\tau}\right\}\\ &=-\lim_{\alpha\rightarrow 0}\mathrm{Tr}\left\{\int_0^{\infty}\frac{d\tau}{\tau}\tau^{\alpha}e^{-\mathcal{O}\tau}\right\}. \end{aligned} \qquad (21)$$

Equation (21) corresponds precisely to Schwinger's formula written in a regularized way. Here we regularized by introducing positive powers of $\tau$, but other regularization schemes can also be used, as for example, the one used by Schwinger[10] in the computation of the Casimir effect[11] (for a simple introduction to this effect with some historical remarks see Ref. 12). It is common to write Eq. (21) formally with $\alpha=0$, but in fact, before taking this limit one should get rid of all spurious terms (those with no physical meaning).

In this note we have presented the generalized $\zeta$-function method for computing determinants in a very introductory level. A detailed discussion with a great variety of examples can be found in Ref. 7. One of the greatest advantages of this method is that for almost all differential operators and boundary conditions that are relevant in physics, the corresponding generalized $\zeta$ function (after the analytical extension is made) is a meromorphic function in the whole complex plane which is analytic at the origin. Furthermore, this method can also be applied successfully in many other branches of physics, as for example, statistical mechanics and quantum field theory among others. Of course there are many easier ways of obtaining $F(t_f-t_i)$ for the harmonic oscillator, but our purpose here was to introduce a new method, which is a powerful one and widely used nowadays. In this sense, we think that the harmonic oscillator provided a perfect scenario for the understanding of the three basic steps of the method, since every undergraduate student is somehow familiar with the harmonic oscillator.

## APPENDIX

In this Appendix we shall obtain the analytical extension of the Epstein function $E_1^{\nu^2}(s;1)$, given in the text by Eq. (9). With this goal, we first write down an equation involving the gamma function, which follows directly from its definition, namely:[9]

$$\Gamma(z)A^{-z}=\int_0^{\infty}d\tau\,\tau^{z-1}e^{-A\tau}, \quad \mathrm{Re}(z)>0. \qquad (22)$$

Using Eq. (22) with $A=n^2+\nu^2$, the Epstein function can be written in the form:

$$\begin{aligned} E_1^{\nu^2}(s;1)&=\sum_{n=1}^{\infty}\frac{1}{(n^2+\nu^2)^s}\\ &=\frac{1}{\Gamma(s)}\int_0^{\infty}d\tau\,\tau^{s-1}e^{-\nu^2\tau}\sum_{n=1}^{\infty}e^{-n^2\tau}. \end{aligned} \qquad (23)$$

On the other hand, from the so-called Poisson summation rule,[9] we can write:

$$\sum_{n=1}^{\infty} e^{-n^2\tau} = -\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\pi}{\tau}} + \sqrt{\frac{\pi}{\tau}} \sum_{n=1}^{\infty} e^{-n^2\pi^2(1/\tau)}. \quad (24)$$

Substituting Eq. (24) into (23), we get

$$E_1^{\nu^2}(s;1) = \frac{-1}{2\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-1} e^{-\nu^2\tau} + \frac{\sqrt{\pi}}{2\Gamma(s)} \int_0^\infty d\tau\, \tau^{s-3/2} e^{-\nu^2\tau} + \frac{\sqrt{\pi}}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^\infty d\tau\, \tau^{s-3/2} e^{-\nu^2\tau - n^2\pi^2/\tau}. \quad (25)$$

Using (22), the first and second integrals of the right-hand side of Eq. (25) can be written directly in term of Euler gamma functions. For the last term, we use the integral representation for the modified Bessel function of second kind:

$$\int_0^\infty dx\, x^{\alpha-1} x^{-\beta/x - \gamma x} = 2\left(\frac{\beta}{\gamma}\right)^{\alpha/2} K_\alpha(2\sqrt{\beta\gamma}), \quad \mathrm{Re}\,\beta, \mathrm{Re}\,\gamma > 0. \quad (26)$$

Therefore, we finally obtain for Eq. (25):

$$E_1^{\nu^2}(s;1) = -\frac{1}{2\nu^{2s}} + \frac{\sqrt{\pi}\Gamma(s-1/2)}{2\Gamma(s)\nu^{2s-1}} + \frac{\sqrt{\pi}}{\Gamma(s)} \sum_{n=1}^{\infty} \left(\frac{n\pi}{\nu}\right)^{s-1/2} K_{s-1/2}(2\pi n\nu), \quad (27)$$

which is precisely Eq. (9). Some comments are in order here. (i) To say that Eq. (27) corresponds to the analytical extension of $E^{\nu^2}(s;1)$ to a meromorphic function in the whole complex plane means that this expression is an analytical function in the whole complex plane except for an enumerable number of poles (which can be infinite) and coincides with the original sum in the region where the sum was defined. (ii) It is worth emphasizing that the above expression is analytic at the origin; in fact, the structure of simple poles of this function is dictated by the poles of the Euler gamma function. It is easy to see that the poles are located at $s = 1/2, -1/2, -3/2, -5/2$, etc.

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## A CONTINUAL SOURCE OF JOY

It is not the least of the triumphs of physics in the present century to have penetrated so deeply behind the veil of our everyday perceptions as to reveal beyond doubt that our first-hand experience of the universe is at best a narrow and distorted view of whatever structure it is of which we are a part; yet we have this assurance also, that our universe, in so far as we have been able to probe it, is a marvellously ordered creation whose fuller understanding is a continual source of joy, in which intellectual satisfaction is mingled with wonder and humility.

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