Customising a tennis racket by adding weights

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Abstract

Most elite tennis players customise their rackets by adding weights at various points to enhance the performance of the racket and to improve its balance and feel. The additional weights also alter the swing weight, vibration frequency, amplitude and node location, the centre of percussion, apparent coefficient of restitution and racket power. In this paper, experimental data is presented on the effects of adding a 30-g mass at several points on the racket frame. The data is consistent with a one-dimensional, nonuniform beam model of the racket, with a few minor exceptions. Mass added at the tip of a racket is more effective in increasing racket power than mass added at any other location, and has the additional advantage of shifting the point of maximum power towards the tip of the racket. For a serve, the point of maximum power can be shifted to a point near the tip of the racket, giving the player a significant height advantage. For a groundstroke, the added mass will shift the point of maximum power to a point near the centre of the strings, where players normally impact the ball.

Keywords: tennis racket, balance point, swing weight, racket power

Introduction

Tennis rackets can be classified as either neutrally balanced, head heavy or head light, depending on whether the balance point (i.e. the centre of mass) is, respectively, in the middle of the racket, closer to the head or closer to the handle. The average recreational player is probably unaware of the location of the balance point of his or her racket, since it has no obvious significance to the player. Professional players usually add lead weights to the head and/or handle, partly to make the racket heavier but also to give it the correct feel and to improve the performance of the racket. The effect of the additional weights is not as straightforward as one might expect. As well as shifting the balance point and changing the total weight of the racket, the addition of mass also affects the swing weight (i.e. the moment of inertia about an axis near the end of the handle), the polar moment of inertia (for rotation about the long axis), the vibration frequency and amplitude, the location of the vibration nodes, the location of the centre of percussion (COP) and the apparent coefficient of restitution (ACOR). The ACOR is defined as the ratio of the normal components of the rebound to incident speed of a ball in a reference frame where the racket is initially at rest. All of these parameters affect the feel of the racket and they also affect racket power and control. In this paper, the more obvious physical effects of additional weights are examined, but the effects on racket feel, power and control are more difficult to quantify. The swing weight and the polar moment both affect the speed at which a racket can be swung or manoeuvered and therefore affect both racket power and the ability of a player to control the racket. The relation between swing weight and racket speed is not well known. Studies of swing speed have been made for golf clubs (Daish 1972) and baseball bats (Watts &

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Bahill 1990), but the equivalent study for tennis rackets is still in a preliminary stage (Mitchell *et al.* 2000).

The dynamic behaviour of a tennis racket can be modelled theoretically at various levels of sophistication, depending on whether the model is one, two or three dimensional and depending on whether the player is considered as part of the dynamic system. In this paper, a simple theoretical model is considered where the racket is treated as a one-dimensional flexible beam with a nonuniform mass distribution in a direction along the beam. Numerical solutions are compared with experimental data for an actual tennis racket, which was modified by adding a 30-g mass at various locations. A one-dimensional model can be used to predict with certainty the theoretical shift in the balance point, the change in swing weight and the shift in the centre of percussion. As shown below, the one-dimensional model also provides good agreement with measurements of the ACOR as well as the vibration frequency and node locations.

Centre of mass, moment of inertia and centre of percussion

It is not easy to determine the mass distribution of any given racket by physical measurements, apart from cutting it into small pieces and measuring the mass of each piece. However, a useful approximation is to assume that the racket consists of two uniform beam sections, a handle of length L_1 and mass M_1, and a head of length L_2 and mass M_2, as shown in Fig. 1. The head itself has a more complex structure than a uniform beam, but the dynamics can usefully be modelled in terms of beam theory, at least for impacts along the major (long) axis of the racket. For this model, it is easy to show that the centre of mass (CM) is located at a distance h from the end of the handle, where

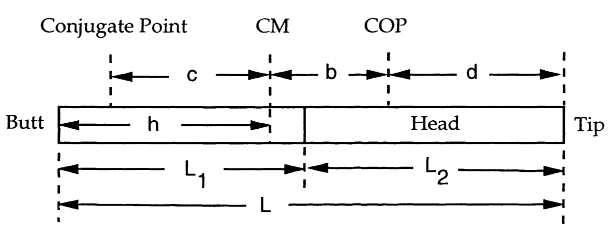

Figure 1 Two segment beam model of a tennis racket, with a handle of length L_1 and a head of length L_2.

$$h = \frac{M_1L_1 + M_2L_2 + 2M_2L_1}{2(M_1 + M_2)} \tag{1}$$

The moment of inertia of the racket about an axis through the CM, for rotation in the direction normally used to strike a ball incident at right angles to the string plane, is given by

$$I_{\text{cm}} = \frac{(M_1L_1 + M_2L_2)^2 + 4M_1M_2(L_1^2 + L_1L_2 + L_2^2)}{12(M_1 + M_2)} \tag{2}$$

and the moment of inertia about a parallel axis through the end of the handle is given by

$$\begin{aligned} I_h &= I_{\text{cm}} + M_{h^2} \\ &= \frac{1}{3}(M_1L_1^2 + M_2L_2^2) + M_2L_1(L_1 + L_2) \end{aligned} \tag{3}$$

where $M = M_1 + M_2$ is the total mass of the racket. For any given M, L_1 and L_2, both h and I_h increase as M_2 increases, i.e. as the racket is made more head heavy. In principle, measurements of both h and I_{cm} or I_h could be used to determine the mass and length of each segment uniquely. Allowable solutions include a zero length but finite mass segment, corresponding to a point mass located at one end of a uniform beam. In practice, it is difficult to measure I_{cm} or I_h with sufficient accuracy to determine the segment masses and lengths with good precision. Solutions that give the correct value of h all have the same value of I_{cm} or I_h within a few percent. Similarly, the moment of inertia of a racket is determined primarily by its mass, its length and the location of the balance point, and it is not otherwise very sensitive to the precise details of the mass distribution. For example, suppose that the head section is modelled as a circular ring of mass M and radius R. The moment of inertia (MOI) of the ring about an axis through any diameter is $MR^2/2$. If a racket is approximated as a uniform beam of mass $M/2$ and length $2R$ (the handle) attached to a circular ring of mass $M/2$ and diameter $2R$ (the head), then the MOI about an axis through the butt end of the handle is $65MR^2/12 = 5.417MR^2$. If the racket is approximated as a uniform beam of mass M and length $4R$, the MOI about one end is $16MR^2/3 = 5.333MR^2$. The circular head model therefore has a MOI that is only 1.6% larger than a uniform beam model.

The above expressions simplify if $L_1 = L_2 = L/2$, where L is the total length of the racket. In this case,

$$h = \frac{L(M_1 + 3M_2)}{4(M_1 + M_2)} \tag{4}$$

and

$$I_h = \frac{1}{12}(M_1 + 7M_2)L^2 \tag{5}$$

indicating that the head mass is seven times more significant than the handle mass in determining I_h. If the beam is completely uniform, then $M_1 = M_2 = M/2$, $h = L/2$, $I_{cm} = ML^2/12$ and $I_h = ML^2/3$.

If a ball impacts on the head of a racket at normal incidence and at a distance b from the CM, then the impact point is the centre of percussion (COP) with respect to a conjugate point in the handle. If the conjugate point is located a distance c from the CM, then $b = I_{cm}/(Mc)$. The location of the COP therefore depends on the assumed location of the conjugate point, as well as the mass distribution of the racket. Figure 2 shows the COP location for a conjugate point at the far end of the handle, where $c = h$. The COP location is given in terms of its distance, d, from the tip of the racket, where $d = L - h - b$. Also shown in Fig. 2 are dimensionless plots of h/L, $I_{cm}/(ML^2/12)$ and $I_h/(ML^2/3)$ for the simplified beam model where $L_1 = L_2$. These plots are independent of the racket mass or length and provide a useful indication of the effects of varying the ratio of the head mass to the handle mass.

Effects of adding a point mass m at a distance p from the tip of the racket are shown in Fig. 3. If the racket itself is modelled as two uniform segments of masses M_1 and M_2 and of equal length (i.e. $L_1 = L_2 = L/2$) then

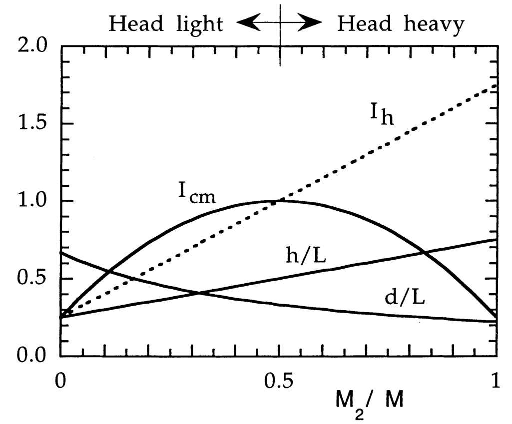

Figure 2 Variation of h/L, d/L, $I_h/(ML^2/3)$ and $I_{cm}/(ML^2/12)$ vs. M_2/M for a racket, where M_2 is the head mass, M is the total racket mass, L is the racket length, h is the distance of the CM from the butt end of the handle and d is the distance of the COP from the tip of the racket.

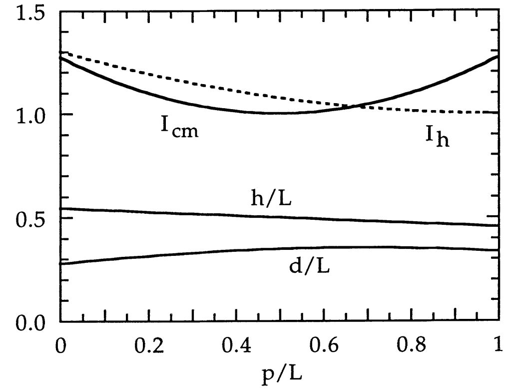

Figure 3 Variation of h/L, d/L, $I_h/(ML^2/3)$ and $I_{cm}/(ML^2/12)$ vs. p/L for a uniform racket of mass M, when a mass $m = 0.1M$ is added at a distance p from the tip of the racket, where L is the racket length, and h and d are shown in Fig. 1.

$$\frac{h}{L} = \frac{M_1 + 3M_2}{4(M + m)} + \frac{m(1 - p/L)}{(M + m)} \tag{6}$$

where $M = M_1 + M_2$ is the original racket mass before the point mass is added. The moment of inertia about the end of the handle is increased to

$$I_h = \frac{(M_1 + 7M_2)L^2}{12} + m(L - p)^2 \tag{7}$$

I_{cm} can be obtained by means of the parallel axis theorem, $I_{cm} = I_h - (M + m)h^2$, where h is given by

Eq. (6). The plots in Fig. 3 are shown for a case where $M_1 = M_2$, $L_1 = L_2$ and where a mass $m = M/10$ is added at various points along the racket from the tip to the end of the handle. There is a significant increase in both I_{cm} and I_h when the mass is added near the tip of the racket, but no increase in I_h if the mass is added at the far end of the handle. The addition of the small mass shifts the COP significantly towards the tip of the racket when the mass is added near the tip, but it shifts the COP slightly towards the throat of the racket if the mass is added near the throat or in the handle. There is no change in the location of the COP if the additional mass m is located at the original COP. Without the added mass, $h/L = 0.5$ and $d/L = 1/3$ for a completely uniform racket. The addition of a mass $m = M/10$ at the tip of the racket shifts the CM to $h/L = 0.545$, it shifts the COP to $d/L = 0.278$ and it increases I_h by 30%. For example, if $L = 70$ cm, the CM is shifted by 3.1 cm and the COP is shifted by 3.9 cm, both towards the tip.

The effects of removing mass at various points along the racket are also described by Eqs (6) and (7) if m is taken to be negative. For example, one can shift the COP towards the tip of the racket by removing mass from the region near the top of the handle (i.e. near the throat of the racket) as pointed out by Frolow (1983). A player cannot easily remove mass from a finished racket, but the manufacturer can choose this option when designing a racket.

Swing weight

The swing weight of a racket, or any other similar item of sporting equipment, is sometimes defined by the product of M and h, since this is easier to measure than I_h (Daish 1972; Jorgensen 1999). The two are related, but I_h is of more direct relevance in determining the angular momentum or acceleration or the rotational kinetic energy of a racket. Swing weight is sometimes defined as the moment of inertia about an axis within the handle, at a finite distance from the end of the handle, but the distance can be chosen differently by different authors depending on the experimental arrangement used to swing the racket. In any case, the actual axis of rotation of a racket, when it is swung towards a ball, is typically 5–20 cm beyond the end of the handle (Brody 1997), depending on the amount of wrist action and the type of stroke. In terms of the biomechanics of a stroke, separate swing weights could be identified for rotation about a number of other axes including the wrist, elbow and shoulder, each of which will generate a different feel of the racket depending on the particular player and stroke. For example, a player using a lot of wrist action could find that a head heavy racket is difficult to control, especially if the player has a relatively weak wrist. Similarly, the addition of mass to the handle of a racket has essentially no effect on I_h but it will alter the balance point and the swing weight about any other axis. Mass added to the handle may therefore be used to increase the swing weight about an axis through the elbow or the shoulder without changing the swing weight about an axis through the wrist.

Given that L, M and h are more easily and accurately measured than the swing weight I_h, an interesting practical question is whether a simple calculation of the first moment Mh is sufficient to characterize a racket, and whether a measurement of I_h provides any new or fundamentally different information. An alternative but equivalent question is whether one can estimate or calculate I_h reliably just from measurements of L, M and h. In theory, the answer is no, since Mh and I_h provide intrinsically different measures of the mass distribution. The quantity Mh is a measure of the first moment of mass and I_h is a measure of the second moment. In practice however, one can estimate I_h reliably from L, M and h, since the mass distribution of a typical racket is not sufficiently pathological to introduce a significant error. Figure 4 shows a plot of I_h vs. h for a uniform (neutrally balanced) racket of length 71 cm and mass 320 g, modified by adding a 30-g mass at various positions along the racket from one end to the other. The actual values of h and I_h are given by Eqs (6) and (7). Alternatively, if one were to measure L, M and h for such a

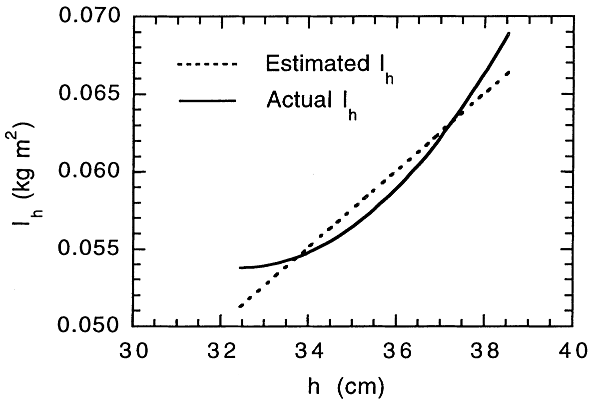

Figure 4 Actual and estimated swing weight, I_h, vs. balance point, h, for a racket of mass 350 g and length 71 cm.

racket, where $M = 350$ g is the total mass of the racket, then one could model the racket as two equal length segments with a head to handle mass ratio given, from Eq. (4), by

$$\frac{M_2}{M_1} = \frac{4h - L}{3L - 4h} \tag{8}$$

One can then estimate I_h for the racket using Eq. (5), which is also plotted in Fig. 4. The difference between the actual and estimated values of I_h is typically less than 2%, which is comparable to the accuracy with which I_h can usually be measured. Furthermore, a difference of 2% in I_h would not be noticed by most players (Brody 2000). Beak *et al.* (2000) obtained a similar result, although in their experiment the balance point was not held constant when the swing weight was varied. One can therefore characterize a racket, with some degree of confidence, purely in terms of its mass, length and balance point, all three measurements being required to calculate the swing weight about any given axis. An interesting exception to this general rule was used by Brody (2000) to vary I_h by up to 10% even though M, L and h were all kept constant. This was achieved by adding about 50 g at the balance point to minimize the increase in I_h or by adding half the mass at the tip and half at the butt end of the handle to maximize the increase in I_h. Brody (1985) also describes a method of measuring I_{cm} to better than 1% using a torsional pendulum.

Vibration modes and apparent coefficient of restitution for a nonuniform beam

Despite the relatively complex structure of a tennis racket, the vibration modes of the frame and the location of the vibration nodes can be described with reasonable accuracy by treating the racket as a simple one-dimensional beam (Brody 1995; Cross 1998a). For a uniform, freely suspended beam of length L, the fundamental vibration nodes are located at a distance 0.224 L from each end of the beam. If $L = 0.7$ m, the nodes are located 15.7 cm from each end. One of the two nodes in a tennis racket is usually located near the centre of the strings, typically about 16 cm from the tip of the racket, the other being located about 15 cm from the end of the handle. A racket will also support higher frequency modes, but these are excited with significant amplitude only if the racket is subject to a relatively short duration impulse, typically 2 ms or less. In practice, the impact with a ball has a duration of about 4 or 5 ms, which effectively damps all vibrations other than the fundamental mode. The fundamental vibration period varies from about 5 ms for stiff, wide-body rackets to about 10 ms for relatively flexible rackets.

In this section, solutions for a nonuniform beam are examined in order to obtain more accurate estimates of the node locations and vibration frequencies of head heavy and head light rackets. The beam model is easily extended to include calculations of the vibration amplitude and the ACOR, by considering the impact of a ball on the beam.

The equation of motion for a beam subject to an external force, F_o per unit length, has the form (Cross 1999)

$$\rho A \frac{\partial^2 y}{\partial t^2} = F_o - \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) \tag{9}$$

where ρ is the density of the beam, A is its cross-sectional area, E is Young's modulus, I is the area moment of inertia and y is the transverse displacement of the beam at coordinate x along the beam. For a rectangular beam with cross-sectional dimensions $a \times b$, $I = ba^3/12$ where b is the width of the beam and a is the dimension in the direction of vibration. Equation (9) neglects the shear force which is of minor significance for short wavelength modes but is negligible for long wavelength modes.

The quantity EI represents the beam stiffness or the flexural rigidity of the beam (Timoshenko & Young 1968). If the material properties, expressed by the parameters ρ and E, remain constant, but the beam dimensions are functions of x, then both A and I will vary with x. However, in this paper, we are interested mainly in the effects of adding lead weights at various points along the beam, in which case the primary effect is to alter the local value of the density, ρ. The lead weights added to a racket normally take the form of thin, self-adhesive strips. If the weights are added to a uniform beam, then EI remains essentially constant along the beam. Numerical solutions of Eq. (9) for a uniform beam of mass M and length L can be obtained by dividing the beam into N equal segments each of mass $m = M/N$ and separated in the x direction by a distance $s = L/N$. If a lead weight is added to any segment, then m can be modified for that segment to include the mass of the lead. Similarly, if extra weight is added to the handle in the form of a grip or an actual lead weight, this can also be treated as a change in the local density. The grip may also alter the local value of EI, but this effect was not included in the calculations presented below since changes at the handle end of a racket have a negligible effect on the ACOR and on the location of the node in the head region. For example, the ACOR is unaffected even if the handle is clamped in a vice (Cross 1999).

An impacting ball may exert a force acting over several adjacent segments, depending on the ball diameter and the assumed number of segments. For simplicity it was assumed that the ball impacts on only one of the segments, exerting a time-dependent force, F. The equation of motion for that segment (the nth segment) is obtained by multiplying all terms in Eq. (9) by s, in which case

$$m\frac{\partial^2 y_n}{\partial t^2} = F - (EIs)\frac{\partial^4 y_n}{\partial x^4} \tag{10}$$

assuming that E and I are independent of x. The equation of motion for the other segments is given by Eq. (10) with $F = 0$. In fact, a ball impacting on the strings of a racket exerts an almost simultaneous force acting over the whole head region of the frame, since the strings transmit the force to the frame with a propagation delay of only about 0.5 ms. The force is not transmitted equally to all sections of the head and tends to be concentrated by the cross strings, in a direction along the major axis, near the impact point. The one-dimensional model ignores this detail, but it includes the fact that a force is rapidly transmitted to segments adjacent to the impact point, via the second term on the right of Eq. (10). This term represents the force exerted on the nth segment by the adjacent segments.

The ball and the strings can both be modelled as simple springs, with a combined spring constant k_b. The equation of motion for the ball is then given by

$$m_b \mathrm{d}^2 y_b/\mathrm{d}t^2 = -F = -k_b(y_b - y_n) \tag{11}$$

where m_b is the ball mass, y_b is the displacement of the ball, and $y_b - y_n$ is the compression of the ball and the strings. The spring constant was taken as $k_b = 3 \times 10^4$ N m^{-1} to be consistent with the observed impact duration, about 5 ms. It was assumed that at $t = 0$, $y_b = 0$, $y = 0$ for all beam segments, the beam was initially at rest and that $\mathrm{d}y_b/\mathrm{d}t = v_1$. The subsequent motion of the ball and the beam was evaluated numerically as described by Cross (1999). These results were used to determine the rebound speed of the ball, v_2, and the apparent coefficient of restitution (ACOR), $e_A = v_2/v_1$.

Solutions of Eqs (10) and (11) over-estimate the ACOR due to the neglect of energy dissipation in the ball. A correction factor for ball losses was obtained from measurements of the COR when the ball impacted on the strings with the head clamped. At the low impact speeds used in this paper, $e = 0.86 \pm 0.01$. Values of e_A quoted below are the theoretical values obtained from Eqs (10) and (11) multiplied by 0.86.

The vibration modes of a beam with uniform E and I but with a nonuniform mass distribution can be obtained by numerical solution of Eq. (10), assuming that $F = 0$, and that $\partial^2 y/\partial t^2 = -\omega^2 y$ where ω is the vibration frequency. In this case, Eq. (10) reduces to

$$\frac{\partial^4 y}{\partial x^4} = \left(\frac{\mu\omega^2}{S}\right) y \tag{12}$$

where $\mu = \rho A$ and $S = EI$. Numerical solutions of Eq. (12) show that the the modes of a nonuniform beam are qualitatively similar to those of a uniform beam and that the node locations are shifted in a direction that one would expect from a localized increase or decrease in wave speed along the beam. For example, if the wave speed is locally small, then the wavelength is locally small and the nodes are relatively close together. For a uniform beam, one can assume that $\partial^2 y/\partial x^2 = -k^2 y$, in which case the dispersion relation has the form $k^2 = \omega(\mu/S)^{1/2}$. For any given ω, k increases and the wavelength decreases as μ increases. Consequently, if the left end of a beam has a higher μ than the right end, the vibration nodes of the beam are shifted, relative to those for a uniform beam, towards the left end of the beam.

Solutions of Eqs (10)–(12) are presented below only for cases where the model racket is freely supported at both ends. The handle of a racket is subject to the force exerted by the player, but this force can be neglected when calculating the ACOR since the impulse reflected off the handle arrives back at the impact point after the ball leaves the strings (Brody 1997; Cross 1999). As a result, the ACOR is independent of any force that may be applied to the handle. Furthermore, the vibration frequencies of a hand-held racket are essentially the same as those for a freely supported racket. The boundary conditions at a freely supported end are given by $\partial^2 y/\partial x^2 = 0$ and $\partial^3 y/\partial x^3 = 0$. For a uniform, rectangular cross-section beam, these boundary conditions yield analytical solutions for the mode frequencies for transverse vibrations, given by

$$\omega = \frac{G^2 a}{L^2}\sqrt{\frac{E}{12\rho}} \tag{13}$$

where $G = kL = 4.730$, 7.853, 10.996 for the first three modes. For a nonuniform beam, numerical solutions of Eq. (12) can be found by guessing both the frequency and $\partial y/\partial x$ at one end of the beam, and then iterating until both boundary conditions are satisfied at both ends of the beam.

Experimental results

The racket studied in this paper was a head light, Topspin 660 Powerlite racket manufactured since the late 1990s, strung at a tension of 260 N. The string area is 660 cm^2. The racket is shown in Fig. 5. It was constructed from a long, hollow tube of graphite composite material in the manner commonly used to construct most modern rackets. The throat section of the head, used to support the main strings in the centre of the racket, was

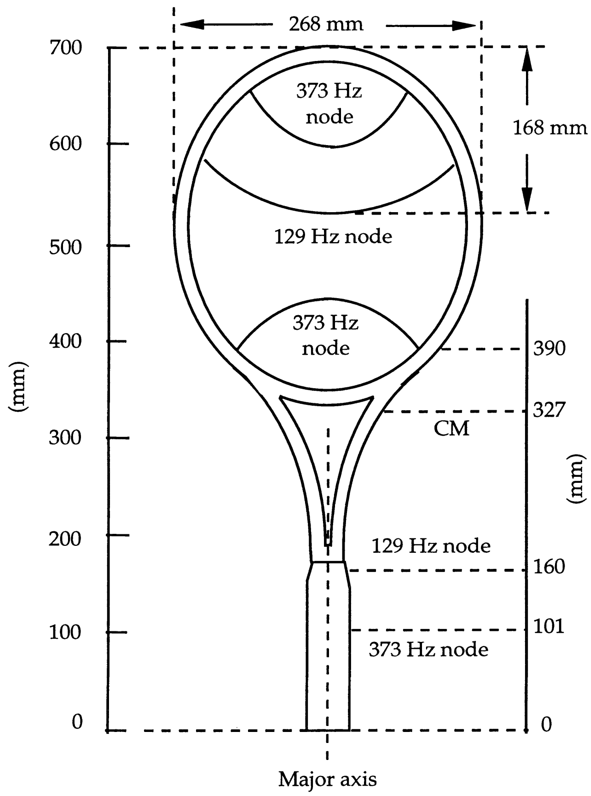

Figure 5 Topspin 660 Powerlite racket parameters, showing the node locations of the fundamental vibration mode at 129 Hz, and the second mode at 373 Hz, with no added mass.

constructed separately and moulded with some reinforcing material into the tubular section forming the rest of the head and extending down to the end of the handle. An interesting feature of the racket is a tight-fitting, removable plastic handle which allows the player to insert additional weights in a cavity inside the handle. The total mass of the racket, 332 g without any additional weights, consisted of the complete graphite frame (236.4 g), the removable handle (60.6 g), grommet strips for the strings (19.6 g) and the strings (15.4 g). Some of the properties of this racket are shown in Table 1, together with the corresponding properties when a 30-g lead mass was added to the tip and when 15 g masses were added at the 3 and 9 o'clock positions. Normally, masses smaller than 30 g would be added at any particular point to customise a racket, but relatively large masses were used in this experiment so that the effects could be more clearly observed. A mass of 30 g at the tip changed the racket from a head light to a slightly head heavy racket.

The balance point was measured simply by balancing the racket on the edge of a ruler. The swing weight was measured by swinging the racket as a pendulum about an axis near the butt end of the handle, and timing the period of oscillation, as described by Brody (1985). A small correction was made, using the parallel axis theorem, to calculate I_h about the axis through the butt end. The parallel axis theorem was also used to calculate I_{cm} from the measured values of h and I_h.

The location of the COP is commonly determined by swinging the racket as a pendulum in order to calculate both I_{cm} and the COP point. In this paper, a more direct method was used, in order to compare the actual and predicted locations of the COP. The COP was measured by suspending the racket by a 1.0-m length of string attached to the butt end of the handle and impacting the strings with a ball, along the major axis, to find the location on the strings at which the butt end of the handle did not deflect. It was not possible to adequately resolve rapid motion of the butt end by eye, although the motion can be captured on film as illustrated by Brody (1987). The COP was located using a small piezo disk mounted at the butt end of the handle as an accelerometer. The piezo signal was integrated to obtain the velocity waveform, which provides a much clearer picture of motion of the handle than does the acceleration waveform. The COP was identified by the change in sign of the DC velocity component for impacts either side of the COP, as described by Cross (1998a, 1998b).

The vibration frequency was measured by taping a small piezo disk to the frame and recording the output with a spectrum analyser. The racket was suspended by a 1.0-m length of string and vibrations were excited by tapping at various points on the frame with a soft hammer. The node points on the frame and on the string plane were located with the same apparatus, by searching for points where the fundamental mode was not excited. Table 1 shows the distance N, from the tip, of the node point on the major axis. For the unweighted racket, the node point 168 mm from the tip lies on the same cross string joining the 3 and 9 o'clock locations on the frame. A two-dimensional plot of the node line in the string plane is shown in Fig. 5. As described by Kawazoe (1997), the node line

Table 1 Measured and predicted racket parameters (1% error in I_h, 5% error in I_{cm})

			No added mass		15 g at 3 & 9 o'clock		30 g at tip	
Parameter	Sym.	Unit	Measured	Predicted	Measured	Predicted	Measured	Predicted
Mass	M	g	332 ± 0.1		362 ± 0.2		362 ± 0.2	
Bal. point	h	mm	327 ± 1	327	345 ± 1	344	357 ± 1	356
Swing wt	I_h	kg m^2	0.049	0.049	0.058	0.057	0.064	0.063
MOI	I_{cm}	kg m^2	0.0133	0.0133	0.0153	0.0145	0.0178	0.0168
COP	d	mm	220 ± 5	250	220 ± 5	240	205 ± 5	213
Frequency	f_1	Hz	129 ± 1	129.0	129 ± 1	129.2	117 ± 1	116.3
Node point	N	mm	168 ± 2	166	168 ± 2	166	120 ± 2	130

forms an approximately circular arc joining node points in the frame near the 2 and 10 o'clock positions.

The measured values of the balance point and fundamental vibration frequency of the unweighted racket indicate, from Eqs (8) and (12), that the racket can be modelled as two uniform beam sections with $L_1 = L_2 = 350$ mm, $M_1 = 188$ g, $M_2 = 144$ g and with $EI = 147$ Nm^2 in both sections. Using this model, the effects of an additional 30 g mass can be predicted, as outlined above, and are shown in Table 1. The predicted results for the balance point, swing weight and fundamental vibration frequency are in excellent agreement with the observed results in Table 1. As predicted, there is no change in the vibration frequency or node location when the additional mass is located at the 3 and 9 o'clock positions, i.e. in line with the vibration node in the middle of the string bed. There was also no observable shift in the location of the COP with added mass at these positions. Given that the node location on the frame, for the unweighted racket, was actually located close to the 2 and 10 o'clock positions, one might expect a small shift in the node location when masses are added at 3 and 9 o'clock. However, within experimental error, the node location along the major axis did not change. This is consistent with theoretical predictions that the shift in vibration frequency and node location is very small when mass is added at or near a node point.

Numerical solutions of Eq. (12) indicate that the fundamental node, with an additional 30 g at the tip, should be located 130 mm from the tip, rather than the observed 120 mm. The shift in the node location was therefore in the correct direction but larger than predicted. The COP was also found to be closer to the tip than expected, by about 20 mm, as described in more detail below.

Additional experiments were undertaken to measure changes in the vibration amplitude of the handle, for a fixed impact speed and impact location, when the 30 g mass was added at various locations on the racket frame. For this purpose, the racket was suspended by a length of string, and a piezo taped close to the butt end of handle was used to measure the acceleration waveform. A reduction in the acceleration amplitude of up to 20% was observed at most locations, but this is insignificant when compared with a factor of three reduction observed when the handle was held by one hand. Consequently, extra mass in the handle or at any other point would be of little use as a vibration dampener, unless the added mass is used to stiffen the frame or has energy absorbing properties.

Comment on node location

The vibration nodes of a racket along the major axis are located in essentially the same positions as those for a straight beam. However, this result is somewhat coincidental since the nodes in the frame are not at this position at all, but near the 2 and 10 o'clock locations, as shown in Fig. 5. It was observed that the locations of the nodes on the frame were not altered by stringing the racket, although the vibration frequency was lowered significantly when the racket was strung. The fundamental frequency dropped from 148 Hz to 141 Hz when the grommets were added and it dropped further, to 129 Hz, when the racket was strung. The nodes in the frame each moved 12 mm closer to the tip of the frame when the grommets were added to the frame, but the node locations did not shift any further when the strings were added.

Intuitively, one might expect that the strings would stiffen the frame, leading to an increase in the vibration frequency. However, the frequency decreased by 8.5% when the strings were added. The additional mass of the strings would account for about a 2% drop in frequency, given that the frequency is inversely proportional to the square root of the total racket mass. The additional decrease in frequency can be explained in terms of beam bending. If one applies a transverse force to the tip of an unstrung racket, then the racket will bend by an amount that depends on its stiffness. If the racket is then strung, and the experiment is repeated, the racket will bend further due to the component of string tension acting in a direction perpendicular to the tip. Consequently, the transverse stiffness of a racket is reduced when it is strung.

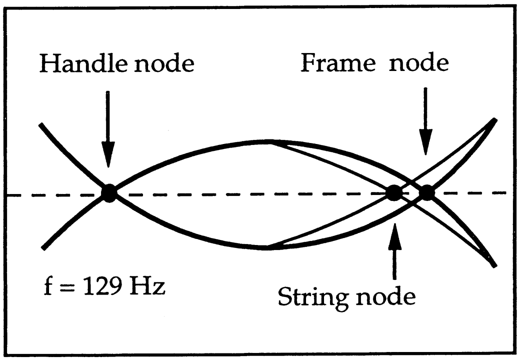

Figure 6 Schematic diagram showing the fundamental vibration mode of the racket frame and the resulting location of the main string along the major axis.

The location of the vibration nodes in the frame cannot be explained in terms of the one-dimensional beam model, but a qualitative picture of the fundamental mode and node locations, consistent with the above results, is shown in Fig. 6. The picture is also consistent with the fact that the vibration amplitude of the frame at the 6 o'clock position was observed to be essentially the same as that at the 12 o'clock position. The location of the node near the middle of the strings differs from that in the frame since the strings remain under tension and will tend to stretch along a straight line path between points of attachment to the frame, the cross strings preventing the mains being exactly straight. The string plane itself has a fundamental vibration frequency of about 500 Hz, so motion at about 129 Hz represents an almost quasi-static stretch of the strings. If the frame has a node in line with the 2 and 10 o'clock positions, the centre main string will have a node point closer to the centre of the string plane, as illustrated in Fig. 6.

Comment on the centre of percussion location

If one assumes that the racket is completely free of all external forces, then an impact anywhere along the major axis will cause the CM to recoil at speed V_{cm} and the racket to rotate at angular frequency ω. The resulting speed at the butt end of the handle is $v_B = V_{cm} - h\omega$. For an impact at the COP, $v_B = 0$, at least for a short period during and immediately after the impact. At any time t after the impact, the racket will have rotated through an angle $\theta = \omega t$ and the component of the rotational velocity, in a direction parallel to the incident ball, will be $h\omega \cos\theta$. Consequently, the butt end of the racket will start to move in the direction of the incident ball after the racket has rotated a few degrees. In the experiment described above, the butt end of the racket was tied to a length of string, which exerts a restoring force on the handle as soon as the butt end begins to move. In order to identify the location of the COP, it was necessary to wait several vibration cycles, about 20 ms, in order to determine if the DC component of v_B was zero. During this time, a restoring force can arise from displacement of the handle, in which case the axis of rotation of the racket is shifted slightly. This effect was not studied in detail, but estimates of the magnitude of this effect are consistent with a shift of 20 mm in the expected location of the COP.

Of more direct significance is the possible shift in the COP resulting from the much larger force on the handle that would be exerted by the hand, as described by Hatze (1998). This effect warrants further careful experimental investigation. However, it is not easy to locate the COP precisely, since motion of the handle in this impact region is dominated by effects due to vibration of the handle (Cross 1998a; 1998b)

Measurements of apparent coefficient of restitution

The arrangement used to measure the ACOR is shown in Fig. 7. The racket was suspended horizontally, with the string bed in a vertical plane, using two lengths of string tied to the racket, one at the butt end of the handle and the other at the 3 o'clock point on the frame. A tennis ball was mounted, as a pendulum bob, at the apex of a V-shaped string support, so that it could impact the strings horizontally, at right angles to the string plane and at selected points along the major axis. This arrangement provided good reproducibility as well as a simple and accurate means of controlling

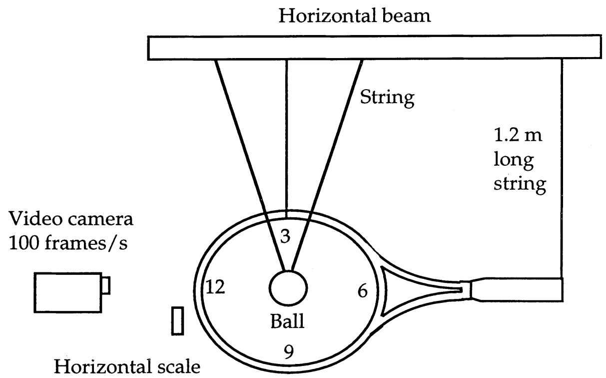

Figure 7 Experimental arrangement used to measure the ACOR along the major axis.

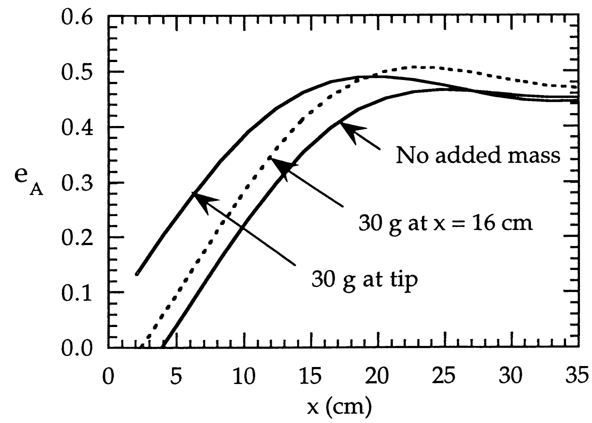

Figure 8 Numerical solutions of Eqs (10) and (11) for the Topspin racket with and without the addition of a 30-g mass.

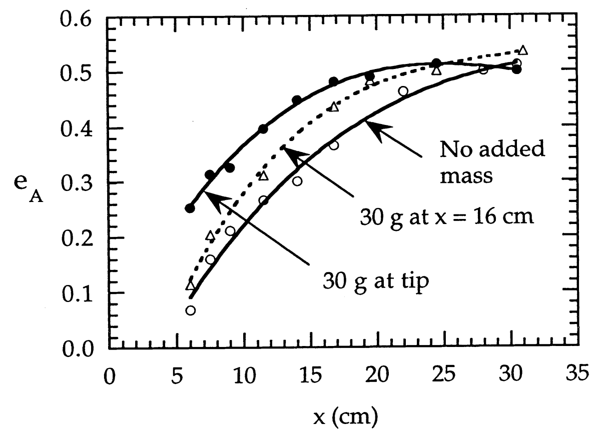

Figure 9 Experimental data showing e_A vs. x for the Topspin racket. The curves are third order polynomial fits to the data, not the theoretical curves shown in Fig. 8.

the impact point. The impact was filmed using a JVC 9600 video camera recording at 100 frames s^{-1}, with 2 ms exposure, in order to capture at least four frames close to the string plane before and after each impact. The centre of the ball facing the camera was marked with a felt pen to help track the speed of the ball. The ball impact speed was held constant at 1.6 ms^{-1}, much lower than the usual speed in a game of tennis, but perfectly adequate to compare with theoretical predictions and to observe the effect of adding weights to the racket frame. Off-axis impacts were not studied in this paper, although it is recognized that the addition of weights at points off-axis significantly increases the polar moment of inertia and will therefore increase the ACOR and reduce the tendency for the racket to twist in the hand for an off-axis impact (Brody 1985).

Measurements of the ACOR, and the corresponding theoretical estimates of the ACOR, are shown in Figs 8 and 9. The unweighted racket was modelled, as described above, with $M_2/M_1 = 0.77$ and $EI = 147$ Nm^2. Agreement between the theoretical and experimental values of e_A is very good, apart from the region near the throat where the ACOR is slightly higher than predicted. The latter effect may be associated with the additional mass used in the construction of the throat section. The most interesting result is that additional mass has the largest effect on the ACOR when it is located at the tip of the racket. One might expect that the ACOR near the middle of the strings might be maximized if the additional mass were located in line with the middle of the strings. However, it is maximized when the additional mass is located at the tip. A simple explanation of this effect is that additional mass at the tip gives the largest increase in swing weight or rotational inertia. Extra mass at the tip also shifts the CM of the racket closer to the centre of the strings than mass added at any other location. The ACOR is a maximum for impacts at the CM of the racket and decreases with distance from the CM due to the energy chanelled into rotational motion of the racket. The maximum possible ACOR of a racket is obtained when the head is clamped, in which case there is no energy transferred to rotation or translation of the racket and the ACOR is then equal to the COR (0.86 in this case).

Comment on racket power

Given that the ACOR is increased significantly by adding a small mass to the tip of a racket, an obvious question is whether or not this acts to increase racket power. In the case of a serve, the ball speed, v, is given by $v = (1 + e_A)V$ where V is the speed of the racket at the impact point and e_A is the ACOR. The serve speed is therefore directly proportional to V and depends only weakly on e_A. For example, a 20% increase in V leads to a 20% increase in serve speed, but a 20% increase in e_A, from say 0.40–0.48, leads to only a 5.7% increase in serve speed.

Any useful definition of racket power must therefore include both the ACOR and the swing weight of the racket. If the swing weight was infinite, the racket head speed would be zero. If the swing weight was zero, the racket head speed would not be infinite since a player cannot swing his or her arm that fast. However, within the normal range of racket weights from about 250 g to about 400 g, it is reasonable to assume that racket head speed will be proportional to some inverse power of swing weight, I. The head speed itself depends on the impact point on the racket. If the racket is swung at angular velocity ω about an axis at a distance A beyond the butt end of the handle, and the ball strikes the strings at a distance x from the tip of the racket, then the velocity of the impact point is $V = (A + L - x)\omega$. The angular velocity will depend on the swing weight about that axis, given by $I = I_{cm} + (A + h)^2$. If $\omega = k/I^n$, then the serve speed is given by

$$v = k\frac{(1 + e_A)(A + L - x)}{\left[I_{cm} + (A + h)^2\right]^n} \quad (14)$$

where k is a constant that depends on the strength of any given player and n is a number, yet to be determined, that describes the relation between head speed and swing weight for any given player. This number is not well known for tennis rackets. Nevertheless, it is of interest to examine solutions of Eq. (14) for typical values of n that may be relevant. For example, if one assumes that the total energy of the racket, $0.5I\omega^2$, remains constant when the swing weight is varied, then $n = 0.5$. If the total energy of the arm remains constant, then $n = 0$. If the total energy of the racket plus the arm remains constant, then n will be less than 0.5. Studies of golf clubs and baseball bats (Daish 1972; Watts & Bahill 1990) are more consistent with the latter assumption, and indicate that n is typically about 0.2 for these implements.

Plots of v vs. x for the e_A profiles in Fig. 9 are given in Fig. 10. It was assumed that the racket could be swung to generate a serve speed v = 200 km h^{-1} for an impact at $x = 15$ cm, when $A = 0.05$ m, regardless of the additional mass or its location. There is no logical reason for such an assumption, but (a) it provides a convenient if somewhat arbitrary means of comparing the three profiles and (b) the resulting value of n is within the range of interest between 0 and 0.5. Conversely, if one were to arbitrarily assume other values of n, then the three profiles would intersect at different points. The results in Fig. 10 were

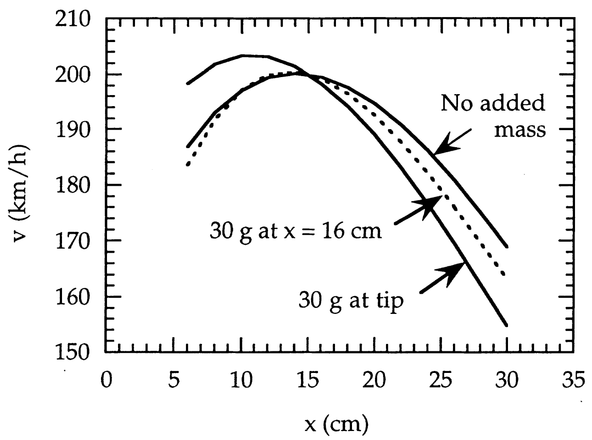

Figure 10 Serve speed v vs. impact distance from the tip, x, when the angular velocity of the racket is chosen to give $v = 200$ km h^{-1} at $x = 15$ cm and when the axis of rotation is 5 cm beyond the end of the handle.

obtained using $\omega = 69.00$ rad s^{-1} with no added mass, $\omega = 65.86$ rad s^{-1} with 15 g at both the 3 and 9 o'clock positions, and $\omega = 63.63$ rad s^{-1} when 30 g is added to the tip. For these parameters, ω is given to a good approximation by $\omega = 28.64/I^{0.314}$. The exponent n here is roughly consistent with the initial study reported by Mitchell *et al.* (2000).

It is also of interest to consider the effect of additional mass on groundstrokes. If the ball is incident normally at speed v_{in}, then the speed of the ball off the racket is given by $v = e_A v_{in} + (1 + e_A)V$. Figure 11 shows an example where $v_{in} = 78$ km h^{-1}, $\omega = 30$ rad s^{-1}, $A = 0.2$ m and where e_A is given by the experimental data in Fig. 9. It was assumed for simplicity that ω is independent of swing weight. For a groundstroke, this is a reasonable approximation since (a) the player does not normally swing as fast as possible and (b) the variation in swing weight is not as significant when the rotation axis is further from the butt end of the handle.

The results in Fig. 10 suggest that when weights are added to the racket head there may be only a slight or no change in power when serving from the middle of the strings, but there may be a significant increase in power when serving from a point near the tip. Even though power from the tip is generally a fraction lower than power from a point near the middle of the strings, the added height is an advantage to a server since the serve angle window is increased (Brody 1987). Adding mass at the tip also shifts the point of maximum power closer to the tip, particularly when the axis of rotation is close to the end of the handle. The actual axis of rotation is likely to vary from one player to another, but the general conclusions here will not be affected, provided the axis of rotation is not a large distance from the butt.

If a player chooses to hit an incident ball with zero racket speed, then $v = e_A v_{in}$ and the ball will rebound with maximum speed from a point near the throat of the racket where e_A is a maximum. At higher racket speeds, and/or when mass is added to the tip of the racket, the maximum power point shifts towards the tip of the racket. In Fig. 11, $v_{in} = 20$ m s^{-1} and V varies from 27 m s^{-1} at the tip of the racket to 18 m s^{-1} at the throat. Figure 11 shows that the maximum power point can be shifted to the centre of the strings when $V \sim v_{in}$ and when additional mass is added at the tip. Furthermore, the maximum power is also increased. Mass added at the 3 and 9 o'clock positions is not as effective in this respect. The primary reason for adding mass at 3 and 9 o'clock is to increase the polar moment of inertia and hence improve the rotational stability of the racket.

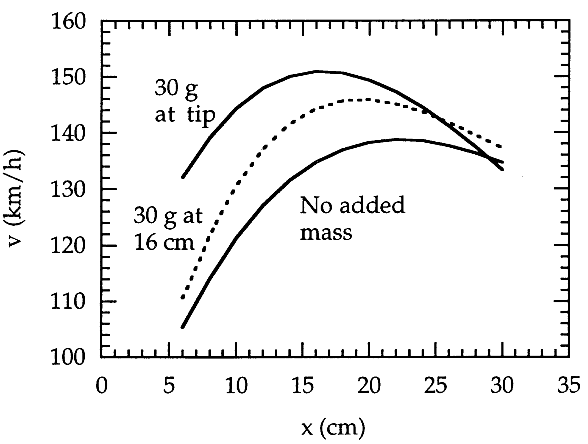

Figure 11 Ball speed v vs. impact distance from the tip, x, for a groundstroke. The incident ball speed is 20 m s^{-1} (78 km h^{-1}), the angular velocity of the racket is $\omega = 30$ rads^{-1} and $A = 0.2$ m.

Conclusions

The addition of weights to a tennis racket has an effect on almost every physical parameter of the racket. Noteable exceptions include the fact that (a) a weight located at the balance point does not shift the balance point, (b) a weight located at the centre of percussion does not shift the centre of percussion, (c) a weight at a vibration node does not shift the location of the node or the vibration frequency and (d) a weight added at any point in the handle does not change the swing weight about that point. A one-dimensional model of the racket provides a good description of all these effects, but a two-dimensional model would provide a better description of the vibration modes of the racket head. Further experimental studies would be desirable to unravel the complications introduced by the force of the player's hand on the handle, in relation to the effect on the COP. Further studies are also required to determine the relationship between racket speed and swing weight so that the concept of racket power can be more properly assessed. However, it is clear that mass added at the tip of a racket is effective in increasing racket power and that the point of maximum power is shifted towards the tip of the racket. For a serve, this locates the maximum power point near the tip, which gives the player an added height advantage. For a groundstroke, the maximum power point is shifted to a point near the centre of the strings, where players normally impact the ball. The effect of additional weights on the feel of a racket was not studied directly in this paper, but this will be determined in part by the location of the two sweet spots formed by the vibration node and the COP. The biomechanics of this would make an interesting study (see, for example, Beak *et al.* 2000 and the references therein).

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