The trajectory of a ball in lawn bowls

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The main objective in lawn bowls is to bowl a ball along a curved trajectory on a smooth grass surface so that it stops closer to the jack (a small white ball) than the opponent's ball. The curvature of the path is induced by shaping the ball so that one side is heavier than the other. Some of the properties of the trajectory are described to illustrate this interesting example of precessional motion.

I. INTRODUCTION

Lawn bowls is a popular outdoor sport for both men and women, especially with more senior citizens, since the game does not require a particularly high level of physical fitness. The game is played at a leisurely pace on an immaculately manicured lawn and requires only an empirical knowledge of the laws of precession. Because the ball is weighted on one side, it travels in a curved path whose radius of curvature decreases with time until the ball eventually stops. The ball has a mass typically of about 1.5 kg, and a diameter of about 12 cm, with small variations allowed to suit the individual bowler and the playing surface conditions. The ball is launched with a speed of about 4 ms^{-1}, at an angle of about 10° to the direct line of sight to the desired end point, then rolls at a walking pace for about 14 s over a total distance of about 25 m.

A weighted ball precesses like a gyroscope or a spinning top[1,2] but its axis of rotation is not anchored to a fixed point. As a result, the precessional motion is combined with linear motion to generate a curved trajectory. In this respect, the path of the ball is similar to that of a wheel or a coin[3,4] that rolls along a surface of its own accord. As the ball or the wheel slows down, the rate of precession increases and the radius of curvature decreases. A spinning top, or a rolling wheel or a coin precesses as a result of the torque associated with a tilt away from the vertical. A ball in lawn bowls tends to remain upright throughout its motion since the center of mass is shifted only slightly from the geometric center and since the surface is indented slightly by the ball, and the tendency of the ball to fall over as it comes to rest is resisted by a sideways reaction force from the ground. The ball will fall over when it is placed on a horizontal solid surface, and may also fall over at the end of its trajectory on a smooth grass surface, depending on its profile and the condition of the surface.

The ball used in lawn bowls is not perfectly spherical and it is not deliberately weighted by any additional mass. The weighting or ''bias'' is achieved by removing mass so that the shape remains circular in a cross section normal to the axis of rotation [as shown in Fig. 1(b)] and is slightly elliptical in a cross section that includes the axis of rotation [as shown in Fig. 1(a)]. In the elliptical cross section, the minor radius on one side of the ball is slightly larger than the other, with the result that one side of the ball is heavier than the other. The essential physics can be adequately modeled, both theoretically and experimentally, using a solid spherical ball that is weighted by grinding a flat surface on one side. Such an experiment was described in this Journal by Guest[5] in 1965 as an interesting undergraduate experiment in precession. Guest used a small steel ball bearing on a horizontal glass or rubber surface and photographed the trajectory with the aid of a strobe light to measure the linear and angular velocities. However, he did not derive a formula for the precessional velocity and did not relate the experiment to lawn bowls. A simpler laboratory technique is to roll a ball or disk on carbon paper over a sheet of white paper to leave an imprint of the trajectory on the sheet.

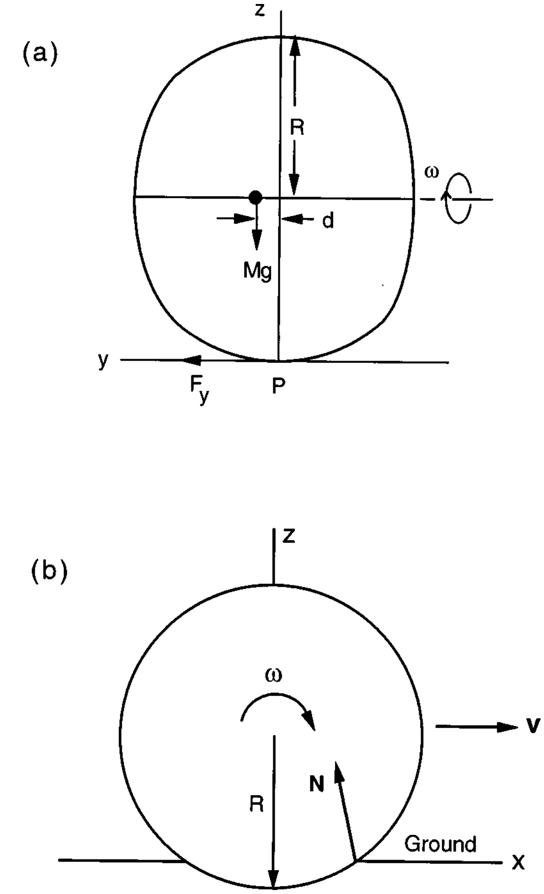

Fig. 1. Cross sections of a lawn bowl in (a) the $y-z$ plane and (b) the $x-z$ plane. The z axis is vertical, and the ball rolls in the horizontal $x-y$ plane, initially in the x direction. For a rolling ball that indents the surface, the ground reaction force, N, acts as shown in (b) to decrease both v and ω so that $v=R\omega$ at all times.

II. EQUATIONS OF MOTION

The situation modeled in this paper is shown in Figs. 1 and 2. A spherical ball of mass M and radius R is weighted by removing mass from one or both sides so that the center of mass is shifted a distance d from the geometric center, as shown in Fig. 1(a). The ball is launched in the x direction on a horizontal surface, in the $x-y$ plane, with an initial velocity v_0, and subsequently travels in a curved path from O to K as shown in Fig. 2. The main object of the following calculations is to determine the correct initial velocity and direction of the ball so that it finally stops at point K, a distance D from O and located at an angle Δ to the x axis. The problem has previously been examined by Brearley and Bolt[6] and by Brearley,[7] but their advanced mathematical treatment obscures the essential physics of the problem and is not suitable for any elementary presentation.

The linear motion of a rolling ball can be described by the relation

$$v=v_0-\mu gt, \tag{1}$$

where v is velocity of the center of mass of the ball, μ is the coefficient of rolling friction, g is the acceleration due to gravity, and t is the time. This relation follows from the fact that the frictional force acting to decelerate the ball can be expressed as $F=\mu Mg$. Experimentally, it is found that $\mu\sim0.032$ on most bowling greens, independent of the mass or speed of the ball, but it can be as large as 0.038 on a slow green or as small as 0.025 on a fast green.[6] In the following treatment, we ignore the initial sliding component of the motion since the ball starts rolling almost immediately after it is launched. An unbiased ball will move in a straight line path of length $S=v_0^2/(2\mu g)$ and come to rest at time $T=v_0/(\mu g)$. The coefficient of friction can be measured easily, without having to measure the initial velocity, from the relation $S=\mu gT^2/2$. The same relation holds for the curved trajectory of a weighted ball, as shown below. It is interesting to note that the ball takes longer to arrive at its

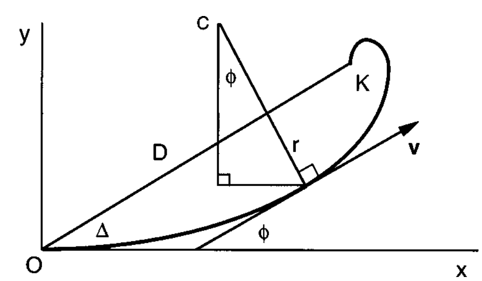

Fig. 2. Trajectory of a ball from O to K in the horizontal $x-y$ plane. The ball is launched along the x axis at angle $\triangle$ to OK.

destination on a fast green than it does on a slow green, since the ball must be launched at lower speed.

The coefficient of rolling friction[8–10] is not simply related to any other coefficient of friction, for the following reason. If a ball of radius R starts to slide without rolling, a friction force, F, acts horizontally at the point of contact, decreasing the linear velocity, v, of the center of mass and increasing the angular velocity, ω, via the torque applied to the surface. Rolling commences when $v=R\omega$, at which point there is no relative motion at the point of contact with the ground. It is observed that a rolling ball will eventually come to rest. One might assume that this is simply due to a friction force acting horizontally on the ball. While such a force would act to decrease the velocity, v, it would increase the angular velocity, ω, and the ball would end up spinning on the spot like the wheel of a car stuck in the mud. Rolling friction arises from the fact that the ball or the surface is slightly rough or deforms in such a way that the reaction force, N, from the ground does not act at a point below the center of rotation, but it acts forward of the center as shown in Fig. 1(b) to reduce both v and ω while maintaining the rolling condition $v=R\omega$. Since the deformation and hence the reaction force is approximately proportional to the weight of the ball, μ in Eq. (1) is essentially independent of M.

In order for a weighted ball to roll smoothly along a curved path, the axis of rotation must pass through the center of mass (CM), as shown in Fig. 1(a). If the axis of rotation does not pass through the CM, then the CM will rise and fall, generating a wobble in the motion. For simplicity, it is assumed that the axis of rotation remains horizontal throughout the trajectory. In fact, the ball will tilt slightly but this has a negligible effect on the trajectory since the ball tilts only near the end of its trajectory and usually only by a small amount.

A frictional force $F_y=Mv^2/r$, shown in Fig. 1(a), is necessary if the ball is to follow a curved path. In the absence of this component, the ball would travel in a straight line, rotating freely about a vertical axis through P due to precession. Such a result might be expected, for example, on a slippery ice surface. The torque component, $\tau_x=F_yR-Mgd$, acting about the geometric center acts to change the direction but not the magnitude of the angular momentum of the ball. This point is discussed in some detail in most elementary physics texts, in connection with the precession of a gyroscope or spinning top. The angular momentum is given by $L=I_{CM}\omega$ where I_{CM} is the moment of inertia about the rotation axis passing through the center of mass. At any point along the trajectory, a tangent to the path makes an angle ϕ with the x axis, and the change in angular momentum in the x direction, as a result of a small rotation $d\phi$ in time dt is $dL_x=-L\,d\phi$. The positive x direction is into the page in Fig. 1(a). The x component of the torque is therefore given by

$$\frac{Mv^2R}{r}-Mgd=L\,\frac{d\phi}{dt}=-I_{CM}\omega\omega_p, \tag{2}$$

where $\omega_p=d\phi/dt$ is the angular velocity of precession. Since $v=r\omega_p=R\omega$ when the ball is rolling, Eq. (2) yields

$$\omega_p=d\phi/dt=MgdR/(I_0v), \tag{3}$$

where $I_0=I_{CM}+MR^2$ is the moment of inertia about a horizontal axis through an edge of the ball. This relation differs from the usual expression for the precessional velocity of a gyroscope or spinning top in that the relevant moment of inertia is I_0 rather than I_{CM}. Equation (3) accounts, within 1%, for the experimental data given by Guest.[5] The radius of curvature is given by

$$r=\frac{v}{\omega_p}=\frac{v^2I_0}{MgdR}, \tag{4}$$

so

$$\frac{dr}{dt}=-pr\,\frac{d\phi}{dt} \tag{5}$$

where $dv/dt=-\mu g$ and $p=2I_0\mu/(MdR)$. Equation (5) can be integrated to give

$$r=r_0e^{-p\phi}, \tag{6}$$

where $r_0=pv_0^2/(2\mu g)$ is the initial radius of curvature. The path length of the ball is therefore

$$s=\int_0^\phi r\,d\phi=(1-e^{-p\phi})r_0/p. \tag{7}$$

The variation of ϕ with time can be obtained by integrating Eq. (3), using Eq. (1), to give

$$\phi=(2/p)\ln(v_0/v), \tag{8}$$

indicating that $\phi\rightarrow\infty$ as $v\rightarrow 0$. From Eq. (7), the total path length is $S=r_0/p=v_0^2/(2\mu g)$ which is independent of p and is therefore the same as that for an unbiased ball. The detailed behavior of the ball at the very end of its trajectory is not considered in this paper. Equations (3) and (4) indicate that $\omega_p\rightarrow\infty$ and $r\rightarrow 0$ as $v\rightarrow 0$, but the ball is likely to tilt or topple in the last few mm of the trajectory.

III. THE TRAJECTORY

The x and y coordinates of the trajectory can be obtained, as a function of time, from the relations $dx/dt=v\cos\phi$ and $dy/dt=v\sin\phi$, so

$$x+iy=\int_0^t ve^{i\phi}\,dt=\int_0^\phi re^{i\phi}\,d\phi, \tag{9}$$

where $v\,dt=ds=r\,d\phi$. Equating real and imaginary parts of Eq. (9) gives

$$x=\frac{r_0}{1+p^2}\,(p-p\lambda\cos\phi+\lambda\sin\phi) \tag{10}$$

and

$$y=\frac{r_0}{1+p^2}\,(1-\lambda\cos\phi-p\lambda\sin\phi), \tag{11}$$

where $\lambda=e^{-p\phi}$. The end point of the trajectory, where $\lambda=0$, therefore has coordinates $Y=r_0/(1+p^2)$ and $X=pY$, so the range of the ball is given by $D=(X^2+Y^2)^{1/2}=r_0/(1+p^2)^{1/2}$ and the angle Δ in Fig. 2 is given by $\tan\Delta=1/p$. A measurement of p can therefore be obtained by measuring Δ or X/Y, and the path length can be calculated from the expression $S=(1+1/p^2)^{1/2}D$. Since Δ is independent of v_0, a ball must be launched at the same angle to the line of sight, regardless of the desired range, D. However, it is against the rules and spirit of the game to use a protractor.

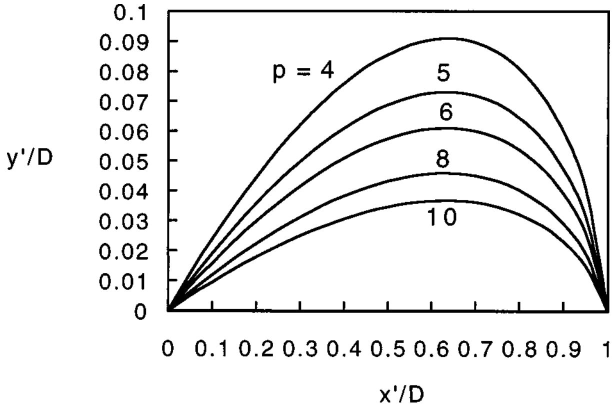

Fig. 3. Calculated trajectories of a ball of mass $M=1.5$ kg and radius $R=6$ cm for p values from 4 to 10. The coordinate system was rotated so the x' axis is along the line OK in Fig. 2. The x' and y' coordinates are normalized to the range, D, so the trajectories shown are independent of the initial velocity.

The radial distance, ρ, from any point (x,y) to the end point (X,Y) is given by $\rho=De^{-p\phi}$, indicating that the trajectory is a logarithmic spiral, as noted by Guest.[5] The ball reaches its maximum displacement in the x direction and moves purely in the y direction when $\phi=\pi/2$, at which point $v=v_0 \exp[-p\pi/4]$. The nature of the trajectory is determined primarily by the parameter, p. If I_{CM} is taken to be essentially that of a sphere, i.e., $I_{CM}=0.4MR^2$, then $p=2.8\mu R/d$, which, for $d\sim 1$ mm, is typically about 5 in lawn bowls, in which case $r/r_0\sim 0.0004$ and $v/v_0\sim 0.02$ when $\phi=\pi/2$. Consequently, the ball approaches the end point in a relatively gentle arc and does not spiral inwards by orbiting the end point more than once, as it would if p were less than about 1. Typical trajectories are shown in Fig. 3 for cases where $p=4$ to $p=10$, which are representative of normal playing conditions. The rules of lawn bowls place no specific limit on p, since it depends on μ, but balls are tested periodically against a master bowl. To pass the test, the bias of the ball must not be less than that of the master bowl. One of the challenges in lawn bowls is to allow for the fact that p can change during the course of a game since the grass can dry out during the afternoon and change the coefficient of rolling friction.

The results shown in Fig. 3 were obtained by rotating the coordinate system through an angle Δ so that all trajectories start and end at the same points (O and K, respectively). In this rotated $x'-y'$ coordinate system, where the x' axis is along the line OK in Fig. 2, one can show that the maximum y' displacement is given by $y'_{max}=D\lambda/(1+p^2)^{1/2}$ and it is located at $x'_{max}=D(1-py'_{max})$, where $\lambda=\exp(-p\Delta)$. For example, if $p=5$, $y'_{max}=0.0731D$ at $x'_{max}=0.634D$ and if $p=8$, $y'_{max}=0.0459D$ at $x'_{max}=0.633D$.

Almost identical results are obtained if the ball is allowed to tilt during the motion. The tilt angle, θ, tends to remain small for lawn bowls, but the trajectory when θ varies with time can be obtained by numerical integration of the equations $dx/dt=v\cos\phi$, $dy/dt=v\sin\phi$ and $d\phi/dt=A/v$ where v is given by Eq. (1) and $A=MgdR\cos\theta/I_0$. The result of such a calculation is that ω_p differs from the tilt-free case only near the end of the trajectory, but the trajectory itself is almost indistinguishable from the tilt-free case. Discussions of the trajectory of a ball in ten-pin bowling, which slides rather than rolls for most of the trajectory, can be found in Refs. 11 and 12.

From a teaching point of view, a rolling ball or coin is perhaps easier to understand than a precessing top or gyroscope. A gyroscope or top appears to have the somewhat mysterious property that it does not fall in response to the tilting torque, but moves in a direction normal to the applied force.[1,2,13] The same situation applies to a rolling ball or coin, but a student is more likely to be at ease with the fact that the ball or coin does not fall, because its momentum carries it forward.

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ARE SCIENTISTS INTERESTING?

Musing on my reaction, I came to a somewhat unnerving conclusion. The fact of the matter is, we scientists are simply not all that interesting. If I may generalize wildly, we are usually dull people with interesting ideas—as distinguished from artists (interesting people with dull ideas) and dancers and athletes (dull people with dull ideas and magnificent physical skills). The more a story focuses on what scientists actually do, I fear, the less interesting it will be.

James S. Trefil, in a review of *The Book Gambit* (by Carl Djerassi) and *Good Benito* (by Alan Lightman), in Scientific American **237**(5), 104–107 (1995).