Center of percussion of hand-held implements

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The center of percussion is commonly regarded as a sweet spot when referring to a baseball bat or a tennis racquet because it is assumed that there will be no sudden motion of the handle with respect to the hand if the corresponding axis of rotation passes through the hand. A problem with this interpretation is that the hand extends over a finite length of the handle and exerts an opposing reaction force on the handle. The hand also changes the total mass and moment of inertia of the system, while the arm restricts free motion of the hand. Experimental results are presented showing that the axis of rotation passes through the hand or the wrist for all the usual impact points on a hand-held implement. As a result, the impact point that feels best is usually the node of the fundamental vibration mode, not the center of percussion.

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I. INTRODUCTION

If an extended object such as a baseball bat is subject to an impulsive force near one end, and if no other force acts on the object, then the object will rotate about an axis toward the other end. The impact point and the rotation axis are known as a conjugate pair, the impact point being the center of percussion (COP) with respect to the rotation axis (and vice versa: if one point is the impact point, then the other is the rotation axis). As the impact point moves closer to the center of mass, the rotation axis moves closer to the opposite end and then moves to a point in space beyond the opposite end. If the impact point coincides with the center of mass, the object translates without rotation, in which case the rotation axis is at an infinite distance from the object. For a baseball bat, an impact about 7 in. from the barrel end of the bat will cause the bat to rotate about an axis located near the end of the handle. Such an impact point is commonly regarded as a sweet spot on the bat because the rotation of the bat about an axis under the hands implies that the handle does not jerk forward or backward during the impact.[1–4] Modern tennis racquets have a much larger head than old wood racquets, in part to ensure that the vibration node and free-racquet COP are near the middle of the strings rather than being close to the frame.[5]

The location of the COP with respect to an axis at or near the far end of a bat is easily calculated if the bat is subject only to an impulsive force acting at the impact point. Given that there is no translation of the bat at the axis of rotation, it might be expected that there will be no reaction force exerted on the bat by the hands if the impact occurs at the corresponding COP. This expectation would indeed be the case if the hands were of zero extent and located only at the axis. In fact, the hands extend over a finite length of the handle and exert a reaction force at the handle end of the bat.[6] The question arises as to how the force exerted by the hands will affect the location of the COP and how it affects the location of the axis of rotation. Similarly in tennis, the impact of a ball on the strings will cause the racquet to rotate about an axis located somewhere along the handle. For a double-handed backhand, a reaction force will be exerted by the hands over a considerable length of the handle. Hatze[7] concluded that the COP was a concept of limited significance when an implement is hand held.

A single-handed forehand or serve or volley is different. Even though one hand extends over about a 10 cm length of the handle, the wrist forms a natural axis of rotation located close to the far end of the handle. The hand and the racquet are located on one side of the wrist axis and the forearm is located on the other side. There is no discomfort if the racquet and the hand rotate as a single unit about an axis through the wrist, provided the hand exerts no net force on the wrist and forearm. Consequently, the COP for this axis is a potential sweet spot on the racquet. At any other impact point, the racquet and the hand will jerk forward or backward during the impact, resulting in a sudden impulsive force or shock on the forearm. A question of interest in this case is whether the racquet and the hand can indeed be treated as a single object, more massive than the racquet alone, or whether the racquet and the hand should be treated dynamically as two separate objects when calculating the impulsive forces on the racquet and the hand. There is a certain amount of cushioning in the hand and in the grip of a racquet handle, allowing for slight motion of the handle with respect to the hand. Soft cushioning and a relaxed grip might allow the racquet to move freely during the impact period, with the result that the handle will slam into the hand for impacts well removed from the COP. If the handle is gripped more firmly, the hand may rotate at essentially the same speed as the handle, in which case it is only the wrist and the forearm that will experience a shock force for impacts removed from the COP.

In this paper, measurements are presented of the recoil speed of the handle of an extended object when it is struck by a ball at various points along the object and when the object is either free to rotate or when the handle end is held in one hand. The location of the rotation axis was also measured. The object chosen was a 72-cm-long, rectangular cross-section wood beam, representing a cross between a tennis racquet and a baseball bat. The wood beam had the same mass and length as a conventional tennis racquet, but it was stiffer at the impact point than a tennis racquet because the beam had no strings. The beam stiffness was about the same as that of a racquet frame alone. The beam was lighter than a baseball bat, but the effect of the hand was more easily measured because the beam had a long flat surface on which to impact the ball and to attach two accelerometers.

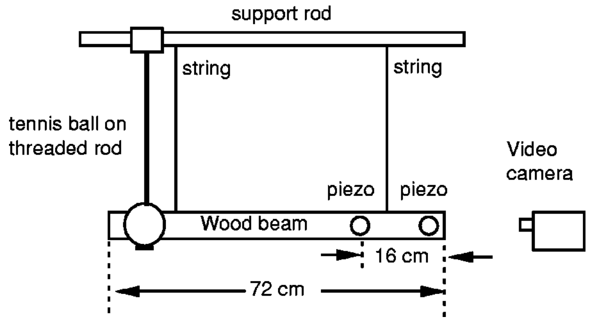

Fig. 1. The freely supported beam, with a tennis ball mounted as a pendulum to impact the beam at 1.20 m/s and two piezo disks to measure the acceleration at points 2 and 16 cm from the end of the beam.

II. EXPERIMENTAL ARRANGEMENT

The arrangement used to impact a freely supported wood beam is shown in Fig. 1. The beam was suspended in a horizontal position by two 50 cm lengths of string attached to a support rod. The beam was 72 cm long, 40 mm wide, and 19 mm thick, and its mass was 328 g. A tennis ball was suspended from the rod as a pendulum and released from a fixed distance from the beam so that it would impact the beam at the same low speed, 1.20 m/s, regardless of the impact point along the beam. Two piezo accelerometers were attached to the beam, one located 2 cm from one end and the other located 16 cm from the same end. These locations were chosen to be on opposite sides of the hand when the beam was held by hand near one end. Two accelerometers were used rather than one in order to locate the rotation axis.

The accelerometers were each constructed from a ceramic piezoelectric disk, 25 mm in diameter and 0.3 mm thick, as used in piezo buzzers. The disks were taped firmly to the beam and the output voltage signals were transmitted via fine wire leads soldered to the disks and taped to the beam to prevent any force being applied to the piezo disks by motion of the leads. A 0.047 μF capacitor was connected in parallel with each piezo and the output voltages were measured with 10 MΩ voltage probes fed to a Pico Technology ADC 212 data acquisition system.[8] The RC time constant of this arrangement was 0.5 s, so that motion of the beam could be recorded reliably[9] for times up to about 0.1 s.

The digitized piezo output voltages were integrated numerically to obtain a measurement of the velocity of the beam at the two disk locations. An absolute calibration of the sensitivity of each of the piezo accelerometers was obtained by filming the impact of the ball with a digital video camera to measure the incident and rebound speed of the impacting ball and the recoil speed of the beam at each end. The same instrumented beam was used to record impacts of a tennis ball when the beam was hand-held. The beam was gripped firmly by one hand between the two piezo disks at one end of the beam in such a way that the beam could have been used to hit a forehand tennis shot. However, the beam was held at rest in a horizontal position so that a tennis ball could be dropped onto the 40-mm-wide face from a height of 52 cm at selected positions along the beam (at an impact speed of 3.2 m/s). The arrangement is shown in Fig. 2. Each drop also was recorded on video film to measure the incident and rebound speeds of the ball so that the vertical impulse on the beam could be determined. Even though the incident speed was the same for all bounces, the impulse varied along the beam because the ball bounced to different heights along the beam. The bounce height was smallest at the free end and largest at the hand-held end.

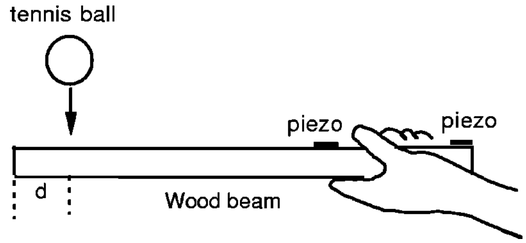

Fig. 2. Hand-held beam.

III. COP FOR A FREE BEAM

An impulsive force F applied at right angles to a free beam at a distance b from the center of mass will cause the center of mass to translate at a speed $V_{\text{c.m.}}$ given by

$$F = M_b \frac{dV_{\text{c.m.}}}{dt}, \tag{1}$$

where M_b is the mass of the beam. The force F exerts a torque about the center of mass given by

$$Fb = I_0 \frac{d\omega}{dt}, \tag{2}$$

where I_0 is the moment of inertia of the beam for rotation about the center of mass and ω is the angular velocity of the beam. Consider a point P on the beam located a distance A from the center of mass; P and the impact point are on opposite sides of the center of mass. The speed, v, of point P is given by $v = V_{\text{c.m.}} - A\omega$, so

$$\frac{dv}{dt} = \left(\frac{1}{M_b} - \frac{Ab}{I_0}\right) F. \tag{3}$$

If the beam is initially at rest, then v is given by

$$v = \left(\frac{1}{M_b} - \frac{Ab}{I_0}\right) \int F\, dt. \tag{4}$$

The axis of rotation coincides with a point where $v = 0$ and hence the corresponding COP is located at a distance b given by

$$b = \frac{I_0}{AM_b}. \tag{5}$$

For a uniform beam of length L, $I_0 = M_b L^2/12$. In this experiment the beam was uniform and of length 72 cm. An impact at one end of the beam (where $b = 36$ cm) causes the beam to rotate about an axis located 24 cm from the other end of the beam. Conversely, an impact at a point 24 cm from one end causes the beam to rotate about an axis through the other end. The two piezo disks were located 2 and 16 cm from one end of the beam. The COP for rotation about an axis through the disk at 2 cm is located 23.3 cm from the opposite end of the beam. The COP for rotation about an axis through the disk at 16 cm is located 14.4 cm from the opposite end of the beam. If v_2 is the velocity of the disk at 2 cm and v_{16} is the velocity of the disk at 16 cm, then

$$\frac{v_2}{v_{16}}=\frac{(1-0.0787b)}{(1-0.0463b)}. \tag{6}$$

This ratio is zero for an impact at the COP for the disk at 2 cm, infinite for an impact at the COP for the disk at 16 cm, and 1.0 for an impact at the center of mass.

The above results are valid for both rigid and flexible beams. Real beams are flexible and vibrate when subject to an impulsive force. Immediately after the impact, the velocity of any point on the beam has a steady or dc component given by Eq. (4), and an ac component that depends on the flexibility or stiffness of the beam. The amplitude of the ac component also depends on the impact point and the impact duration, being zero for any given vibration mode if the impact occurs at a vibration node. An impact of duration T will excite all vibration modes up to a frequency of about $f=1.5/T$ because the frequency spectrum for a half-sine impulse of duration T extends only to about $f=1.5/T$. Modes with a frequency $f>1.5/T$ are excited only weakly. In the present experiment, the only mode of significance for the impact of a tennis ball on the beam was the fundamental mode at 193 Hz. This mode has two nodes, located 15.8 cm from each end of the beam when the beam is freely supported. Large amplitude, high frequency modes at 506 and 962 Hz were observed when a golf ball was dropped onto the beam, because the impact duration (1.5 ms) was much shorter than that for the tennis ball (8 ms). However, the response of the beam and the arm to a golf ball impact was more complex, making identification of the COP more difficult. Consequently, the only results reported in this paper are those for the impact of a tennis ball.

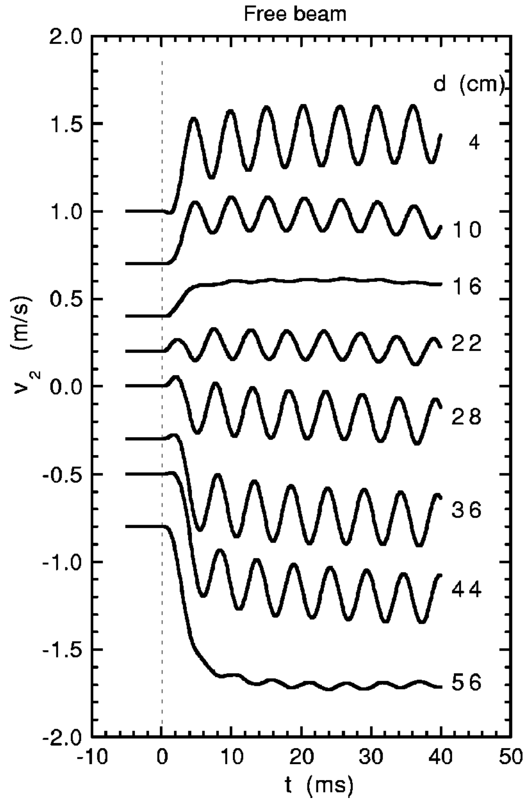

Fig. 3. Velocity of the freely supported beam 2 cm from the right end of the beam for a tennis ball incident at 1.2 m/s; d is the impact distance from the left end of the beam.

IV. EXPERIMENTAL RESULTS

Velocity waveforms for the freely supported beam at the two accelerometer locations are shown in Figs. 3 and 4. The waveforms are shown for impacts at intervals of about 6 cm along the beam, although measurements were actually taken every 2 cm. To prevent the traces from overlapping, they have been shifted vertically, which may give the impression that the velocities are much larger than the actual velocities. In fact, the velocity before impact was zero at every impact location. The time axis also was shifted slightly for every waveform so that $t=0$ corresponds to the initial displacement of the beam rather than the time at which the ball first impacts the beam. The impact generates a transverse bending wave that takes about 5 ms for a round trip up and down the beam for the fundamental vibration mode. Consequently, there is a delay of up to 2.5 ms between the initial impact and the time at which the bending wave arrives at each accelerometer location.

The ac and dc components of the beam velocity are clearly evident. The impact duration was about 8 ms, varying slightly with impact position along the beam. Impacts at the vibration nodes at $d=16$ cm and $d=56$ cm resulted in essentially zero vibration of the beam. The accelerometers were mounted at the right end of the beam, and the impact distance d is measured from the left end of the beam. The accelerometer located 16 cm from the right end was located close to a vibration node, so the vibration amplitude at this point was much smaller than that near the far right end of the beam. It can be seen from Fig. 3 that the COP with respect to an axis 2 cm from the right end of the beam is located near the $d=22$ cm impact point, as predicted. Impacts at $d<22$ cm caused the right end of the beam to deflect toward the incoming ball, while impacts at $d>22$ cm caused the right end of the beam to deflect away from the incoming ball. An impact at $d=22$ cm caused the beam to vibrate, but the dc component of the velocity waveform was essentially zero. Figure 4 shows the velocity of the beam at a point 16 cm from the right end of the beam, for the same impacts as those in Fig. 3. In this case, the COP is located near the $d=14$ cm impact point, as predicted. The velocity ratio v_2/v_{16} was found to be in excellent agreement with Eq. (6) for all impact points along the beam (see Fig. 5). The corresponding results for the hand-held beam are shown in Figs. 6 and 7. The following can be seen.

(a) Beam vibrations are strongly damped by the hand and the vibration frequency is reduced from 193 to 171 Hz.

(b) There is a vibration node 16 cm from the free end, as for the freely supported beam.

(c) There is no vibration node 16 cm from the hand-held end (at the $d=56$ cm impact point). This node point shifts to a point underneath the hand, as described previously.[4]

(d) The velocity of the handle end of the beam is reduced when it is hand held, particularly for impacts near the handle end. This effect is to be expected, given that the hand restricts free motion of the beam at the handle end. The effect is larger than the results in Figs. 3–7 might suggest, given that the ball impacted the free beam at 1.2 m/s, and it impacted the hand-held beam at 3.2 m/s.

(e) Impacts near the free end of the beam cause the hand-held end to move upward (in Fig. 6), while impacts on the beam near the hand cause the hand-held end to move downward. On a longer time scale the beam returns to its original

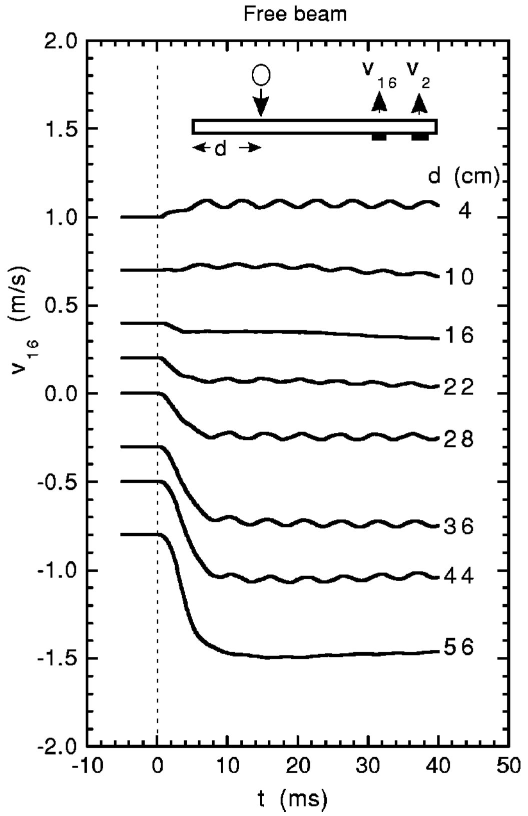

Fig. 4. The velocity of the freely supported beam 16 cm from the right end of the beam.

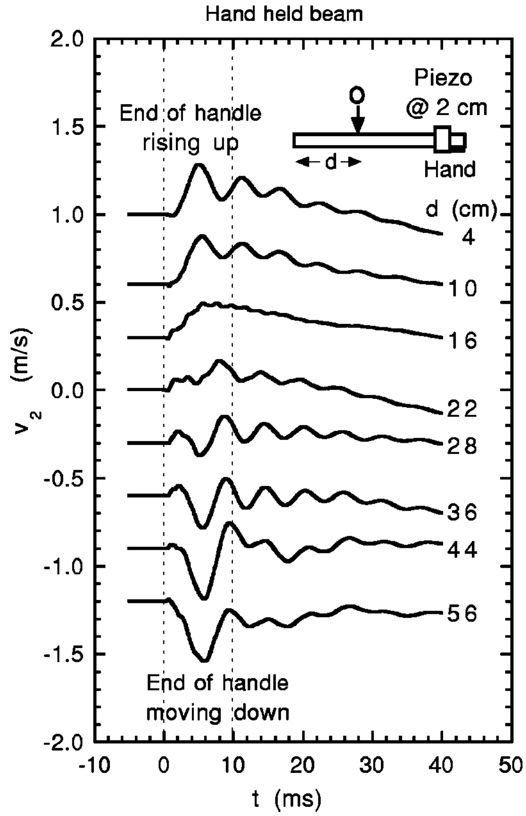

Fig. 6. Velocity, v_2, of the hand-held beam 2 cm from the right end of the beam for a tennis ball incident at 3.2 m/s.

horizontal position, the hand and the arm responding in the manner of a damped spring when a firm grip is maintained on the beam. One might expect to find an impact point near the middle of the beam where the dc velocity at the handle end of the beam remains zero during and for a short period after the impact. In fact, there is no steady or dc component because the hand exerts a time-varying reaction force on the handle in response to the torque exerted on the beam during the impact. Nevertheless, the motion of the handle at the point 2 cm from the end of the beam is significantly reduced in the impact region extending from about $d=28$ cm to about

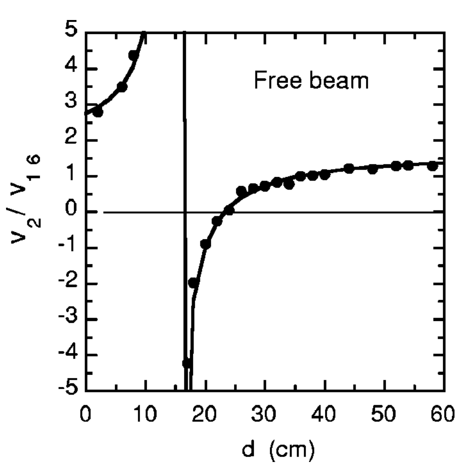

Fig. 5. Experimental values of v_2/v_{16} vs d for the freely supported beam. The smooth curve is given by Eq. (6). This ratio is 1.0 for an impact in the middle of the beam at $d=36$ cm.

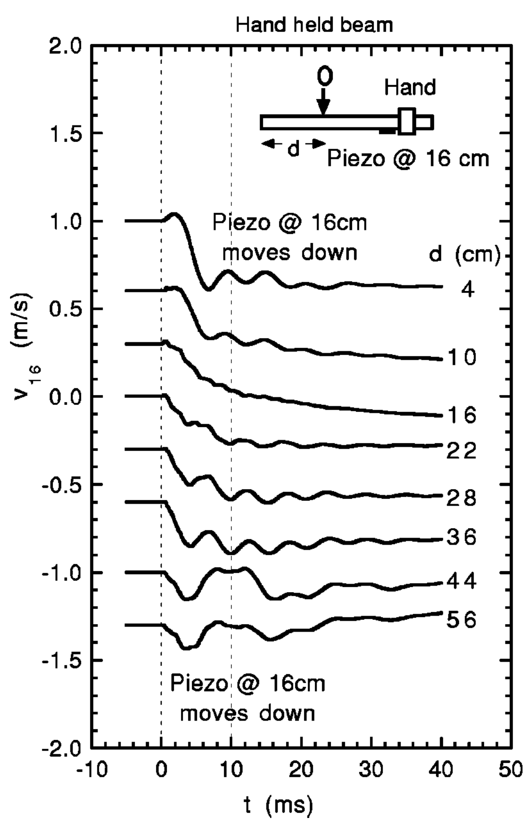

Fig. 7. Velocity, v_{16}, of the hand-held beam 16 cm from the right end of the beam.

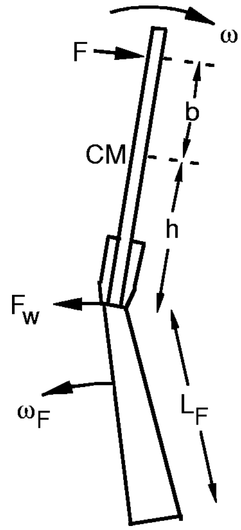

Fig. 8. Model used to determine the COP for a hand-held beam.

$d = 34$ cm. The COP in this case is not well defined, but can be described as an impact region where the velocity at the end of the handle is reduced by a factor of about 3 compared with impacts near the free end of the beam. By contrast, the COP for the free beam is well defined and is located close to the impact point at $d = 22$ cm. Part of the difficulty in identifying the hand-held COP experimentally is due to the fact that the vibration amplitude is relatively large for impacts around 30 cm from the free end of the beam. Furthermore, the vibration period is comparable to the impact duration, making it difficult to distinguish between beam vibration and beam rotation.

(f) As shown in Fig. 7, the point located 16 cm from the hand-held end of the beam deflects downward at all impact locations along the beam. For impacts with $d < 28$ cm, the far end of the handle moves upward while the point 16 cm from this end moves downward at a similar speed. Consequently, the axis of rotation is located under the hand for all such impacts. In the impact region $28 < d < 34$ cm, the rotation axis passes through the wrist.

V. EFFECT OF THE HAND AND ARM

The results obtained for the hand-held beam can be modeled by considering the situation shown in Fig. 8. The impact will be considered in a frame of reference where the beam and the forearm are both initially at rest. Prior to an impact, a bat or racquet is usually swung by applying a force to the handle, but this force is ignored during the impact because it is much smaller than the impulsive force generated during the impact. Similarly, the force exerted by and on the hand after the impact is also ignored, because this force is relatively small and acts over a longer time scale than the impact itself. To simplify the analysis further, the beam is regarded as being perfectly rigid, in which case we can ignore beam vibration.

A ball incident from the left at speed v_1 will bounce off the beam at speed v_2, exerting a normal force F on the beam. The beam is held in the hand which is assumed to be rigidly attached to the beam, both being free to rotate as a single unit about an axis through the wrist. The beam and the hand rotate at angular velocity ω, and the hand exerts a force F_w on the wrist joint, causing the forearm to rotate at angular velocity ω_F about an axis through the elbow. The wrist joint exerts an equal and opposite force F_w on the hand. We assume for simplicity that the elbow remains at rest.

The torque acting on the forearm about an axis through the elbow is given by $F_w L_F = I_F(d\omega_F/dt)$, where L_F is the length of the forearm and I_F is the moment of inertia of the forearm. If the forearm has mass M_F and a uniform mass distribution, then $I_F = M_F L_F^2/3$. If the mass distribution is not uniform, then M_F can be regarded as an equivalent mass rather than the actual mass. The wrist translates at speed $v_w = L_F \omega_F$, and hence $F_w L_F = (I_F/L_F) dv_w/dt$. We can write this relation as

$$F_w = M_e \frac{dv_w}{dt}, \tag{7}$$

where $M_e = M_F/3$ is the equivalent mass of the forearm. The forearm can therefore be regarded simply as a mass $M_F/3$ attached to the wrist which is pivoted about an axis through the wrist.

In the present case, the beam and the hand are regarded as a single rigid object of total mass $M = M_b + M_h$, where M_b is the mass of the beam and M_h is the mass of the hand. The ball impacts a distance b from the center of mass of the beam and hand, where the center of mass is a distance h from the handle end of the beam. We will assume that the handle end of the beam is adjacent to the wrist. The total force on the beam and hand is given by

$$F + F_w = M \frac{dV_{\text{c.m.}}}{dt}, \tag{8}$$

where $V_{\text{c.m.}}$ is the velocity of the center of mass. The torque about the center of mass is given by

$$Fb - F_w h = I_{\text{c.m.}} \frac{d\omega}{dt}, \tag{9}$$

where $I_{\text{c.m.}}$ is the moment of inertia of the beam and the hand about an axis through their combined center of mass. Because the wrist translates at speed $h\omega$ with respect to the center of mass, the wrist velocity is given by $v_w = h\omega - V_{\text{cm}}$, and hence

$$\frac{dv_w}{dt} = h \frac{d\omega}{dt} - \frac{dV_{\text{c.m.}}}{dt}. \tag{10}$$

If we combine Eqs. (7)–(10), we find that

$$\frac{F_w}{F} = \frac{(Mbh - I_{\text{c.m.}})}{[Mh^2 + I_{\text{c.m.}}(1 + M/M_e)]}. \tag{11}$$

The force F_w acting on the wrist and forearm is experienced by a player as an impulsive shock of the same duration as the impulsive force F and is measurably different from the force transmitted to the arm due to beam vibrations. A player may not be able to distinguish the difference between shock and vibration, although some players might notice that vibration persists for a short period after the impact is over. The force transmitted to the forearm depends on a number of parameters, including the mass of the implement and its mass distribution. An interesting question is how this force can best be minimized, but we will first determine the axis of rotation and the location of the COP.

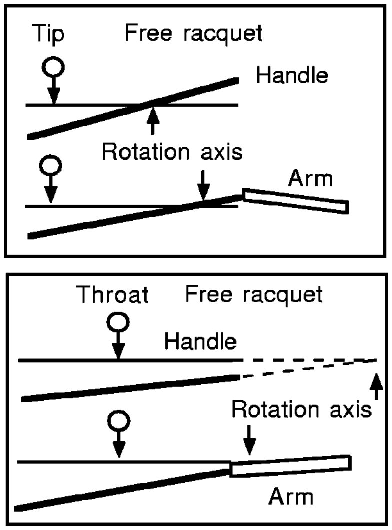

Fig. 9. Displacement of a beam or racquet and forearm.

A. Axis of rotation

The beam rotates about an axis that is located at a distance A from the center of mass of the beam and hand system. The axis itself remains at rest, and hence the center of mass rotates at speed $V_{\text{c.m.}}=A\omega$ about this axis. The location of the axis can be determined from Eqs. (8)–(11) together with the relation $dV_{\text{c.m.}}/dt=A(d\omega/dt)$. The result is that

$$A=\frac{I_{\text{c.m.}}+M_e h(h+b)}{Mb+M_e(h+b)}. \tag{12}$$

If the beam is free rather than hand-held, then $M_e=0$ and $A=I_0/(Mb)$. In this case, the rotation axis will coincide with the far end of the handle if $A=h$, and hence $b=I_0/(M_b h)$. For most modern racquets, this value of b corresponds to an impact near the center of the strings or a few centimeters closer to the throat. If the racquet is hand held, then the hand and the forearm restrict the motion of the racquet handle and the rotation axis shifts to a point close to the hand, as illustrated in Fig. 9. If the axis of rotation coincides with the end of the handle, then $A=h$, and we find from Eq. (12) that $b=I_{\text{c.m.}}/(Mh)$, where $I_{\text{c.m.}}$ and M both include the mass of the hand.

B. Center of percussion

According to Eq. (11), the force F_w at the wrist is zero when

$$b=I_{\text{c.m.}}/(Mh), \tag{13}$$

corresponding to an impact at the COP. Impacts closer to the tip of a racquet cause the forearm to jerk in the same direction as the outgoing ball, while impacts closer to the handle cause the forearm to jerk in a direction away from the outgoing ball.

The effect of the hand mass is to shift both the center of mass and the COP of the hand and racquet system closer to the handle, thereby shifting the COP away from a point near the center of the strings and into the throat area of a racquet. Consider the situation shown in Fig. 10 where the center of mass of a free racquet or a free beam is located at a distance c from the end of the handle. For the $L=0.72$ m, $M_b=0.328$ kg uniform beam used in this experiment, $c=0.36$ m, and $I_{\text{c.m.}}=I_0=M_b L^2/12=0.0142\ \text{kg m}^2$ for the beam alone. If the beam is freely supported and rotates about an axis through one end of the beam, then $b=I_0/(cM_b)=0.12$ m, which locates the COP 0.24 m from the other end of the beam (essentially as observed, except that the observation point was 2 cm from the end of the beam).

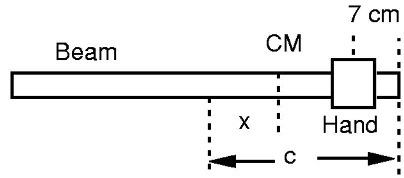

Fig. 10. Location of the center of mass for a hand-held beam.

The beam in the above experiment was held with the center of mass of the hand 0.07 m from the end of the beam. If the center of mass of the free beam is shifted a distance x when the mass of the hand is added to the beam, then $M_b x=M_h(c-x-0.07)$ so

$$x=\frac{M_h(c-0.07)}{(M_b+M_h)}. \tag{14}$$

If $M_h=0.5$ kg, then the center of mass shifts by a distance $x=0.175$ m to a point located a distance $h=0.185$ m from the end of the handle, and $I_{\text{c.m.}}$ increases to $0.0309\ \text{kg m}^2$. In this case, $b=0.20$ m from Eq. (13) and the COP is 0.33 m from the free end of the beam. The effect of the hand is therefore to shift the COP 9 cm closer to the handle. If $M_h=0.4$ kg, then $x=0.159$ m, $h=0.20$ m, $I_{\text{c.m.}}=0.0294\ \text{kg m}^2$, $b=0.20$ m, and the COP is 0.32 m from the free end of the beam. These estimates are consistent with the experimental results presented above, although the COP is not as well defined experimentally as it is from the analysis presented in this section. The difference is presumably due to the fact that hand, wrist, and forearm are subject to many different muscle forces and do not behave in as simple a manner as assumed in this section.

Equation (12) can be used to determine the location of the COP for rotation about an axis 16 cm from the handle end of the beam, where $h=0.185$ m and $A=0.025$ m when $M_h=0.5$ kg. If we assume that $M_e=0.5$ kg, we find that $b=-0.77$ m, which accounts for the fact that the beam did not rotate about this axis for any impact point along the beam when it was hand held. The location of the rotation axis as a function of the impact distance from the free end of the experimental beam is shown in Fig. 11 for a freely supported beam and for a hand-held beam with $M_h=0.5$ kg and $M_e=0.5$ kg. The rotation axis for the hand-held beam is located under the hand or near the handle end of the beam for all impact points along the beam, as was observed. Solutions for the free and the hand-held beams coincide when the impact point is 22.8 cm from the free end of the beam, in which case the rotation axis is located 3.2 cm from the handle end of the beam. The coincidence of solutions in this impact region may help to explain why the effect of the hand on the location of the COP has not previously received much attention and why previous measurements[4,6] of the COP for a hand-held implement appeared to be consistent with calculations that ignore the effect of the hand.

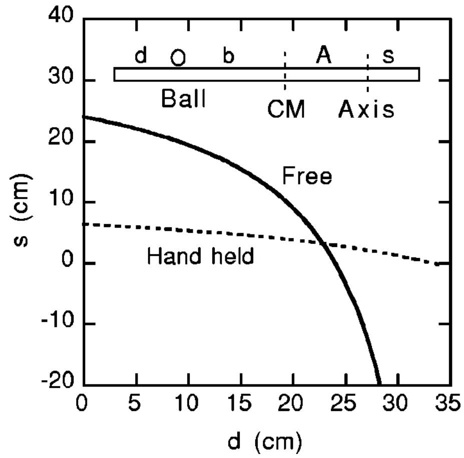

Fig. 11. s vs d where $s=h-A$ is the distance from the right end of the beam to the rotation axis and $d=L-h-b$ is the distance from the left end of the beam to the impact point.

VI. MINIMIZING THE SHOCK FORCE

Apart from hitting a ball at the COP, there are other ways of minimizing the shock force transmitted to the forearm. The simplest way, given that F_w is proportional to F, is not to hit the ball too hard. But suppose that a player hits the ball as hard as possible and misses the COP point by a big margin. In order to minimize the shock force in that case, it helps to have a light forearm and hand, a heavy implement, and an implement with a large moment of inertia about an axis through the end of the handle. The shock force would be zero on a massless hand and arm. Table I shows the peak value of F_w for a 70-cm-long tennis racquet, assuming that the ball impacts 5 cm from the tip (free end) of the racquet and the incident speed of the ball relative to the impact point on the racquet is $v_1=50$ m/s. The force, F, acting at the impact point depends on the speed, v_2, at which the ball bounces off the racquet. When integrated over the duration of the impact, $\int F\,dt=m(v_1+v_2)$, where m is the mass of the ball. The force varies with time approximately as $F=F_0\sin(\pi t/T)$, where F_0 is the magnitude of the force and T is the duration of the impact.

In a frame of reference where the racquet is initially at rest, the ball will bounce at a speed $v_2=e_Av_1$, where e_A is the apparent coefficient of restitution (ACOR)—typically about 0.4 in the middle of the strings and about 0.1 near the tip of a racquet. The coefficient of restitution (COR) is defined as the ratio of the relative speeds of two objects after and before the collision, while the ACOR is defined in terms of the speed of one object when the other is initially at rest.[10] The peak force on the strings is given from the above relations by

$$F_0=(1+e_A)\pi m v_1/(2T), \tag{15}$$

which is typically about 1000 N when $v_1=50$ m/s, greater than the weight of most players. For the results in Table I, e_A was calculated from the relation[10]

$$e_A=\frac{(eM_{\text{eff}}-m)}{(M_{\text{eff}}+m)}, \tag{16}$$

where e is the COR at the impact point and M_{eff} is the effective mass of the racquet at the impact point given by[10]

$$M_{\text{eff}}=\frac{M_b}{1+M_b b^2/I_{\text{c.m.}}}. \tag{17}$$

M_b is the actual mass of the racquet, b is the distance from the impact point to the center of mass of the racquet, and $I_{\text{c.m.}}$ is the moment of inertia of the racquet about an axis through its center of mass. The bounce speed is independent of the mass of the hand or the forearm because the ball bounces before the transverse bending wave reflected off the hand or the handle end of the beam returns to the impact point.[11,12] In Table I, e was taken as 0.6 for a flexible racquet or 0.8 for a stiff racquet, the difference arising from the fact that more energy is lost in racquet vibrations when the racquet is more flexible.[13]

Table I shows calculations for a ball of mass 57 g impacting on a light or heavy racquet ($M_b=0.25$ or 0.4 kg, respectively), each racquet being either flexible or stiff. One can also influence the impact force by changing the string tension; lower string tension acts to reduce the impact force by increasing the impact duration. Consequently, results are also given in Table I for soft strings and stiff strings, assuming that $T=6$ or 4 ms, respectively. In the present experiment T was about 8 ms because a relatively soft tennis ball was used to reduce the amplitude of the beam vibrations and because the impact duration increases at low ball speeds. An additional factor included in Table I is the weight distribution of the racquet. A racquet can be regarded as having a head section of mass M_1 and a handle section of mass M_2, each of equal length and each with a uniform mass distribution. Calculations are given for a head heavy racquet with $M_1/M_2=1.2$ and a head light racquet with $M_1/M_2=0.8$. These values were used to calculate the location of the center of mass, the impact parameter b, and the moment of inertia

Table I. Factors affecting the force on the racquet and the force on forearm. A boldface entry denotes a change from the first row. The units of mass, force, and time are kilograms, newtons, and milliseconds, respectively.

Racquet mass M_b	Head/handle mass ratio M_1/M_2	COR e	Impact duration T	Hand mass M_h	Forearm mass M_e	Impact force F_0	Force on arm F_w
0.25	0.8	0.8	4.0	0.5	0.5	1117	568
0.40	0.8	0.8	4.0	0.5	0.5	1341	524
0.25	**1.2**	0.8	4.0	0.5	0.5	1198	521
0.25	0.8	**0.6**	4.0	0.5	0.5	993	504
0.25	0.8	0.8	**6.0**	0.5	0.5	744	378
0.25	0.8	0.8	4.0	**0.4**	0.5	1117	550
0.25	0.8	0.8	4.0	0.5	**0.4**	1117	511
0.40	**1.2**	**0.6**	**6.0**	**0.4**	**0.4**	837	250

of the racquet either alone [in Eq. (17)] or with a hand attached [in Eq. (11)]. In Table I, the top row represents conditions where the force on the arm is maximized. Each subsequent row shows how the force can be reduced by varying one parameter at a time (the parameter in bold) and the bottom row shows the result of varying all parameters simultaneously to minimize the force on the arm. The largest single effect in Table I is obtained by reducing the string tension, although this result obviously depends on the magnitude of the assumed change for each parameter.

Experienced tennis coaches sometimes comment that the incidence of arm injuries increased in the 1980s when graphite racquets replaced wood racquets. They also comment that young players rarely experienced elbow problems in the wood racquet era, but this is no longer the case. Modern racquets are lighter, stiffer, and have a larger head, allowing the player to swing the racquet faster without mis-hitting the ball. Compared with tennis players of 30 years ago, today's players employ a much more aggressive type of shot, almost all use a western grip to impart topspin, and many launch themselves off the court with the ferocity of the upward swing of the racquet. The above calculations indicate that a light racquet with a stiff frame, strung at high string tension and swung at high speed will maximize the shock force transmitted to the arm. In all of these respects, modern racquets seem to be specifically designed to maximize the shock force on the arm, despite the fact that the increased stiffness of modern racquets has led to a big reduction in frame vibration.

VII. DISCUSSION

The location of the COP is usually determined without consideration of the mass of the hand, as if the hand exerted no force on the handle at all.[14] This might be a reasonable assumption if the hand and the grip covering the handle were very soft, allowing the handle to rotate freely, but on the time scale of the impact, the handle presses firmly against the hand. For example, suppose that a ball is incident at 20 m/s on a racquet moving toward the ball at 30 m/s. The ball is then incident at 50 m/s relative to the racquet. If we neglect the hand force and assume an impact near the racquet tip, then it is easy to show that the handle will rotate at or soon after the end of the impact at a speed of around 24 m/s in the racquet frame of reference. At an average speed of 12 m/s and over the 5-ms period of the impact, the handle will move a distance of 6 cm against the hand. There is insufficient cushioning in the grip and the hand to allow for free motion of the handle over a distance of more than a few millimeters. Consequently, the hand and the handle will move together during most of the impact. One might expect that the mass of the forearm should also be added to the mass of the hand in Eq. (13), but an impact at the COP results in rotation of the racquet and the hand about an axis through the wrist, without translation of the wrist or the forearm. Consequently, there is no force on the wrist for an impact at the COP, and the forearm has no effect during the impact on motion of the racquet. It is only for impacts away from the COP that the forearm has an effect on motion of the handle.

A separate issue is whether the force of the hand acting on the handle has any effect on the post-impact speed of the ball. In general, the hand force does not have any significant effect on the ball, because the ball usually rebounds before the reflected transverse wave off the hand gets back to the impact point.[10–12] The authors of Ref. 14 could have used this argument in order to ignore the hand force, rather than assuming an impact at the COP.

A few years ago, Adair and I exchanged comments in this Journal regarding the location of the sweet spot of a baseball bat.[15] Adair's view was that the sweet spot is located at the node of the fundamental mode, while my view was that the sweet spot is a small region encompassing both the node and the COP, given that the total energy transmitted to the hand involves rotation and translation of the handle as well as vibration. Furthermore, my experimental data on both a free and a hand-held bat suggested that I was correct.[6] I was therefore surprised to find that the sweet spot of the uniform beam was located at or close to the fundamental node, judging by the feel of the beam in my hand, exactly where Adair[15] would predict it to be. For a free bat, the fundamental node in the barrel and the COP are only about 1 cm apart when the axis of rotation is at the knob end of the handle. For the uniform beam in this experiment, the fundamental node and the COP are 8 cm apart when the beam is freely supported and when the rotation axis passes through the far end of the handle. When the beam was hand held, the COP shifted even further away from the fundamental node. It was therefore relatively easy to identify the fundamental vibration node, rather than the COP, as the sweet spot of a uniform beam. For a hand-held bat, the velocity of the handle, during and after an impact, is minimized for impacts close to both the fundamental node and the COP of a free bat. The proximity of the node and the free bat COP makes it difficult to determine whether the node or the COP is the more significant sweet spot or whether both spots contribute to a sweet spot zone of finite extent. In the bat experiments, the motion of the handle was difficult to interpret because the rotational and vibrational components of the handle motion appeared to be similar in magnitude and frequency. In fact, during the first 2 ms of the impact, I found that the handle rotated locally in the opposite direction to the rest of the bat. Rotation of a bat handle during and shortly after the impact period is therefore due almost entirely to excitation of the first few vibration modes. In other words, the handle rotates mainly as a result of vibrational bending of the bat rather than rigid body rotation. As a result, the impact point that minimizes the velocity of a bat handle is not the free bat COP or the hand-held bat COP, but it is at or near the vibration nodes of the first few modes.

The inference from the bat and the beam experiments is that the hand and arm are more sensitive to vibrations than to a sudden, temporary change in speed of the handle. Nevertheless, if I catch a cricket ball or a baseball in my bare hands without moving my hands backward to absorb the shock, I feel a sting in my hands that has nothing to do with vibration of the ball. The hand and the arm are sensitive to both shock and vibration but the shock force transmitted to the handle of a bat is presumably sufficiently weak that the subjective feel of an impact is determined mainly by the vibration amplitude. A tennis racquet is different because a tennis ball is much softer than a baseball, the string plane is much softer than a bat, the impact duration is much longer, and hence the shock force is much smaller and so is the vibration amplitude. For a very stiff racquet frame and with strings at low tension, it is possible to eliminate all vibrations in the frame of a racquet, even the fundamental mode.[10] The fundamental mode and all high frequency modes are suppressed if $f > 1.5/T$, a situation that can arise in practice if $f > 200$ Hz and if $T > 7.5$ ms. Under these conditions, I find that the sweet spot of a tennis racquet is located in the throat region of the racquet, judging by its feel. In fact, the whole impact region between the center of the strings and the throat feels relatively "dead" when compared with more flexible racquets, especially when compared with old wood racquets. With more flexible racquets and a stiffer ball or strings, I find from the feel of the racquet that the sweet spot is located in a small region near the middle of the string plane, coincident with the fundamental node. This is the region identified by almost all tennis players, not just the author, that feels "best." An interesting experiment described by Hatze[16] showed that tennis players tend to hit the ball in the middle of the strings near the node point rather than at the COP. Nevertheless, elite players tend to serve from a point closer to the tip of the racquet due to the added height advantage.

VIII. CONCLUSIONS

Measurements of the location of the center of percussion of a simple wood beam indicate that the COP shifts by a large distance toward the hand when the beam is hand held rather than freely supported. Consequently, the location of the COP of a free beam is of little relevance to the impulsive force that is transmitted to the hand. The location of the fundamental vibration node at the free end of the beam is unaffected, although the vibration node at the handle end of the beam shifts to a point under the hand. The subjective feel of the beam indicates that the sweet spot coincides with the fundamental vibration node. Impacts at this or any other spot causes the beam to rotate about an axis through the hand or the wrist. The resulting impulsive force on the hand and arm varies with impact position along the beam, but it appears that this force is not as significant as that due to vibration of the beam in terms of the perceived "feel." The sweet spot of a baseball bat is located at or near the nodes of the first few vibration modes, about 15 cm from the barrel end of the bat. This position happens to be close to the COP for a freely supported bat, but the COP for a hand-held bat is shifted closer to the hands, away from the sweet spot region. Similarly, the sweet spot of a tennis racquet is usually near the middle of the strings, at or close to the fundamental vibration node of the racquet frame. However, some modern racquets are so stiff and light that all vibration modes are suppressed, even the fundamental mode. In that case, the sweet spot can be subjectively identified as a region near the throat of a racquet, close to the COP for rotation about an axis near the wrist.

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[8]⟨www.picotech.com⟩ has details of the unit used and the accompanying software that allows one to convert a PC into an excellent digital storage oscilloscope or spectrum analyzer.

[9]Piezo disks generate an output voltage directly proportional to the force on the disk. If the disk is stationary, an equal and opposite force on each side of the disk generates the voltage. If the disk is attached on one side to a moving object, then the force acting on the side of the disk attached to the object is given by the mass of the disk times its acceleration. In the latter case, the output voltage is proportional to the acceleration of the object to which it is attached. Commercial accelerometers often have an additional small mass attached to the piezo element to increase its sensitivity, but this was not needed in the present experiment. A piezo element behaves electrically as a capacitor with an induced charge proportional to the force on the piezo. If the force remains constant, then the charge remains constant unless one attempts to measure it. The charge decays with a time constant RC, where R is the resistance of the measuring device and C is the capacitance of the piezo plus any capacitor connected in parallel with the disk.

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