A double pendulum model of tennis strokes

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The physics of swinging a tennis racquet is examined by modeling the forearm and the racquet as a double pendulum. We consider differences between a forehand and a serve, and show how they differ from the swing of a bat and a golf club. It is also shown that the swing speed of a racquet, like that of a bat or a club, depends primarily on its moment of inertia rather than on its mass.

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I. INTRODUCTION

It is well known that the swing of a golf club can be accurately modeled as a double pendulum,1,2 Animations illustrating the basic physics are available.3 The swing of a baseball bat can also be modeled as a double pendulum.4,5 In this paper, the double pendulum model is used to describe the swing of a tennis racquet. We consider only the forehand and serve, and show how they differ from each other and from the swing of a club and a bat.

A conventional double pendulum consists of two segments joined end-to-end at a pivot point, the upper segment also being pivoted about a fixed axis near its upper end. The two segments normally swing in a vertical plane, driven by gravity alone. A tennis stroke can be described in a simplified manner by treating the forearm as one segment of a double pendulum and the racquet as the other segment, pivoted at the wrist. In a tennis stroke, gravity plays only a minor role; the two segments are driven primarily by muscles in the arm and the hand, and can swing in a horizontal or a vertical plane or in any other plane that the player chooses. The forearm does not act as a fixed pivot point because the upper arm rotates about an axis in the shoulder and because the upper arm translates as a result of forward motion of the player during the stroke. The conventional model of a double pendulum is modified in this paper to take these added complications into account in a manner similar to that described previously in relation to the swing of golf clubs1,2 and baseball bats.4,5

Some features of real strokes are not captured by the double pendulum model, such as rotation of the hand and the forearm or any other body segment about its long axis, nor is the three-dimensional nature of many tennis strokes fully described. Nevertheless, the double pendulum model provides useful insights into the forces and torques required to swing a racquet, the manner in which the forearm and the racquet are mutually coupled, and the dependence of swing speed on the physical properties of a racquet.

It is shown that the swing speed of a racquet, like that of a club or a bat, depends on its moment of inertia rather than on its mass. It is not obvious that this dependence should be the case, given the complex and nonlinear interaction between the arm and the implement in the hand. A common assumption is that light clubs, bats, and racquets can be swung faster from the swing of a club and a bat, depends primarily on its moment of inertia rather than on its mass.

II. EQUATIONS DESCRIBING A DOUBLE PENDULUM

The geometry of the double pendulum model, as it applies to the swing of a racquet, is shown in Fig. 1. The elbow translates in the x direction at velocity $v_{E,x}=dx_{E}/dt$ and in the y direction at velocity $v_{E,y}=dy_{E}/dt$. The forearm and racquet rotate in a clockwise direction in the x-y plane at angular velocities $\omega_{1}=-d\theta/dt$ and $\omega_{2}=-d\phi/dt$, respectively; the angles $\theta$ and $\phi$ are defined in Fig. 1.

Normally, the elbow moves down and to the left in Fig. 1 if the ball approaches from the left, in which case, both $v_{E,x}$ and $v_{E,y}$ are negative. If the elbow is moving vertically down in Fig. 1(a) and is decelerating, then the effect is equivalent to a vertical force acting upward at the elbow, with the result that a clockwise torque is exerted on the forearm. An additional clockwise torque arises if the elbow is moving to the left and is decelerating. Both effects are significant when swinging a racquet or any other implement. An identical effect occurs at the wrist. If the angular velocity of the forearm decreases, the resulting force on the racquet at the wrist joint acts to increase the angular velocity of the racquet.

The forearm has mass $M_{1}$, length $L_{1}$, and its center of mass (point $G_{1}$) is a distance $h_{1}$ from the elbow. The hand is rigidly attached to the racquet handle when swinging a racquet, and both rotate at the same angular velocity. The racquet-hand system has mass $M_{2}$, length $L_{2}$, and the center of mass (point $G_{2}$) is at a distance $h_{2}$ from the wrist. The coordinates of $G_{1}$ and $G_{2}$ are $x_{1}=x_{E}+h_{1} \sin \theta$, $y_{1}=y_{E}-h_{1} \cos \theta$, and $x_{2}=x_{E}+L_{1} \sin \theta+h_{2} \sin \phi$, $y_{2}=y_{E}-L_{1} \cos \theta-h_{2} \cos \phi$. Differentiation with respect to time yields the ve-
The center of mass of the forearm and racquet, respectively. The quantity $G$ is no net force exerted by a couple. If $\omega_1$ rotates at angular velocity $\omega_1$, the racquet and the hand exert a force on the forearm at the wrist joint, as indicated in Fig. 1. The force components $F_{x,1}$ and $F_{y,1}$ on the forearm at the elbow joint, and the forearm exerts a force with components $F_{x,2}$ and $F_{y,2}$ on the racquet at the wrist joint. The racquet and the hand exert a force on the forearm at the wrist joint, as indicated in Fig. 1(d), which is equal and opposite to the force of the forearm on the racquet and hand. In addition to the forces at each joint, muscles in the upper arm exert a couple $C_1$ on the forearm and muscles in the forearm exert a couple $C_2$ on the hand and racquet. A couple consists of equal and opposite forces, and thus there is no net force exerted by a couple. If $C_2=0$, the wrist acts as a free hinge and is described as “relaxed,” although the hand must still maintain a firm grip on the handle. The hand and racquet exert an equal and opposite couple $C_2$ on the forearm. The equations of motion for the double pendulum are derived in the Appendix in terms of the torques acting about $G_1$ and $G_2$ and are given by Eq. (A10). The effect of gravity can be ignored if the racquet and forearm swing in a horizontal plane. The effect of gravity was included in serve calculations, including those shown in Fig. 2.

### III. Qualitative Features of a Double Pendulum

The behavior of a double pendulum is nonlinear and not intuitively obvious because the motion of one segment normally has a strong effect on the motion of the other segment. It is useful to summarize briefly the essential features as they pertain to the swing of a racquet. The strength of the coupling between the two segments of a double pendulum depends on their relative mass. The term “coupling” here refers to the influence of one pendulum segment on the other segment. When swinging a light tennis racquet (or a squash or badminton racquet) of mass less than about 100 g, the coupling between a 1.5 kg forearm and the racquet is weak. A constant couple applied to each segment results in a monotonic increase in the angular velocity of each segment, at least for the duration of a typical stroke. A different result is obtained when the racquet mass is about 250 g or more because the coupling between the forearm and the racquet is then stronger. In that case, a constant couple applied to each segment results in an initial monotonic increase in the angular velocity of the forearm, followed by a monotonic decrease. As the angular velocity of the forearm decreases to a minimum, the angular velocity of the racquet increases to a maximum. The latter process can be described as a transfer of rotational energy or angular momentum from the arm to the racquet. The process can be enhanced, and the racquet speed increased if the couple applied to the forearm is decreased shortly before the angular velocity of the racquet reaches its maximum value. In that case, the angular velocity of the forearm decreases more rapidly and the forearm transfers a greater fraction of its rotational energy to the racquet.

Some of these features are illustrated in Fig. 2(a) for a 1.5 kg forearm with length of 0.3 m swinging either a 100 or a 500 g racquet. Both racquets were assumed to have a uniform mass distribution, were swung with a 500 g hand, with constant couples $C_1=60$ N m and $C_2=12$ N m applied throughout the swing, and with $\beta=\delta-\theta=90^\circ$ at $t=0$. Elbow motion was ignored for the calculations shown in Fig. 2. The couple applied to each racquet was not chosen to ensure that the racquet was correctly aligned on impact with the ball. Rather, the same couple $C_2$ was used for both racquets to illustrate the behavior of a double pendulum when the only parameter that is altered is the racquet mass. Figure 2(b) illustrates the effect of decreasing $C_2$ to zero, with the surprising result that the angular velocity of each racquet increases as $C_2$ decreases. The same effect has been observed in experiments with a mechanical double pendulum.

The parameters chosen in Fig. 2 and in later calculations are typical of those that apply to tennis. Forearm and hand mass data are given in most biomechanics textbooks as a percentage of total body mass and is typically 1.8% and 0.6%, respectively, for adult males. The couple applied to a racquet or the forearm has not been measured as far as we know, although the net torque (which includes the couple) is commonly measured when studying the biomechanics of ten-
nise strokes. The peak elbow torque generated in a serve was found to be 78.3 N m averaged over eight male players competing at the 2000 Olympic Games, and the average elbow torque during the serve was 67.6 N m for the same players.12 In a similar study by Bahamonde and Knudson,13 the peak elbow torque in a forehand stroke was found to vary from 27 to 62 N m for 15 different players.

The elbow couple was taken as 60 N m in Fig. 2 for the purpose of illustrating the qualitative features of the double pendulum motion. The resulting angular and linear velocities of the forearm and racquet are typical of those measured in a fast serve by male tennis players.12,14 Most tennis racquets weigh between 250 and 350 g. The racquet mass assumed for the results of the calculations shown in Fig. 2 was extended well beyond the normal range to exaggerate differences between light and heavy racquets and to highlight that the swing speed of a racquet depends only weakly on its mass or its moment of inertia.

For the 100 g racquet, the angular velocity of the forearm is still increasing when the angular velocity of the racquet is a maximum near $t=0.09$ s [see Fig. 2(a)]. The behavior of the 100 g racquet after 0.1 s is not of practical significance because a player would reverse the direction of $C_1$ after impacting a ball. The 100 g racquet does not swing five times faster than the 500 g racquet, as might be expected. It swings five times faster at the start of the swing, but the maximum swing speed is only 22% greater because the transfer of energy from the arm to the racquet is less efficient with a light racquet and the 500 g racquet is accelerated over a longer period of time. For the 500 g racquet, the forearm is almost stationary when the angular velocity of the racquet is a maximum. An increase in the coupling between the forearm and the racquet can be achieved if the couple applied to the racquet is reduced, as indicated in Fig. 2(b). As a result, a greater fraction of the energy in the arm is transferred to the racquet and the maximum angular velocity of the racquet is increased.

The results in Fig. 2(b) highlight the point made by Jorgensen1 that golfers can expect lower swing speeds if the wrists are used deliberately in an attempt to swing the club faster. However, the result depends sensitively on the timing of the wrist couple. If a wrist couple is applied late in the swing, then the swing speed is increased.2,15

When using a heavy racquet or a baseball bat, the coupling between the forearm and the racquet or the bat is so strong that the racquet or the bat tends to rotate too fast to line up correctly at the intended impact point. It may then be necessary to reduce the angular velocity of the racquet so that it arrives in the correct orientation to meet the ball. The latter result can be achieved by increasing the couple applied to the forearm just before impact or by decreasing the couple applied to the racquet just prior to impact with the ball. When swinging a baseball or a softball bat, it is necessary to apply a negative couple to the bat, using both hands, prior to impacting the ball.2,15

IV. A TYPICAL FOREHAND

A forehand in tennis is often struck with topspin by swinging the racquet upward as well as forward. We will ignore vertical motion of the racquet and assume that both the forearm and the racquet swing in a horizontal plane. Forward motion of the racquet usually commences when the forearm is pointing in a direction approximately toward the back fence ($\theta=90^\circ$ in Fig. 1) and the racquet is about $50^\circ$ further around, with $\phi=140^\circ$. Players do not always begin forward motion of the racquet from that position, but the values given here are common and will be used in the following. It is known that a double pendulum is chaotic and sensitive to the initial conditions, but good players do not swing a racquet or a golf club in a chaotic manner. The reason is not that all players start their swing from exactly the same position, but that chaos becomes important only after a few complete cycles of oscillation of the pendulum. The first half cycle of oscillation of a double pendulum is reproducible.4

In a tennis forehand, the ball is normally struck in front of the body with the forearm extended forward while the racquet is aligned nearly parallel to the net at the point of impact. We will assume that the ball is struck when $\theta=-40^\circ$ and $\phi=0$ so that the outgoing ball heads toward the net. This type of swing differs from that in golf or baseball, where $\beta=\phi-\theta$ is approximately $90^\circ$ at the start of the swing and remains close to $90^\circ$ while the wrists remain locked. Soon after the start of a golf or baseball swing, the wrists relax and the striking implement then rotates rapidly to align with the arms at the point of impact so that $\beta=0$.

A realistic forehand can be modeled by assuming that the couples $C_1$ and $C_2$ remain constant in time. A realistic golf swing can also be modeled when the couple $C_1$ remains constant.1 If the wrist remains locked, then the forearm and the racquet swing at the same rotational speed at the beginning of the swing, and $C_2$ decreases with time. If $C_2$ remains constant in time, then the forearm swings faster than the racquet at the beginning of the stroke, while the racquet swings faster than the forearm toward the end of the stroke. Both types of forehand are common and can be viewed in slow motion.13 The effects of allowing $C_2$ to vary with time are considered in Sec. V.

Calculations for a medium pace forehand are shown in Figs. 3 and 4 for a racquet with length of 70 cm, mass of 300 g, swing weight of 310 kg cm², and with center of mass 35 cm from the butt end of the handle. The mass of the forearm was taken as 1.5 kg, and the mass of the hand was 0.5 kg. The swing weight of a racquet is an industry term referring to its moment of inertia about an axis in the handle located...
10 cm from the butt end. The racquet was swung with $C_1 = 25 \text{ N m}$ and $C_2 = 2.5 \text{ N m}$, resulting in good alignment of the racquet at the nominal impact point.

For the calculations in Figs. 3 and 4, the elbow was assumed to translate in the $x$ direction with a velocity $v_{E,x} = -33t + 69t^2$ and in the $y$ direction with a velocity $v_{E,y} = -42t + 174t^2$. $v_{E,x}$ and $v_{E,y}$ are zero at the start of the swing, $v_{E,x}$ has a maximum value of $-4 \text{ m/s}$ at $t = 0.24 \text{ s}$ (the impact time), and $v_{E,y}$ has a maximum value of $-2.5 \text{ m/s}$ at $t = 0.12 \text{ s}$ and is zero at $t = 0.24 \text{ s}$. These values are typical of those measured by the author for a medium pace forehand but are not critical to the outcome of the swing. Small changes in $v_{E,x}$ or $v_{E,y}$ or the starting position of the racquet require small variations in $C_1$ or $C_2$ to align the racquet, but do not result in large changes in the rotational speed of the racquet.

The angular velocities of the forearm and the racquet are shown in Fig. 4. Despite the fact that $C_2$ was held constant, rotational energy transferred from the arm to the racquet during the swing, with the result that $\omega_1$ decreased to a minimum and $\omega_2$ increased to a maximum near the end of the swing. The latter effect is well documented in the biomechanics literature but rarely explained in terms of double pendulum mechanics. In an efficient tennis (or golf or baseball) swing, rotational energy is first transferred from the upper arm to the forearm and is subsequently transferred from the forearm to the racquet after a short time delay.$^{1,2,4-6,12-14}$

Calculations similar to those in Figs. 3 and 4 were repeated for a range of different racquets with different values of mass, swing weight, and center of mass location, all swung with the same forearm couple $C_1$, starting position, and elbow speed as in Figs. 3 and 4. The magnitude of the couple $C_2$ applied to each racquet was altered so that each racquet swung into correct alignment with the incoming ball, with $\phi = 0$ when $\theta = -40^\circ$. Racquets with a larger moment of inertia require a larger couple for alignment. For example, the 100 g racquet with its center of mass at 35 cm was swung with $C_2 = 1.38 \text{ N m}$, and the 400 g racquet with its center of mass at 35 cm was swung with $C_2 = 2.54 \text{ N m}$.

Racquets with a 35 cm center of mass were assumed to have a uniform mass distribution with a moment of inertia $ML^2/12$ about the center of mass. The calculations were repeated for racquets having a center of mass at 37 cm by increasing the swing weight of each racquet by 20% (compared with the uniform mass distribution value) and also for a group of racquets having a center of mass at 32 cm by decreasing the swing weight of each racquet by 20%. The results of the calculations are shown in Fig. 5, giving the linear velocity of the tip of the racquet at impact versus the mass of the racquet [Fig. 5(a)] and the swing weight of the racquet [Fig. 5(b)]. In terms of the outgoing ball speed, the parameter of greatest interest is the linear velocity of the racquet at the impact point on the strings. The velocity is a maximum at the tip of the racquet.

The results in Fig. 5 show that the swing speed of a racquet depends primarily on the swing weight of the racquet rather than on its mass. Figure 5(a) shows a strong correlation between swing speed and racquet mass if the center of mass remains constant and also shows that the center of mass can vary over a relatively wide range in practice. For an arbitrary selection of racquets having different centers of mass, there is only a weak correlation between swing speed and racquet mass, especially when the mass is restricted to commercially available racquets [indicated by the square box in Fig. 5(a)]. Figure 5(b) shows a much stronger correlation with swing weight, given that Fig. 5(b) includes the same 15 racquets as those in Fig. 5(a) and therefore includes racquets with a wide range of centers of mass. The results in Fig. 5(b) indicate that the tip velocity is proportional to $1/(\text{swing weight})^n$, where $n = 0.17 \pm 0.01$, indicating that the speed at which a racquet can be swung depends only weakly on its swing weight.

The latter result was found to be insensitive to the choice of initial parameters. If the initial racquet position is changed from $\phi = 140^\circ$ to $\phi = 130^\circ$, keeping the forearm couple fixed at 25 N m, then a slightly larger racquet couple is required to swing the racquet into alignment, resulting in a slight decrease in the velocity of the racquet tip. Results with $\phi = 130^\circ$ for the same racquet selection, as indicated in Fig. 5, gave the same $n = 0.17$ power law dependence of tip speed on swing weight as that shown in Fig. 5(b).

The results in Fig. 5 were obtained by keeping $C_1$ fixed because it is easier for a player to vary $C_2$ to align each racquet when swinging different racquets. A player could also vary $C_1$, while keeping $C_2$ fixed, to achieve the same result. However, major changes in $C_1$ are then required to align both the arm and each racquet due to the high angular velocity acquired by low mass racquets. The results shown in
Fig. 6. Calculated positions of the forearm and racquet at 0.02 s intervals for a 300 g racquet swung in a vertical plane to impact a ball when the racquet is aligned with $\phi=180^\circ$ and $\theta=195^\circ$. The dashed line at the right shows the motion of the elbow.

Fig. 5 better represent the variation of swing speed under conditions where the effort exerted by the player remains approximately constant for each racquet. The results for a near maximum effort swing are described in the following.

V. A TYPICAL SERVE

A serve in tennis is similar to a golf swing and the swing of a baseball bat in that the server usually starts the swing with the racquet aligned at about 90° to the forearm. Impact with the ball occurs when the racquet is approximately in line with the forearm, as it is in golf and baseball. There is also a preliminary stage prior to the commencement of the forward swing, where the server positions the racquet behind his or her back so that the racquet points vertically down to the court. We will assume that the swing starts with $\phi=0$ (racquet pointing to the court) and the forearm starts in a horizontal position, with $\theta=90^\circ$. Impact with the ball is assumed to occur when the racquet swings through 180° to $\phi=180^\circ$ and when the forearm swings through an angle of 105° to $\theta=195^\circ$ so that the forearm is slightly in front of the server on impact (see in Fig. 6).

The preliminary stage of the swing in a tennis service includes rapid rotation of the upper arm, with the result that the elbow is rising rapidly and moving forward at the instant the racquet is pointing down to the court. The effect was measured by filming an elite player at 300 frames/s when serving a ball at 45 m/s (100 mph). The results of the recorded motion of the elbow are shown in Fig. 6. The vertical speed of the elbow decreased from 9 m/s at the nominal start of the forward swing (when $\phi=0$) to zero at the point of impact with the ball. The horizontal speed of the elbow increased from 5 m/s at the start of the swing to 8 m/s and then decreased to 2.5 m/s on impact with the ball.

Good fits to the elbow data were obtained with $a_x=\frac{dv_E}{dt}=-125+2600t$ and $a_y=\frac{dv_E}{dt}=-150+1250t$. These fits were used in Eqs. (A6) and (A9) for the following calculations, together with the initial conditions $\omega_1=\omega_2=0$. There was negligible rotation of the forearm or the racquet in the $x$-$y$ plane at the start of the swing because the forearm remained locked at right angles to the upper arm and the racquet remained locked at right angles to the forearm. Rotation of the upper arm caused the forearm and racquet to translate as a rigid body at the same speed as the elbow, but the forearm remained horizontal just prior to the start of the swing. Subsequent deceleration of the elbow in both the $x$ and the $y$ directions resulted in a much smaller couple being needed to rotate the forearm than that shown in Fig. 2. Nevertheless, the torque acting on the racquet is about the same in Figs. 2(a) and 7 because the angular acceleration of the racquet is about the same.

If the wrist remains locked at the beginning of a serve, the couple $C_2$ applied to the racquet is determined by the condition that $\omega_1=\omega_2$. As the racquet accelerates, $C_2$ decreases and the wrist can relax completely when $C_2=0$, at least for a short time interval. In a golf swing, $C_2$ can remain zero until impact occurs with the ball, but in a tennis serve, the centrifugal force acting along the forearm is usually too small to swing the racquet into line. Consequently, the server needs to apply a positive couple to the racquet late in the swing to ensure that the racquet impacts the ball in the correct orientation.

Calculations for a typical serve are shown in Figs. 6 and 7 for a 70 cm long, 300 g racquet with a 35 cm center of mass and a swing weight of 310 kg cm². The racquet was swung by applying a constant arm couple $C_1=25$ N m throughout the swing. In Fig. 6, the wrist remained locked until $C_2$ decreased to zero. Then, a constant arm couple $C_2=11.7$ N m was applied at $t=0.05$ s to bring the racquet into alignment. The angular velocities of the forearm and the racquet, as well as $C_1$ and $C_2$, are shown in Fig. 7(a).

Figure 7(b) shows a similar serve for the same racquet parameters, the same elbow velocity and the same forearm couple, but a constant racquet couple $C_2=9.29$ N m was applied throughout the swing to align the racquet. The second serve resulted in a slightly lower swing speed (67.5 rad/s rather than 69.6 rad/s), but the result indicates that it is not necessary to lock the wrist at the beginning of the serve, nor is it necessary to relax the wrist in the middle of the serve. For both serves represented in Fig. 7, the couple $C_2$ provided approximately half of the 20 N m torque required to swing the racquet, as described by Eq. (A4).

The effect of serving with lighter or heavier racquets was examined, as for forehand swings, by swinging each racquet using the same arm couple as in Figs. 6 and 7, and the same elbow velocity. A constant couple was applied to each racquet throughout the swing, heavier racquets requiring a larger value of $C_2$ to align the racquet than lighter racquets. However, 500 g racquets required a lower value of $C_2$ than 400 g racquets due to the increased coupling between the forearm and the racquet. The results are shown in Fig. 8.
again indicating that the swing speed of a racquet depends primarily on the moment of inertia of the racquet rather than on its mass. The speed of the tip of the racquet is plotted in Fig. 8 and is a combination of angular speed and translational speed of the racquet. Two best fit lines are shown in Fig. 8, indicating that the slope varies with swing weight. The racquet tip speed is proportional to \(1/(\text{swing weight})\), where \(n=0.22\pm0.01\) for low swing weight racquets and \(n=0.27\pm0.01\) for high swing weight racquets. The exponent \(n\) is given by the slope of the best fit lines in Fig. 8 because the results are plotted on log-log scales. The exponent \(n\) is larger in Fig. 8 than in Fig. 5(b) even though the same set of racquets was compared and the same couple was used to swing the forearm. The difference is presumably due to the larger couple applied to each racquet and the larger elbow speed, both of which acted to increase the torque acting on each racquet.

The general conclusion that swing speed depends primarily on swing weight is one that is specific to the task. A different result is obtained if we assume that the couple applied to each racquet has the same value for all racquets. In that case, the maximum swing speed does not occur at the same arm position for each racquet nor is each racquet correctly aligned on impact with the ball. In that hypothetical situation, the maximum swing speed decreases as the racquet mass increases or as the swing weight increases, at least for racquets having the same center of mass. When comparing racquets with a range of different positions of the center of mass, the correlation between swing speed and swing weight is weak, as is the correlation between swing speed and racquet mass.

VI. DISCUSSION

Clubs and bats are usually swung in a single plane, while racquet motion is often more complex and can involve rotation of many body segments about all three axes. Nevertheless, the essence of a tennis stroke is captured in the double pendulum model, providing useful insights into the basic mechanics. A racquet is light enough to be swung with a single arm, whereas clubs and bats are normally swung using two arms. The forces and torques used to swing a club or a bat are correspondingly larger, with the result that the wrist plays a small role late in the swing and can relax completely when swinging a bat or a club. Because racquets are generally lighter and have a lower moment of inertia, the wrist plays a more significant role in controlling the swing of a racquet, not only at the beginning of the swing but throughout the whole swing. For a fast serve, the wrist actively rotates the racquet by providing a couple amounting to about half of the total torque required to swing the racquet, the other half arising from the force applied at the wrist joint by the forearm.

Golfers and batters in baseball and softball usually lock their wrists at the beginning of the swing to ensure that the club or bat does not get left behind when they swing their arms forward. The same technique can be used when swinging a racquet, but racquets can also be swung effectively without locking the wrist. A more aggressive approach can be used in a tennis forehand, where the arm swings ahead of the racquet at the beginning of the swing and then slows rapidly just before impact with the ball. Likewise when serving with a racquet, it is not necessary to lock the wrist at the beginning of the swing, although players normally apply a couple to the racquet at the beginning of the serve so that the racquet swings forward. It is possible to swing a racquet forward on the end of a rope without applying a couple to the racquet, but that technique is of academic interest only.

In recent years, it has been shown experimentally that the swing speed of a sporting implement depends primarily on its moment of inertia \(I\) rather than on its mass and that the swing speed for a maximum effort swing is proportional to \(1/I\), where \(n\) is typically about 0.26 for baseball and softball bats and other implements with a swing weight greater than about 300 kg cm\(^2\). It has also been shown experimentally that the exponent \(n\) decreases when swinging implements with a swing weight less than 300 kg cm\(^2\). The experimental data on swing speed are consistent with the calculations presented in this paper, at least with those for a near maximum effort serve. A simple explanation of the low value of \(n\) is that most of the effort of a batter or a tennis player is expended in swinging the arms and the addition of a light implement in the hand does not alter the swing speed in proportion to the weight or swing speed of the implement.

APPENDIX: EQUATIONS OF MOTION FOR A DOUBLE PENDULUM

The velocity components of the center of mass points \(G_1\) and \(G_2\) in Fig. 1 are, respectively,

\[
\begin{align*}
\dot{v}_{x1} &= x1/\dot{t} = v_{E,x} - h_1 \omega_1 \cos \theta, \\
\dot{v}_{y1} &= v_{E,y} - h_1 \omega_1 \sin \theta, \\
\dot{v}_{x2} &= x2/\dot{t} = v_{E,x} - L_1 \omega_1 \cos \theta - h_2 \omega_2 \cos \phi, \\
\dot{v}_{y2} &= v_{E,y} - L_1 \omega_1 \sin \theta - h_2 \omega_2 \sin \phi.
\end{align*}
\]

The net force on the forearm can be calculated in terms of the components \(F_{x1} = F_{x2} = M_1 dv_{x1}/\dot{t}\) and \(F_{y1} = F_{y2} = Mg = M_1 dv_{y1}/\dot{t}\), giving

\[
F_{x1} - F_{x2} = M_1 \left( a_x - h_1 \cos \theta \frac{d\omega_1}{dt} - h_1 \omega_1^2 \sin \theta \right). \quad \text{A2a}
\]
\( F_{y1} - F_{y2} = M_1 \left( g + a_v - h_1 \sin \theta \frac{d\omega_1}{dt} + h_1 \omega_1^2 \cos \theta \right), \)  
\[ (A2b) \]

where \( g \) is the acceleration due to gravity [Fig. 1(a) is upside-down for a serve]. Similarly, the net force on the racquet-hand system has components

\[
F_{x2} = M_2 \left( a_x - L_1 \cos \theta \frac{d\omega_1}{dt} - L_1 \omega_1^2 \sin \theta - h_2 \cos \phi \frac{d\omega_2}{dt} \right.
\]
\[ - h_2 \omega_2^2 \sin \phi \right), \]
\[ (A3a) \]

\[
F_{y2} = M_2 \left( g + a_y - L_1 \sin \theta \frac{d\omega_1}{dt} + L_1 \omega_1^2 \cos \theta \right.
\]
\[ - h_2 \sin \phi \frac{d\omega_2}{dt} + h_2 \omega_2^2 \cos \phi \right). \]
\[ (A3b) \]

The torque about point \( G_2 \) can be calculated from the geometry in Fig. 1(c) and is given by

\[ C_2 + F_{x2} h_2 \sin \phi + F_{y2} h_2 \sin \phi = I_{cm2} \frac{d\omega_2}{dt}, \]
\[ (A4) \]

where \( I_{cm2} \) is the moment of inertia of the racquet-hand system about an axis through \( G_2 \). The substitution of Eq. (A3) into Eq. (A4) gives

\[ I_2 \frac{d\omega_2}{dt} = Q - B \cos \beta \frac{d\omega_1}{dt}, \]
\[ (A5) \]

where

\[ Q = C_2 + B \omega_1^2 \sin \beta + R \left[ a_x \cos \phi + (g + a_y) \sin \phi \right], \]
\[ (A6) \]

and \( I_2 = I_{cm2} + M_2 h_2^2 \) is the moment of inertia of the racquet-hand system about an axis through the wrist, \( B = M_2 h_2 L_1 \), \( R = M_2 h_2 \), and \( \beta = \phi - \theta \) is the angle between the racquet and a straight line extension of the forearm.

From the geometry of Figs. 1(a) and 1(d), the torque about point \( G_1 \) in the forearm is given by

\[ C_1 - C_2 + F_{x1} h_1 \cos \theta + F_{y1} h_1 \sin \theta + F_{x2} (L_1 - h_1) \cos \theta \]
\[ + F_{y2} (L_1 - h_1) \sin \theta = I_{cm1} \frac{d\omega_1}{dt}, \]
\[ (A7) \]

where \( I_{cm1} \) is the moment of inertia of the forearm about an axis through \( G_1 \). Substitution of Eqs. (A2) and (A3) into Eq. (A7) gives

\[ A \frac{d\omega_1}{dt} = P - B \cos \beta \frac{d\omega_2}{dt}, \]
\[ (A8) \]

where

\[ P = C_1 - C_2 - B \omega_1^2 \sin \beta + S \left[ a_x \cos \theta + (g + a_y) \sin \theta \right], \]
\[ (A9) \]

and \( A = I_1 + M_2 L_1^2, I_1 = I_{cm1} + M_1 h_1^2 \) is the moment of inertia of the forearm about an axis through the elbow, and \( S = M_1 h_1 + M_2 L_1 \).

Equations (A5) and (A8) can be combined to give

\[ \frac{d\omega_1}{dt} = \frac{PL_2 - QB \cos \beta}{AI_2 - B^2 \cos^2 \beta}, \]
\[ (A10a) \]

\[ \frac{d\omega_2}{dt} = \frac{QA - PB \cos \beta}{AI_2 - B^2 \cos^2 \beta}. \]
\[ (A10b) \]

Equation (A10) can be solved numerically to find \( \omega_1 \) and \( \omega_2 \) as functions of time for any starting positions of the forearm and the racquet and for the given values of the two couples and the acceleration of the elbow. If the wrist is locked at the beginning of the swing, then \( \omega_1 = \omega_2 \), and Eq. (A10) can be solved to find \( C_2 \) in terms of \( C_1 \). The equations simplify if the forearm and racquet are locked at right angles because then \( \beta = 90^\circ \) so that \( PL_2 = QA \).

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