The polar moment of inertia of striking implements

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Abstract

A sporting implement used to strike a ball functions best when the implement strikes the ball near its sweet spot. If the impact point is well removed from the sweet spot, then the impact is usually described as a "miss–hit". One method of minimising the effects of a miss–hit is to increase the polar moment of inertia of the implement, in which case rotation about the long axis is reduced. It is shown in this paper that the resulting error in the outgoing ball angle is usually negligible, regardless of the value of the polar moment of inertia. The main advantage of increasing the polar moment is that the outgoing ball speed is increased for off–axis impacts.

Introduction

If a racquet or a club strikes a ball at an impact point that does not coincide with its long axis then the torque about the long axis will cause the implement to rotate about that axis. If the implement rotates by say 5 degrees during the impact then one might expect that the ball will also be deflected by about 5 degrees away from its intended path. If the implement itself was misaligned by 5 degrees on impact, then the ball would be deflected by about 10 degrees from its intended path, assuming that the angle of reflection is about equal to the angle of incidence. The error caused by rotation of the implement can be reduced by increasing the moment of inertia about the long axis, commonly known as the polar moment.

Attempts were made by the author to measure the error in the ball exit angle caused by long–axis rotation of (a) a golf putter and (b) a tennis racquet. In both cases, the error in the rebound angle was much smaller than expected, despite the fact that the ball was struck about 6 cm off–axis and both the racquet and the putter head rotated rapidly as a result of the impact. The two "failed" attempts are described in this paper, together with a theoretical analysis. It is shown that in most cases of practical interest, the error in the rebound angle is negligible, even in a high speed impact.

The results reported in this paper differ from those obtained by Nilsson and Karlsen (2006). The latter authors found that the exit angle of a golf ball from a putter was about 1.7° away from the normal when the ball was struck 3 cm off–axis. In the present experiment, the head weight of the putter (150 g) and its polar moment were significantly smaller than the corresponding values for a commercial putter (head weights are typically 300 to 400 g), yet the exit angle was at most 0.2° away from the normal even when the ball was struck 6 cm off–axis. The difference between the two sets of results is likely to be due to curvature of the trajectory of the spinning golf ball when it slides or rolls on a horizontal surface. In the author's experiment, the ball was suspended in the air as a pendulum bob to avoid any interaction with the ground.

Theoretical analysis

Suppose that an implement of mass M approaches a ball of mass m, the ball is initially at rest, and the implement itself is not rotating, as indicated in Fig. 1. V_1 denotes the speed of the impact point on the implement, located at distance b from the centre of mass (CM). The speed of the centre of mass, V_{cm1}, is equal to V_1. We assume for the moment that the impact time is sufficiently short that the rotation angle of the implement during the impact is negligible, and that the ball exits at right angles to the face of the implement. After the impact, the ball exits at speed v_2, the speed of the implement at the impact point decreases to V_2, the speed of the CM decreases to V_{cm2} and the implement rotates counter–clockwise at angular velocity ω_2.

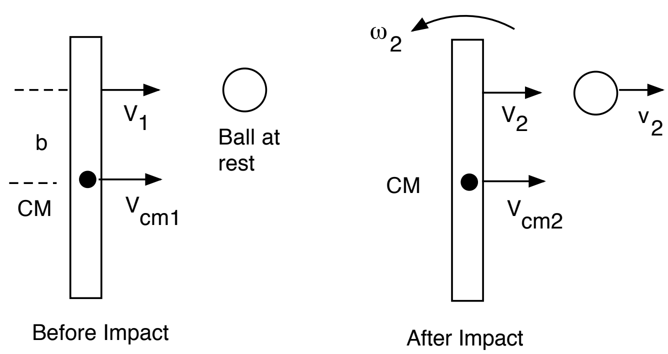

Figure 1: An implement approaches a ball a rest at speed V_1. After the impact, the ball exits at speed v_2 and the implement rotates at angular velocity ω_2.

Conservation of linear momentum indicates that

$$MV_{cm1} = MV_{cm2} + mv_2, \tag{1}$$

while conservation of angular momentum indicates that

$$I_{cm}\,\omega_2 = mbv_2 \tag{2}$$

where I_{cm} is the moment of inertia of the implement about its CM. The coefficient of restitution, e, for the collision is defined by

$$e = \frac{v_2 - V_2}{V_1} \tag{3}$$

From Eqs. (1)–(3), and the relation $V_2 = V_{cm2} - b\omega_2$, we find that

$$v_2 = (1 + e_A)V_1 \tag{4}$$

and

$$\omega_2 = \frac{(1+e_A)mbV_1}{I_{cm}} \tag{5}$$

where

$$e_A = \frac{e - m/M_e}{1 + m/M_e} \tag{6}$$

is the apparent coefficient of restitution, and

$$M_e = \frac{MI_{cm}}{I_{cm} + Mb^2} \tag{7}$$

is the effective mass of the implement at the impact point (Brody et al, 2002). These relations can be used to describe rotation about any axis through the CM, but we are interested primarily in rotation about the long axis of the implement. For an implement of width W in a direction perpendicular to the long axis, I_{cm} is approximately equal to $MW^2/12$. I_{cm} can therefore be increased, in order to reduce ω_2, by increasing M or W or by altering the mass distribution so that a larger fraction of the total mass is located near the perimeter of the implement. The player can minimise ω_2 by striking the ball on axis, where $b = 0$, or by striking the ball at low speed so that V_1 is relatively small. Conversely, the implement will rotate rapidly during the collision if b and V_1 are both large and if I_{cm} is relatively small. The polar moment of inertia is the smallest of the three principal moments, so the potential for an error in the intended ball trajectory is largest when the implement rotates about its long axis.

There are two sources of error when $b > 0$. One concerns the outgoing angle of the ball and the other is due to a reduction in the outgoing ball speed. As b increases, M_e decreases, so e_A decreases and hence v_2 decreases. When $b = 0$, $M_e = M$, but when $b = W/2$ then $M_e \approx M/4$. A significant reduction in ball speed can therefore be expected for impacts near the perimeter of the implement, but the effect on the outgoing ball angle is less certain. In order to quantify the latter effect, we can assume for simplicity that the force, F, at the impact point is given by $F = F_o \sin(\pi t/T)$ where T is the impact duration, and $F = 0$ when $t > T$. In that case, the angular velocity of the implement, ω, is given by

$$\frac{d\omega}{dt} = \frac{Fb}{I_{cm}} = \left(\frac{F_o b}{I_{cm}}\right) \sin(\pi t/T) \tag{8}$$

and the ball speed, v, is given by

$$\frac{dv}{dt} = \frac{F}{m} = \left(\frac{F_o}{m}\right) \sin(\pi t/T) \tag{9}$$

Since $v = 0$ and $\omega = 0$ at $t = 0$, we find that

$$F_o = \frac{m\pi v_2}{2T} \tag{10}$$

and

$$\omega = \frac{d\theta}{dt} = \left(\frac{F_o bT}{\pi I_{cm}}\right) [1 - \cos(\pi t/T)] \tag{11}$$

where θ is the rotation angle of the implement. Consequently,

$$\theta = \left(\frac{F_o bT}{\pi I_{cm}}\right) [t - \frac{T}{\pi} \sin(\pi t/T)] \tag{12}$$

The angular velocity increases to $\omega_2 = 2F_o bT/(\pi I_{cm})$ at $t = T$ and the rotation angle increases from zero to $\omega_2 T/2$ at $t = T$. At time $t = T/2$, when F is a maximum, the rotation angle is only 18% of its value at $t = T$. The ball therefore compresses to its maximum value while the implement rotates through a relatively small angle and then expands while the implement rotates through a relatively large angle. The situation is shown in Fig. 2. The normal reaction force on the ball can be resolved into a component $F\cos\theta$ perpendicular to the initial position of surface and a component $F\sin\theta$ parallel to the initial position of the surface. Integration of the two components over the impact time then yields the outgoing speed of the ball in the two directions, giving the result that the ball exits at an angle $\theta_2/4$ where

$$\theta_2 = \frac{F_o bT^2}{\pi I_{cm}} \tag{13}$$

is the rotation angle of the implement at time $t = T$. A similar result is obtained for other assumed force waveforms. For example, if F remains constant in time then the ball exits at an angle $\theta_2/3$. The exit angle of the ball is much smaller than the rotation angle of the implement since the implement rotates most rapidly toward the end of the impact (when ω is large) whereas the force on the ball is usually largest about mid way through the impact.

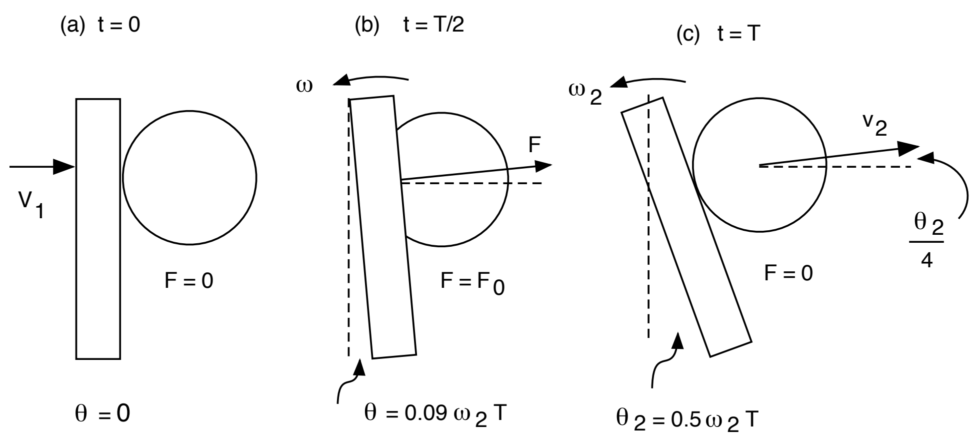

Figure 2: Schematic diagram showing the ball at three instants of time for an off–axis impact. During the first half of the impact the implement rotates through a relatively small angle while the force on the ball, and the ball compression, increase to maximum values. During the second half of the impact, the implement rotates through a relatively large angle away from the ball. The time–integrated effect of the force on the ball is that the ball exits at an angle $\theta_2/4$ where θ_2 is the rotation angle of the implement at the end of the impact.

Experimental details

The situation shown in Fig. 1 is typical of that used by a golfer when putting and was investigated using an experimental putter head of mass $M = 150$ g and width $W = 16$ cm. The putter head was constructed from a 16 cm long uniform aluminium bar, aligned horizontally in the usual manner, and was allowed to rotate freely about a 1 m long vertical shaft inserted through the centre of mass of the head. The shaft itself did not rotate about its long axis. A low mass head was chosen deliberately to reduce its polar moment so that errors in the ball exit angle would be more easily measured. The head and shaft were swung as a pendulum, without rotation of the head, to impact a stationary 47 g superball suspended as a pendulum. The putter head was incident at speed $V_1 \approx 1$ m/s. A superball was used rather than a golf ball to avoid impacting on a dimple which could have affected the ball exit angle. The impact was filmed at 300 fps using a Casio EX-F1 video camera mounted above the apparatus. A large area grid was mounted on a horizontal table below the apparatus to assist with measurements of the speed and angle of incidence of the head, the alignment of the head, and the outgoing speed and angle of the ball. Video film showing impacts at $b = 0$ and $b = 6$ cm can be seen HERE (Putter.mov)

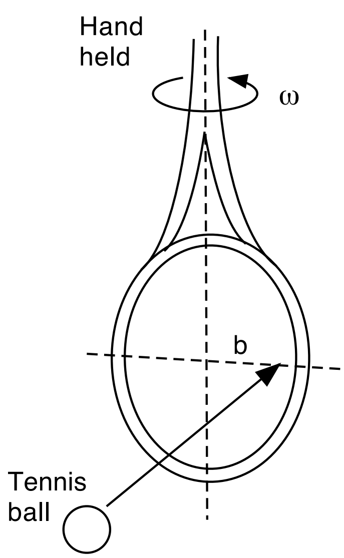

Figure 3: A racquet was supported in a vertical position by holding the handle with two hands. A ball was fired at 29.3 m/s at right angles to the string plane to impact the strings at a point 7.7 cm to the right of the centre of the string plane.

According to Eq. (2), the angular velocity of an implement used to strike a ball, after the impact is over, is directly proportional to b and to the outgoing ball speed. The same result is implicit in Eq. (13), given that v_2 is proportional to the peak force on the ball. Consequently, one might expect a more significant effect on the ball exit angle if the ball is struck at high speed and if the implement rotates through a large angle during the collision. Such an experiment was conducted by firing a tennis ball at a speed of 29.3 m/s to impact a hand held racquet at right angles to the string plane so that the ball impacted near the outer edge of the string plane. The arrangement is shown in Fig. 3. A Casio EX-F1 video camera was mounted directly above the impact point to film the impact at 300 fps. Video film of this experiment can be viewed HERE (Tennis.mov)

Golf putter results

Several impacts were filmed with $b = 0$, resulting in zero rotation of the putter head, zero rotation of the ball, a ball exit speed $v_2 \approx 1.54$ m/s and zero deflection of the ball away from the expected path (in a direction perpendicular to the face of the club head). The experimental data gave $e = 0.94 \pm 0.01$, $e_A = 0.58 \pm 0.01$ and an effective mass $M_e = 206 \pm 2$ g. The effective mass was larger than the mass of the putter head due to the additional mass of the long steel shaft.

Impacts were also filmed with $b = 6$ cm, resulting in rapid rotation of the putter head after the impact, but there was no significant deflection of the ball away from the line of incidence of the putter head. In this experiment, $e = 0.95 \pm 0.01$, $e_A = 0.12 \pm 0.002$, $V_1 = 1.19 \pm 0.01$ m/s, $v_2 = 1.33 \pm 0.01$ m/s, and $\omega_2 = 10.5 \pm 0.1$ rad/s. The impact duration was measured with a small piezo disk attached to the putter head, giving $T = 3.2$ ms. From Eq. (13), the estimated ball exit angle is only 0.2°, which was too small to measure reliably. Misalignment of the putter head by only 0.2° on impact could account for the fact that the observed exit angle was less than 0.2°, which was the uncertainty in both the measured exit angle and the angular alignment of the putter head. The observed ball deflection angle was therefore consistent with the theoretical prediction within experimental error, but the uncertainties in the measured angles were too large to provide a useful comparison with the predicted exit angle.

If used as an actual putter, the main problem with this putter head, apart from its light weight, would be the reduction in ball speed for an impact near the heel or toe. For an impact at $b = 6$ cm, the ball exit speed, given by Eq. (4), is reduced by 29% compared with an impact at $b = 0$. It is unlikely that a good golfer would strike the ball so far from the middle of the putter head, so the reduction in ball exit speed would generally be much less than 29%.

Tennis racquet results

In the tennis racquet experiment, the ball impacted the strings at a point 7.7 cm to the right of the centre of the string plane. The ball bounced at an angle of $0.8 \pm 0.2°$ to the normal, in the expected direction, at a speed of 5.0 ± 0.1 m/s, while the racquet rotated at 63.8 ± 0.6 rad/s about its long axis immediately after the impact. The impact duration was not measured but is known from previous experiments to be about 5 ms. Consequently, the racquet rotated by about 9° during the collision, in which case the rebound angle of the ball would be 2.25° according to Eq. (13). The theoretical results described above were derived in a reference frame where the ball is initially at rest, but the angular velocity of the racquet and the rebound angle of the ball are unaltered if the reference frame is chosen so that the racquet is initially at rest.

The rebound angle of the ball was therefore much less than the rotation angle of the racquet but it was less than half the value predicted by Eq. (13). A plausible explanation is that the ball was deflected closer to the normal by deformation of the string plane. The string plane is stiffer near the edge of the racquet frame than in the middle of the string plane, with the result that the string plane deforms in such a way that the normal to the string plane points toward the centre of the strings. The same effect is observed when a ball is dropped vertically onto the horizontal membrane of a drum. Regardless of the impact point, the ball bounces toward the centre of the drum.

Of much greater concern to a player is the fact that e_A in this experiment was only 5.0/29.3 = 0.17. In the racquet reference frame, e_A is the ratio of the ball exit speed to the incident speed. For an impact in the middle of the strings, e_A was 0.38 ± 0.01. In the ball reference frame, the outgoing ball speed is given by Eq. (4) and is reduced by 15% for an impact near the edge of the racquet frame compared with an impact in the middle of the strings. A better result would be obtained by adding lead tape to the frame of the racquet to increase its polar moment of inertia. The effect would be to reduce rotation of the racquet about its long axis, thereby increasing e_A and the outgoing ball speed for impacts close to the racquet frame.

Discussion

Long–axis rotation of the head of a golf club on impact with the ball is the result of a miss–hit, but the effect is compensated to some extent by the gear effect. When the head makes contact with the ball, it grips the ball and the two rotate in opposite directions in the same manner as two gears (Cross and Nathan, 2007). The ball thereby acquires sidespin. If the ball is struck by a driver and heads to the side of the fairway, then the ball will curve back toward the centre of the fairway as a result of the Magnus force on the ball. The gear effect works best if the centre of mass of the head is located well behind the striking face so that the impact point on the face develops a significant velocity component in a direction tangential to both the face and the ball, as indicated in Fig. 4.

The gear effect will also occur with putters, especially with mallet type putters that extend well behind the face of the putter. The extension in mallet putters is designed to assist a golfer to provide better visual alignment of the putter, rather than to enhance the gear effect. Nevertheless, the ball will be struck with side spin if it is stuck off–axis. If the ball then slides or rolls forward on the green, it will tend to curve away from a straight line trajectory in the manner shown in Fig. 4. A measurement of the rebound angle off the club face could easily be misinterpreted, especially if the angle is measured in terms of the lateral displacement of the ball over its entire trajectory.

A ball that is sliding forward on a horizontal surface will experience a sideways force if it is spinning either about a horizontal axis pointing in the direction of motion, or about a vertical axis. In billiards, a Masse shot is one where a ball is struck so that it slides forward and also spins about about a horizontal axis pointing forward. Friction with the table acts to exert a sideways force on the ball, in which case the ball can be made to curve around another ball that is blocking a straight line path to a third ball (Wallace and Schroeder, 1988).

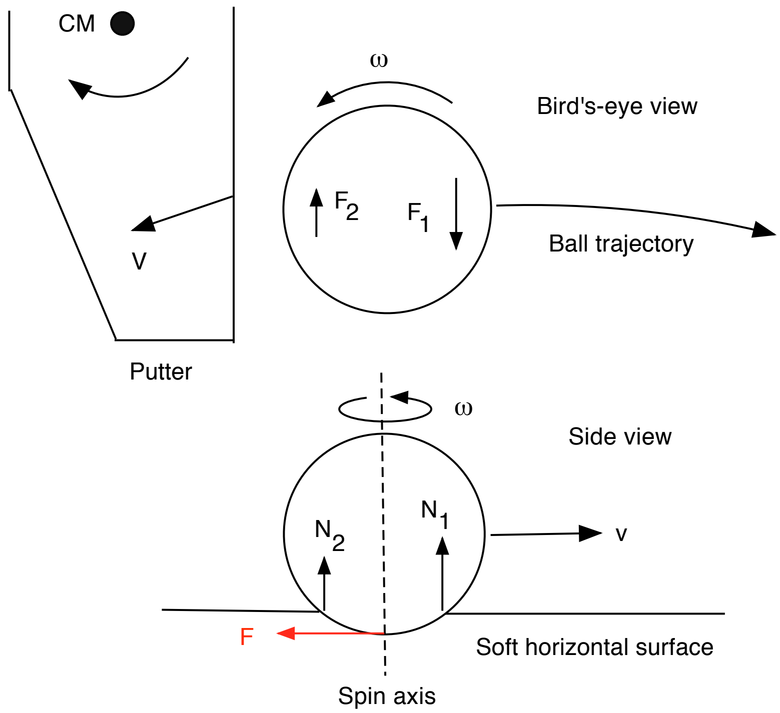

Figure 4: A golf ball struck off–axis causes the putter to rotate with a velocity component V at the impact point. The component of V parallel to the putter face is tangential to the ball and results in sidespin due to the gear effect. The ball curves to the right along a horizontal surface if the spin axis is vertical and if the ball rotates counter–clockwise viewed from above. The ball slides for a short distance and then rolls. If the normal force N_1 at the front of the ball is larger than that at the rear, then the friction force F_1 arising from the ball spin will be larger at the front of the ball than at the rear.

In a putting stroke, the ball can spin about a vertical axis as a result of the gear effect. A significant sideways force on a ball spinning in this manner has not previously been observed as far as the author is aware, but can be expected by analogy with a similar effect that occurs in the game of curling. In curling, a heavy cylindrical rock is pushed forward at low speed on ice and is also given an initial rotational velocity about a vertical axis so that the rock deflects sideways, or curls, from its initial straight line path. Ideally, no sideways deflection would be expected. In practice, the horizontal friction force acting backward on the rock exerts a torque about its centre of mass, causing it tilt slightly forward. As a result, the normal reaction force and the friction force acting at the front of the rock is larger than the corresponding forces at the rear. When the rock is spinning about a vertical axis, the transverse friction force components at the front of the rock are larger than those at the rear, so there is a net transverse force on the rock. The latter effect has been analysed by Voyenli and Eriksen (1985) for a hockey puck sliding on ice and by Penner (2001) for a curling rock sliding on ice.

Lateral deflection of a ten–pin bowling ball occurs if the ball is given initial spin about a vertical axis (Frohlich, 2004). In that case case, the deflection is relatively small unless the ball is suitably weighted so that it can precess. The friction force on the ball is small since the alley is oiled and since the ball does not deform, so it contacts the alley over a very small contact area. The deflection of a golf ball could be significantly larger since the ball slides on a relatively soft green, indents the green, and contacts the green over a relatively large contact area.

The effect was investigated by the author by projecting a golf ball on a horizontal, low pile carpet. The ball was projected by hand, using a finger of one hand and the thumb of the other hand, so that it spun about a vertical axis when it was projected forward on the carpet. The initial spin was about 15 rev/s, the peripheral speed of the ball being slightly larger than the initial forward launch speed. The ball came to a stop typically about 1 m from the launch point, depending on its forward speed, having deflected laterally by about 0.2 m and leaving a visible track about 5 mm wide on the carpet. When the spin direction was reversed, the direction of curvature also reversed.

A video clip showing the curvature can be seen HERE (Spin.mov)

A surprising result of the carpet experiment was that the spin axis remained approximately vertical over the whole trajectory. Friction reduced the translational speed to zero while the ball kept spinning on the spot for a short time afterwards about its vertical axis. The result can be attributed to the high initial spin given to the ball. When the experiment was repeated by projecting the ball forward at low spin rates, the ball curved by a smaller amount and it rolled forward about a horizontal axis before it came to a stop. An additional surprise was that the translational speed of the ball decreased to zero at a rate consistent with rolling friction rather than sliding friction, regardless of the initial spin. The explanation is shown in Fig. 5.

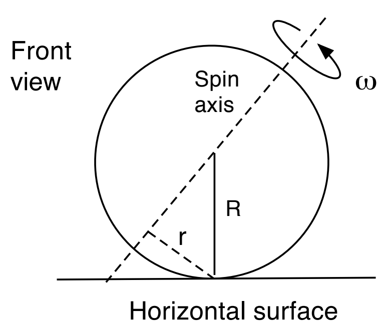

Figure 5: Front view of a ball approaching the reader at speed v and spinning about an axis inclined to the horizontal. The spin axis rotates with time so that the ball rolls on the surface along a circular track around the ball of radius $r = v/\omega$.

A rolling ball normally rolls with its spin axis horizontal. However, if the spin axis is inclined to the horizontal, then it can roll forward along a circular track around the ball of radius

$r = v/\omega$ where v is the translational speed of the ball and ω is its angular velocity. Since the point on the ball in contact with the surface rotates at speed $r\omega$ with respect to the centre of the ball, the contact point is at rest on the surface when $v = r\omega$. If v and ω decrease to zero at different rates then the rolling condition can be maintained if the spin axis tilts, as observed experimentally. When projected with large ω, the spin axis of the golf ball tilted slowly into a vertical position, the ball continuing to spin about a vertical axis even after v dropped to zero. When projected with small ω, the spin axis of the ball slowly tilted into a horizontal position and then commenced rolling in the conventional manner, v and ω decreasing to zero at the same time (Cross, 2011).

An explanation for the curvature of a ball spinning about a vertical axis, offered by Penner (2001), is that the friction force, F, acting backwards at the bottom of the ball exerts a torque on the ball, resulting in a relatively large normal reaction force N_1 at the front edge of the contact area and a relatively small normal reaction force, N_2, at the trailing edge, as indicated in Fig. 4. If the friction force at the front of the ball, F_1, is larger than the friction force F_2 at the rear of the ball (see Fig. 4), then there will be a net sideways force on the ball. An alternative explanation for the curvature is that the ball indents the carpet and the front edge ploughs through the carpet, giving a larger normal reaction force and a larger friction force at its front edge. The latter effect is well known from studies of rolling friction (Domenech et al, 1987). The net transverse force acts to deflect the ball sideways in the observed direction.

Nilsson and Karlsen (2006) measured the lateral deflection of a golf ball over its entire trajectory, finding that mallet putters deflected the ball further than conventional blade-type putters. The latter result suggests that the lateral deflection may have been caused primarily by spin imparted to the ball rather than by errors in the exit angle of the ball as it left the putter. The authors did not measure the spin, and the present author did not attempt to reproduce their results. Such an experiment would be of interest in order to pinpoint the cause of the large lateral deflections observed by Nilsson and Karlsen.

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