BALL TRAJECTORIES

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1. PROJECTING FOR MAXIMUM DISTANCE

In some ball sports, a player will want to project the ball as far as possible. The ball must then be launched as fast as possible, but the interesting physics question concerns the best angle at which to project the ball. For example, should a golfer hit the ball upwards at say 45 degrees to the horizontal or will the ball travel further if it is projected at a lower angle? The same question arises in baseball if a player is trying to hit a home run, or in cricket if a player is trying to hit a six over the fence, or in a soccer throw or in football or when throwing a javelin or shot put. Similarly, an athlete competing in the long jump needs to jump both upwards and forwards, but what is the best launch angle above the horizontal?

There is no single answer here that covers all cases but the problem can be viewed in the following way. When a ball is hit or thrown for maximum distance then it travels in the horizontal direction at speed V and it remains in the air for time T. The horizontal distance (D) travelled before it lands is given by D = VT. For D to be as large as possible, V and T both need to be as large as possible. V is a maximum when the ball is projected as fast as possible in the horizontal direction. But then the ball will fall to the ground quickly and T will be quite small. T is a maximum when the ball is projected straight up in the air so it travels as high as possible, but then the horizontal speed V is zero. The ball will spend a long time in the air but it will travel up and down along the same vertical path and travel zero distance horizontally.

If the only force acting on the ball is the force of gravity and if the ball is projected from ground level and lands at ground level then the ball will travel the greatest horizontal distance when it projected at 45 degrees to the horizontal. That way, the vertical launch speed is the same as the horizontal launch speed and it represents the best compromise between maximising V and maximising T.

There are several reasons why 45 degrees is NOT the best angle in practice. One is that the ball will also be be subject to a drag force acting backwards on the ball due to air resistance. The drag force can be bigger than the gravitational force if the ball is travelling fast enough, although this situation would never arise when projecting a very heavy ball such as the shot put. When the drag force is taken into account, maximum distance requires that the launch angle is less than 45 degrees. As ball speed increases, so does the drag force and the lower is the required launch angle. A launch at 45 degrees would allow the ball to remain in the air for a longer time, but it would then be launched at a lower horizontal speed at the start and it would slow down more because of the longer flight time.

An additional aerodynamic force arises if the ball is spinning. This force is called the Magnus force, it increases as the ball spin is increased, and it acts at right angles to both the path of the ball and to the rotation axis. For example, when a golf ball is projected with backspin then the Magnus force acts upwards on the ball as a lift force and holds the ball in the air for a longer time than it otherwise would if it wasn't spinning. In that case, the ball will travel the maximum horizontal distance before landing if it is launched at an angle of around 10 or 20 degrees to the horizontal. If the ball has enough backspin so that the Magnus force is greater than the force of gravity then the ball will rise up at an angle greater than 20 degrees after it is launched.

In cases such as the shot put where the ball is launched from a certain height above ground level, the best launch angle is also less than 45 degrees. One way to understand this is to imagine that the ball actually started at ground level with a 45 degree launch but after it rises to the actual launch height it will be travelling at a smaller angle to the horizontal. Alternatively, one can consider this situation as one where a ball is launched from ground level but where it lands below ground level. Suppose that someone throws a ball off a cliff or a tall building and wants it to land as far out as possible. The landing point is then well below ground level and the time in the air depends mainly on the time it takes to fall to the bottom. There is no need to throw the ball at 45 degrees to gain extra height if the extra time in the air is only a small fraction of the time taken to fall to the bottom of the cliff or building. In that case, maximum distance is achieved by throwing almost horizontally to maximise the horizontal launch speed. The same reasoning applies if a person wants to jump off a tall cliff or building and land as far out as possible.

The long jump is a case where the best launch angle is about 25 degrees. In this case the athlete takes many running steps to build up horizontal speed. The very last step is used to increase the vertical speed. It is not physically possible to jump vertically in one step at the same horizontal speed as the runup. The athlete could jump at 45 degrees using a much slower runup but the jump distance would then be much smaller. The time in the air is determined by the vertical jump speed, which cannot be increased since only one step is taken to achieve that vertical speed. The only way to increase the jump distance D is to to increase the horizontal launch speed, V, by taking a long runup. Then D = VT where T is the time in the air. Actually, D is a bit larger than this because the feet land ahead of the centre of mass and because the centre of mass starts from a point in front of the feet at the start of the jump.

In athletic events, there is another consideration. That is, the force that can be applied in the horizontal direction is not the same as the force that can be applied in the vertical direction. A ball can be thrown faster in the horizontal direction than in the vertical direction. Consequently, the best launch angle for the shot put or the javelin throw is even lower than one might expect just from the aerodynamics of the problem.

2. DRAG FORCE

If a ball or any other object is moving at speed V through the air, then the air exerts a backwards force on the ball called the drag force. If the ball is moving vertically up then the drag force acts vertically down (and vice versa). If the ball is moving forward in a horizontal direction then the drag force acts backward in a horizontal direction. The formula for the drag force is

$$F = \frac{1}{2} C_D d A V^2$$

where C_D is called the drag coefficient, d is the density of the air and A is the cross-sectional area of the ball. For a ball of radius R, $A = \pi R^2$. The value of d at 20 degrees Centigrade is 1.21 kg/m^3. Unlike the gravity force, the drag force does not depend on the mass of the ball. It depends only on its radius and speed. C_D depends to some extent on the surface roughness of the ball and it also depends on ball speed.

For a circular disk, $C_D = 1.0$. For a streamlined object, C_D can be less than 0.1. The value of C_D is constant and equal to about 0.5 for a sphere at low ball speeds but it can drop dramatically to around 0.2 at high ball speeds. The drag force itself may or may not drop when C_D drops, since F is proportional to V squared. The drop in C_D occurs as a result of a change from smooth or laminar air flow around the ball to a more turbulent flow. The drop in C_D is especially large on very smooth balls, is smaller on slightly rough balls and does not occur at all for a tennis ball since the ball surface is too rough. For a tennis ball, C_D is about 0.55 regardless of ball speed, but it depends slightly on the smoothness or roughness of the cloth surface. A fluffy ball will slow down faster, partly because the drag coefficient is larger and partly because the ball diameter is then larger.

The speed at which C_D drops depends on the diameter of the ball. It occurs at low ball speeds on large balls (such as a soccer ball) and at high ball speeds for small balls (such as a golf ball). Golf balls are slightly dimpled so that C_D drops at a lower ball speed than it would for a perfectly smooth ball. For example, C_D drops to about 0.25 at speeds above 8 m/s for a soccer ball. For a golf ball, C_D drops to about 0.25 at speeds above 15 m/s and remains at 0.25 even at speeds up to 70 m/s. A golf ball is struck at speeds of around 60 m/s and travels further in air than it would in a vacuum since the lift force due to its backspin holds it in the air for a longer time.

3. DRAG FORCE ON A PARTY BALLOON

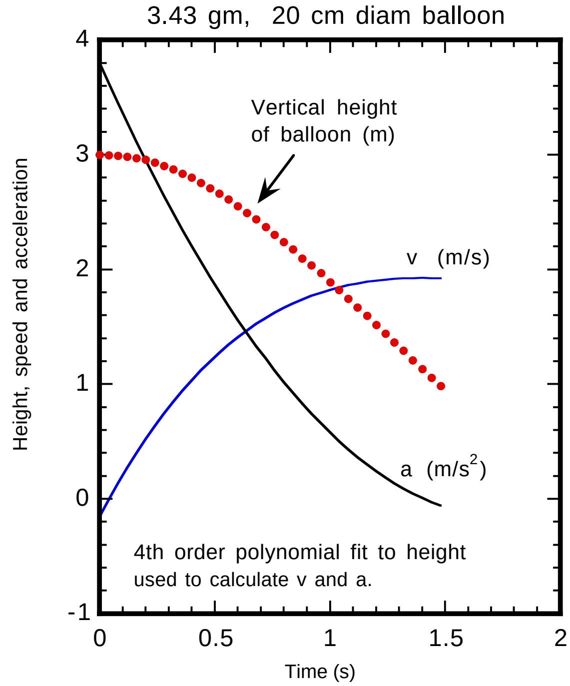
The effect of the drag force is very noticeable on balls or other objects that are relatively light. It is the drag force on a leaf or a piece of paper that causes it to take a longer time to fall to the ground than a heavy object. A 57 gm tennis ball slows down by about 25% from the moment it is struck to the time it lands on the other side of the court. Most players would not even notice that the ball slows down as it travels through the air, but it does. A ping-pong ball would not even make it over the net. The drag force on a ping-pong ball is less than that on a larger tennis ball travelling at the same speed, but the mass of a ping-pong ball is many times smaller and the drag force therefore has a much bigger effect in slowing it down.

The effect of the drag force on a 20 cm diameter balloon dropped from the ceiling in my office is shown in movie A. For comparison, movie B shows the same situation for a baseball dropped from the ceiling. It is obvious that the balloon falls much more slowly than the baseball. The acceleration of the baseball is essentially equal to 9.8 m/s^2 during the whole

time the ball falls since the drag force is much smaller than the gravitational force on the ball, at least at the relatively low drop speed seen in the movie. After falling through a height of 2.5 m, the ball accelerated to a speed v = 7.0 m/s.

It was relatively easy to calculate the drag coefficient for the balloon by measuring its speed as it fell. I wanted to check whether the drag coefficient was about 0.5, as it is for all other spherical objects travelling through the air at low speed. I chose to measure it for a balloon since the effect is much easier to see and much easier to measure than for a heavy ball. I also learnt a few interesting things along the way, such as the fact that the air inside the balloon was a lot heavier than the balloon itself. I also found that adding 7 gm of air to the balloon increased its measured weight by only 0.25 gm since the buoyant force of the air outside the balloon exerted an upwards force on the balloon that almost cancelled the weight of the extra 7 gm of air added inside the balloon.

The balloon accelerated to 1.95 m/s after falling through a height of 1.1 m and then the speed remained constant for the remainder of its fall. It reached a constant speed called the terminal velocity since at that point the total force on the ball dropped to zero. The force of gravity acting down on the balloon was exactly balanced by the drag force acting upwards and by the buoyant force of the air. In order to get a nice vertical drop I needed to tie a 2.1 gm nut onto the bottom of the balloon using a very light cotton thread. Without the nut, the balloon tended to rotate and to veer off to one side. The rubber balloon itself had a mass



of only 1.3 gm but it was inflated by adding 7.0 gm of air, so the total mass of the balloon plus the nut was 10.4 gm.

A tricky part of this experiment was working out how much air I added. The mass of the inflated balloon was measured to be 1.55 gm and its volume was approximately 0.00557 m^3. The balloon was not exactly spherical so I estimated its volume assuming that it was sphere of average radius 0.11 m. The balloon was 20 cm in diameter across the long axis and 25 cm in diameter along its long axis. The mass of the displaced air was therefore 6.75 gm, which gives the buoyant force. In other words, the mass of the inflated balloon was actually 1.55 + 6.75 = 8.3 gm. On the weighing scale it registered only 1.55 gm since the air exerted a buoyant force lifting it up against the force of gravity.

Since the rubber itself had a mass of 1.3 gm, the air inside the balloon had a mass of 7.0 gm. This is slightly larger than the mass of the displaced air since the air inside the balloon was at a slightly higher pressure.

The acceleration of the balloon as it was falling is given by

$$F = ma = mg - F_B - F_D$$

where F_B is the buoyant force (mass of displaced air times g) and F_D is the drag force. At the start of the fall where $V = 0$, the drag force is zero so $a = g - F_B/m$. This works out to be 3.5 m/s^2, which is close to the value 3.8 m/s^2 that I measured.

Terminal velocity is reached when $a = 0$ and then $F_D = mg - F_B$. Using this formula I found that $C_D = 0.50$, which is the value expected for a slowly moving spherical ball. Part of the reason that a balloon falls slowly is that the drag force slows it down, but the buoyant force also plays a big role. Both of these forces are tiny compared to the force of gravity on a baseball, although the drag force is important at higher ball speeds than those shown in the baseball movie.

4. DRAG FORCE ON A BASEBALL

Consider a baseball of mass 145 gm and diameter 73 mm. If we take $C_D = 0.5$ at low ball speeds then with $A = 0.00418$ m^2 and $d = 1.21$ kg/m^3, the drag force is

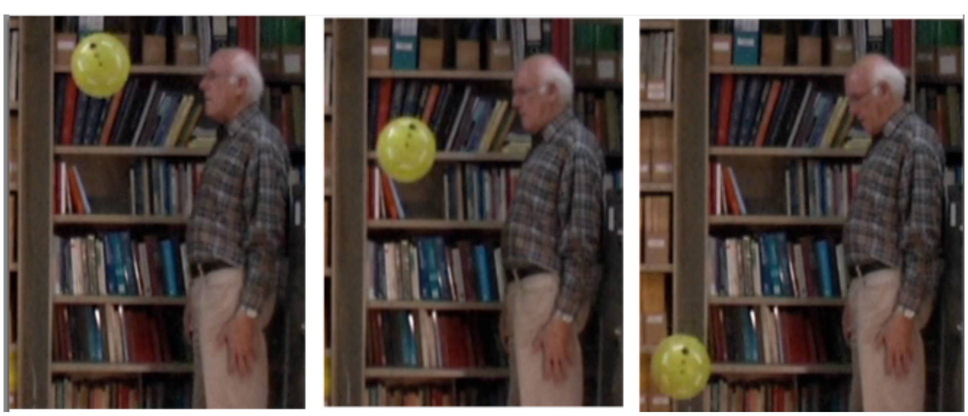
$$F_D = 0.00127V^2$$

while the gravitation force is $mg = 0.145 \times 9.8 = 1.42$ Newton. At a ball speed $V = 7$ m/s, the drag force is only 0.062 Newton, 23 times smaller than the gravitational force. The two forces are equal at a speed V = 33 m/s, which corresponds to the terminal velocity of a baseball dropped from a great height.

5. MAGNUS FORCE ON A BALLOON

Most people who measure the Magnus force do so by spinning a ball in a wind tunnel. The ball spins at a fixed location in the wind tunnel while the air flows past it. An alternative way to measure the Magnus force is to film the trajectory of a spinning ball using several cameras, but a lot of cameras would be needed to view at right angles to a high speed ball travelling over a distance of 50 m or more. At least one of the cameras needs to film the ball at around 200 frames/sec or more to measure the spin rate. Either way, measuring the Magnus force is usually a difficult and expensive operation.

A conventional video camera can be used to film the flight of a ball at low ball speeds and at low spin rates, but the Magnus force is relatively small. To see any significant effect at low ball speeds one needs a ball with a large surface area and a small mass. The obvious choice is a balloon, although I have not seen any previous measurements for a balloon myself. It turns out that a spinning balloon provides an excellent demonstration of the Magnus effect since it is something that anyone can do without needing expensive equipment and it is very easy to observe the effect.



Vertical drop of a balloon spinning clockwise. The Magnus force deflected it to the left

The experiment is shown in Movie A for a case where the balloon was spun clockwise, and in Movie B for a counter–clockwise spin. I removed the nut at the bottom of the balloon for this experiment but I added a strip of white adhesive tape around the circumference of the balloon to give it a bit more rotational inertia and to help balance it. Without the tape, the balloon tended to wobble and to twist around as it fell The extra mass of the tape made it more stable and allowed it to spin for a longer time without slowing down so rapidly.

Spinning a balloon by hand is not as easy as it looks. I found that the best technique was to throw it upwards slightly as I spun it. That way I was able to get the ball to drop vertically at the start so I could more easily see the effect of the sideways Magnus force. The Magnus force increases with both spin rate and with ball speed squared, so there is essentially no sideways force at the start of the fall since the ball speed is too low. As the ball fell towards the floor its spin rate decreased but its speed increased, with the result that the ball deflected sideways in the expected direction. The Magnus force acts on the whole ball in the same direction as the direction of rotation of the leading (bottom) edge.

I was surprised that the spin rate decreased so quickly. On heavier balls there is only a slight decrease in spin rate as the ball travels through the air, although the effect has not been examined very closely by anyone as far as I know. The effect is due to friction between the air and the surface of the ball, and can be described in terms of the viscosity of the air. If a ball starts spinning in a bucket of honey then it will stop spinning very rapidly since honey is very viscous. Water has lower viscosity (it pours more easily) and air has even lower viscosity. Neverthless, the viscous force of the air acting on a ball as light as a balloon is enough to stop it spinning after only a few seconds. The same viscous force would act on any other ball of the same diameter and spin rate, but it would have a much smaller effect on a heavy ball than on a light ball.

The Magnus force FM acting on a spinning ball travelling at speed V is given by

$$F_M = \frac{1}{2} C_L d A V^2$$

where C_L is called the lift coefficient, d is the density of the air and A is the cross-sectional area of the ball. The formula is essentially the same as that for the drag force but the lift coefficient is generally smaller than the drag coefficient. The coefficient is called a lift coefficient since the Magnus force is a vertical lift force on say a golf ball moving horizontally with backspin. In fact, the Magnus force acts horizontally on a ball moving vertically, and it acts vertically down on a ball travelling horizontally with topspin. The Magnus force exists only if the ball is spinning and it increases with the rate of spin. The formula here doesn't show the spin effect, but CL depends on the rate of spin, being roughly proportional to the rate of spin (and is zero when the spin is zero).

6. BASIC PHYSICS

When a ball or any other object is projected through the air it will follow a curved trajectory until it hits the ground. The trajectory can be calculated easily if we ignore air resistance and assume that the only force acting on the ball is that due to gravity. The acceleration due to gravity is g = 9.8 m/s^2 and the vertical force F acting on a ball of mass m is given by $F = mg$. In that case, the horizontal speed of the ball through the air remains constant since there is no horizontal force on the ball. Gravity acts only in the vertical direction. The resulting ball path is then a parabola.

A ball projected vertically upwards at speed V will rise to a height H where the potential energy mgH is equal to the intial kinetic energy $mV^2/2$. Since $mgH = mV^2/2$, we find that $H = V^2/(2g)$. For example, if $V = 1$ m/s then $H = 1/(2x9.8) = 0.051$ m = 5.1 cm. If $V = 10$ m/s then H is 100 times larger, or 5.1 m.

If a ball is projected at speed V at an angle θ to the horizontal, then the horizontal launch speed is $V_x = V\cos\theta$ and the vertical launch speed is $V_y = V \sin\theta$. At any time t after the launch the horizontal (x) and vertical (y) positions of the ball will be given by the formula $s = ut + at^2/2$ where s = distance travelled, u = initial velocity and a = acceleration. In this case,

$$x = (V\cos\theta)t \qquad\qquad (\mathrm{a} = 0)$$

and

$$y = (V\sin\theta)t - gt^2/2 \qquad\qquad (\mathrm{a} = -\mathrm{g})$$

assuming that the ball starts at $x = 0$ and $y = 0$. The vertical speed of the ball at any time t is given by

$$V_y = V\sin\theta - gt \qquad\qquad (\mathrm{v} = \mathrm{u} + \mathrm{at})$$

so the ball reaches its maximum height at time $t = (V \sin\theta)/g$ when $V_y = 0$. The ball takes the same time to fall back to the ground, so the total travel time in the air is $2t = 2(V \sin\theta)/g$ and the total horizontal distance travelled (the range R) is given by

$$R = (V\cos\theta)2t = 2(V\cos\theta)(V\sin\theta)/g = V^2(\sin 2\theta)/g$$

R is a maximum when $\theta = 45$ degrees.