Why bows get stiffer and racquets get softer when the strings are added

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The frame of a string instrument is subject to a large tension force when the strings are installed. Intuitively, one might expect that the frame would be stiffened by the strings. Experimental data and a theoretical analysis are presented to show that this is not generally the case. An archer's bow is much stiffer when it is strung, but a tennis racquet is softened when the strings are added. As a result, the mode frequencies for transverse vibrations increase for a bow and decrease for a racquet when the strings are added. The effect of the strings depends on the extent to which the frame is bent at equilibrium.
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I. INTRODUCTION

There are many types of instruments fitted with strings, including violins, pianos, racquets, and bows used to shoot arrows. The physics of the strings is well understood, but the physics of the support frame is much more complex. The frame has its own characteristic set of vibration modes, which generally enhance the performance of the instrument but sometimes generate annoying resonances with the strings in musical instruments.[1] An interesting question, but one that rarely seems to arise, is whether the mode frequencies of the frame depend on the string tension. One might expect that the frame would be stiffened by the strings, leading to an increase in the mode frequencies of the frame. Experiments conducted by the author indicate that the mode frequencies of a violin frame increase by less than 1% when the string tension is increased from zero to normal tension. However, this is not typical of all devices fitted with strings. When a string is added to an archer's bow the fundamental vibration frequency of the bow increases by a factor of about 3 or 4. An even more surprising result is that the fundamental vibration frequency of the frame of a tennis racquet *decreases* by about 10% when the strings are added, depending on the stiffness of the frame and the string tension. This is not simply due to the additional mass of the strings, about 15 g, which would account for a drop in frequency of only about 2%. The author could find only one article[2] in the literature relevant to this observation. The authors used a finite element model of the racquet and predicted (incorrectly) that the vibration frequency would increase when the strings are added.

When a violin or a racquet is strung, there is no visible bending of the frame. The lowest frequency mode of the frame of a violin or a tennis racquet is a one-dimensional bending mode, with nodes 10–15 cm from each end. The frequency is about 180 Hz in both cases since racquets and violins have about the same mass and stiffness and comparable lengths. The above differences in the two cases are due in part to the factor of about 24 difference in the total tension force. A violin has four strings at an average tension of about 12 lb, but a racquet is strung with about 18 strings parallel to the long axis, each at a tension of about 60 lb. An archer's bow differs from a violin or a racquet in that the bow is bent before it is strung and it bends a lot further when it is strung.

The drop in frequency when a racquet is strung is counterintuitive. However, it is easily explained. In fact, it is the bow that is harder to explain, at least in words. Any displacement of an unstrung bow from its equilibrium position generates a restoring force arising from bending of the bow. The bow will have a certain stiffness or spring constant. When the bow is strung, it assumes a new shape where the tension force in the string is balanced by the stiffness of the bow. Any displacement of the bow from the new equilibrium position involves a change in length of the string. For example, if an external force is applied to *reduce* the curvature of the bow, an extra force is required to stretch the string. If an external force is applied to *increase* the curvature of the bow, then one might think that the string will help here, and the task will be relatively easy. However, in this case the string tension drops so the string offers less assistance, not more. The situation can be compared with the case of a helical spring that is compressed axially by means of a string mounted along the axis of the spring and tied under tension to each end. In that case, the spring constant of the system is the sum of the spring constants of the spring and the string. The string therefore acts to stiffen the spring. The geometry of a strung bow is different, but the effect of the string is to stiffen the bow, as described in more detail below.

When a tennis racquet is strung, there is no change in the equilibrium curvature of the frame since the tension is applied along the main axis and the racquet remains straight. If an external force is applied to bend the frame in a direction perpendicular to the axis, then the strings parallel to the axis are shortened and the tension drops. However, the main effect of the strings is that they assist the external force, resulting in a larger displacement of the frame, in the same way that adding a string to a bow increases its curvature. The fact that the string tension drops is not significant. The main effect is that when the frame bends, the string tension develops a component perpendicular to the axis, and this component enhances the displacement rather than resisting it. As a result, the frame of a racquet becomes softer rather than stiffer when the strings are added. The vibration frequency of the frame therefore decreases. If the string tension is large enough the frequency can even drop to zero, in which case the frame will be statically unstable and will buckle.

II. VIBRATION OF A UNIFORM BEAM

The effect of a stretched string on the vibration frequency of its support structure is easiest to analyze if one considers a simple structure such as a rectangular cross-section beam. The string may be mounted along the axis of the beam, or off axis as in a bow, as shown in Fig. 1. The equation of motion

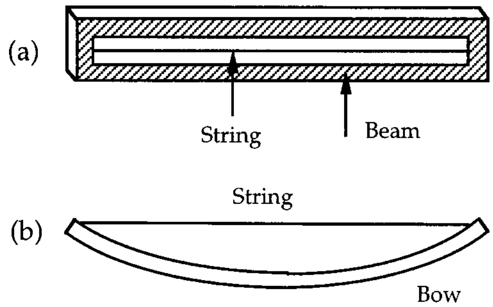

Fig. 1. Geometry of a simple beam tensioned by a string (a) along the beam axis or (b) off axis, as in a bow.

for a straight or a curved beam subject to an external transverse force, F_0 per unit length, has the form[3–5]

$$\rho A \frac{\partial^2 n}{\partial t^2} = F_0 - \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 n}{\partial s^2}\right), \tag{1}$$

where ρ is the density of the beam, A is its cross-sectional area, E is Young's modulus, I is the area moment of inertia, and n is the transverse displacement of the beam at coordinate s along the beam axis. If the equilibrium shape of the beam has the form of a curved bow, then s is the local tangential coordinate and n is the transverse displacement in a direction perpendicular to s. Equation (1) is valid only for small values of n, and is also valid for a curved bow provided the radius of curvature is large compared with the transverse dimensions of the beam.[3,4] For a uniform beam of mass M and length L, numerical solutions of Eq. (1) can be obtained by dividing the beam into N equal segments each of mass $m = M/N$ and separated in the s direction by a distance $h = L/N$. If an external transverse force, $F = F_0 h$, acts on any segment, then the equation of motion for that segment is obtained by multiplying all terms in Eq. (1) by h, in which case

$$m \frac{\partial^2 n}{\partial t^2} = F - (EIh) \frac{\partial^4 n}{\partial s^4}, \tag{2}$$

assuming that the beam is uniform so that E and I are independent of s. If a segment is not subject to an external force, then the equation of motion for that segment is given by Eq. (2) with $F = 0$. The boundary conditions at a free end are given by $\partial^2 n/\partial s^2 = 0$ and $\partial^3 n/\partial s^3 = 0$.

If the beam is perfectly straight and a string is mounted at tension T along the x axis, then $F = 0$ and there is no equilibrium bending of the beam. However, if the beam bends due to vibration, then the string tension in the x direction gives rise to a perpendicular force component $F = T\partial n/\partial s$, as illustrated in Fig. 2. F acts in a direction to assist rather than oppose the displacement of the beam. During any vibration cycle, as n increases, $\partial n/\partial s$ increases, and T decreases

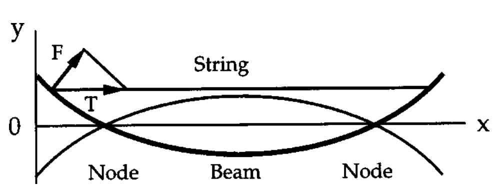

Fig. 2. Fundamental vibration mode of a straight beam tensioned by a string mounted along the axis of the beam.

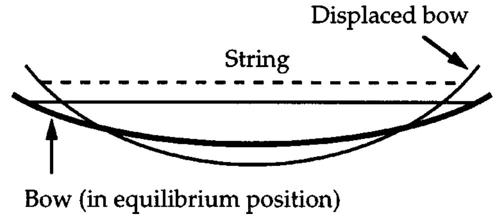

Fig. 3. Fundamental vibration mode of a bow, showing a small displacement of the bow from its equilibrium position.

due to the fact that the string length decreases. If the vibration amplitude is small, and if the beam is straight in the equilibrium position, the variation in T can be neglected in the term $T\partial n/\partial s$. If we let $\partial^2 n/\partial t^2 = -\omega^2 n$ then Eq. (2) can be expressed in the form

$$\frac{\partial^4 n}{\partial s^4} = \left(\frac{m\omega^2}{EIh}\right) n + \left(\frac{T}{EIh}\right)\frac{\partial n}{\partial s}, \tag{3}$$

where T acts on a segment near each end of the beam but $T = 0$ for all other segments. For numerical convenience, the beam was divided into 500 equal segments and the tension was applied at the 5th and 496th segments so that free boundary conditions could be applied to the segments at each end. A finite difference form of Eq. (3) was used to obtain the numerical solutions, as described in Ref. 6. The technique used to obtain a solution was to integrate from $s = 0$ to $s = L$, starting with a small displacement n and with assumed values of both ω and $\partial n/\partial s$. The latter values were iterated until both boundary conditions were satisfied at $s = L$. The boundary conditions at $s = 0$ were used to start the integration.

In the case of a strung bow, the equilibrium shape of the bow is determined by the stiffness of the bow and the string tension. In order to describe small amplitude transverse vibrations of the bow, the external force F in Eq. (2) can be taken as the additional transverse force exerted on the beam by the string, arising from a small displacement of the beam. Such a force arises from (a) the change in length of the string and (b) the change in angle between the string and the bow. The geometry is shown in Fig. 3. Suppose that the string tension decreases from T_1 to T_2 when the attachment point of the string is displaced a distance n in a direction perpendicular to the bow, as in Fig. 4. If the tangent angle of the bow at that point increases from θ_1 to θ_2, then

$$F = F_2 - F_1 = T_2 \sin\theta_2 - T_1 \sin\theta_1 . \tag{4}$$

T_2 is given by

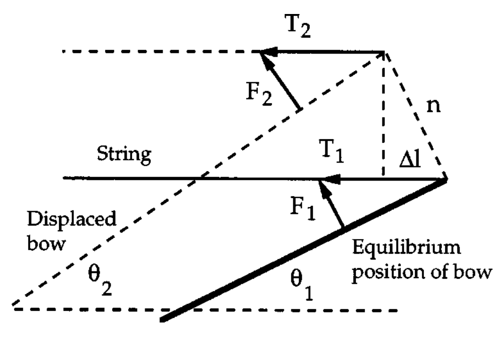

Fig. 4. Displacement of a bow and the string for small amplitude vibrations.

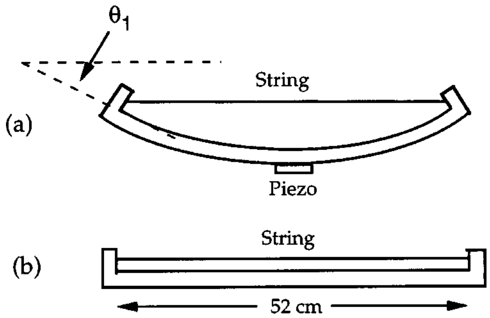

Fig. 5. Experimental bow with (a) $\theta_1>0$ and (b) $\theta_1=0$.

$$T_2=T_1-2k\Delta l, \tag{5}$$

where k is the spring constant of the string and $\Delta l = n \sin \theta_1$ is the decrease in length of the string at each end. When n is small, $\sin(\theta_2-\theta_1)=\partial n/\partial s$, in which case

$$F=T_1 \cos \theta_1 \frac{\partial n}{\partial s}-2kn \sin^2 \theta_1 . \tag{6}$$

For an initially straight beam, $\theta_1=0$ and the transverse force due to the string is given by $F=T_1 \partial n/\partial s$, as described above. For a bow, $k\Delta l$ is typically much smaller than T_1 for small amplitude vibrations, but $T_1 \partial n/\partial s$ is typically much smaller than kn. Consequently, the dominant term in Eq. (6) is normally the term involving k, in which case $F<0$. In that case, both terms on the right-hand side of Eq. (2) are negative, resulting in an increase in the vibration frequency. For a bow with a small but finite value of θ_1, both terms on the right-hand side of Eq. (6) may be similar in magnitude and F may even be zero. If $F=0$ then the vibration frequency of a strung bow is identical to that of the unstrung bow. If the beam is straight in the equilibrium position then $F>0$ and the effect of the string is to reduce the vibration frequency.

III. EXPERIMENTAL RESULTS

In order to test the effect of stringing a beam, a simple experiment was set up as shown in Fig. 5. An aluminum beam was chosen, of length 58 cm, width $b=25.4$ mm, thickness $b=3.0$ mm, and mass $M=118$ g. A 3 cm length at each end was bent at right angles to provide a support to tie a string, giving an effective beam length $L=52$ cm. The fundamental vibration frequency of the beam, without being tensioned by a string, but suspended freely by a length of string, was 48.6 Hz. It was determined by taping a light piezo disk to the middle of the beam, and tapping the beam with a finger to excite beam vibrations. The output from the piezo was fed to a storage oscilloscope to record the transient, high Q, damped oscillations.

The beam was then tensioned using a length of 1.30-mm-diam nylon tennis string, of linear mass density, 1.50 g/m. Under tension, and with a string length of 48 cm, the beam bent into an arc like a bow. As a result, the fundamental vibration frequency of the beam increased to 114.3 Hz. The fundamental vibration frequency of the string was measured separately, by plucking the string, to be 223 Hz.

Measurements at other string tensions were made by plastically deforming the beam to a straighter shape by hand, thereby reducing its curvature and increasing the stretched length of the string. This procedure is relatively easy with a thin aluminum beam, since the beam can be permanently distorted by hand and since a nylon tennis string is relatively easy to stretch. The stiffness of the unstrung beam decreased slightly when the beam was bent into a permanent bow, but this effect was too small to be significant in this experiment. The fundamental vibration frequency of the free, unstrung, straight beam decreased from 48.6 to 46.3 Hz when the beam was permanently deformed into a bow. When the beam was again straightened, the frequency returned to 48.6 Hz.

Table I. Fundamental frequency of a strung, aluminum bow.

θ_1 (deg)	L_s (cm)	T_1 (N)	f_s (Hz)	f_b (Hz)	f (Hz)
25	48.0	66.3	223	114.3	116.8
22	49.0	78.5	240	108.1	106.5
19	49.5	86.1	250	101.3	95.0
17	50.5	114.7	286	85.1	85.2
12	51.0	154.6	330	60.0	56.9
0	52.0	96.9	258	33.3	29.2
0	52.0	0	0	48.6	48.6

The string tension was not measured directly, but it was calculated from the measured vibration frequency of the string, assuming that mass of the string remained fixed, at 0.70 g, when the string was stretched. The angle, θ_1, between the string and the bow decreased when the bow was partially straightened. The string was attached to points 5 mm from the right-angle bends at each end of the bow, as shown in Fig. 5. The angle θ_1 was determined by extrapolating to an intersection point of the string and the bow as illustrated in Fig. 5. A summary of the results obtained by this method is given in Table I, where L_s is the length of the string, f_s is the fundamental vibration frequency of the string, f_b is the observed fundamental vibration frequency of the beam, and f is the theoretically predicted fundamental frequency of the beam. The result at $\theta_1=0$ was obtained by re-stringing the beam a second time at a lower tension, since the original tension was too high and caused the beam to buckle at small θ_1. At sufficiently high tension, a straight beam will become statically unstable against bending. If the right-hand side of Eq. (2) becomes positive, then any small displacement of the beam will grow with time. In practice, buckling ceases when the beam bends far enough to assume a new equilibrium position. This situation is similar to the buckling of a column under compression.[7]

The most interesting results in Table I are that (a) the vibration frequency of the beam increased by a large factor when the beam was strung as a bow, (b) the vibration frequency of the beam decreased back toward the unstrung beam frequency as the beam was straightened, despite the increase in string tension, and (c) the frequency of the strung bow is less than that of the unstrung bow when $\theta_1=0$. These results are consistent with numerical solutions of Eqs. (2), (3), and (6). Solutions obtained for an aluminum beam of length $L=52$ cm, mass $M=118$ g, with $EI=3.09$ kg m^2 are shown in Table I. The value of EI was chosen to fit the data for the unstrung beam. The value of k used in the calculations was 2.1×10^4 N m^{-1}, corresponding to the measured value of the nylon string. A nylon string has a nonlinear stress versus strain curve, but it remains approximately linear at tensions in the range shown in Table I. A nylon tennis string enters the nonlinear region in the normal operating range of tensions from about 250 to 400 N. As shown in

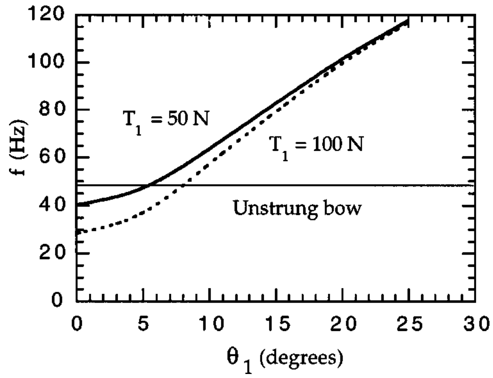

Fig. 6. Numerical solutions showing the vibration frequency of the experimental bow as a function of θ_1 when $T_1 = 50$ N or $T_1 = 100$ N.

Table I, the beam frequency is quite sensitive to θ_1, so that small errors in the measurement of θ_1 would be sufficient to account for the differences between the measured and calculated frequencies.

In Table I, the angle and the tension both change. Figure 6 shows theoretical calculations of the beam frequency as a function of θ_1 at two different tensions. It can be seen that when $T_1 = 50$ N, there is no change in the frequency of the bow if θ_1 is about 6°. At $T_1 = 100$ N, the same result is found at $\theta_1 \sim 8°$. This result follows from Eq. (6), which shows that $F = 0$ at a certain θ_1, where θ_1 depends on the string tension and the stiffness of the string. This effect does not account for the fact that the mode frequencies of a violin frame do not depend significantly on the string tension. The strings are mounted at an angle of about 10° in a violin. Since the strings are relatively stiff and at a relatively low tension, the angle at which $F = 0$ will be less than about 2° in a violin. The relatively small increase in the mode frequencies for a violin can therefore be attributed to the low string tension and the high initial frame stiffness. A calculation for steel strings of length 42 cm, diameter 0.7 mm, and stiffness $k = 2 \times 10^5$ N m^{-1} indicates that the fundamental frequency of a violin might increase from 180 Hz at $T_1 = 0$ to 195 Hz at $T_1 = 200$ N. This is larger than observed, but the geometry is considerably more complex than a simple bow, and is complicated by the additional transverse force applied at the bridge.

Real bows[8] are not as simple as the idealized bow used in this experiment. The shape of most modern bows resembles a single-cycle cos function and the string is mounted almost tangentially at the ends with θ_1 close to zero. Nevertheless, the fundamental vibration frequency increases by a large factor when the bow is strung. This type of bow would require a more sophisticated analysis than presented above to take into account the stress distribution within the bow.

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