Chapter 10

Photons and Neutrinos

In the foregoing development of the theory, it is implicit that the are charged. Uncharged particles may be separated into two classes depending on their properties under under the formal operation that converts particles into antiparticles. One class consists of those uncharged particles that have some property other than charge that changes sign under this operation. An example is the Λ^0 hyperon that has strangeness $S = 1$; its antiparticle $\bar{\Lambda}^0$ has strangeness $S = -1$. Another example is the neutrino, of which there are three flavors (electron, muon and tau), each of which has a specific helicity; antineutrinos are different from neutrinos due to their helicities being opposite. The other class of uncharged particles have no such additional charge-like property that can distinguish the antiparticle from the particle. Such particles are their own antiparticles, and are called strictly neutral particles, or more simply *neutral* particles. Examples of neutral particles include the π^0 meson and the photon.

10.1 Neutral bosons

Neutral bosons obey the Klein Gordon equation with a wavefunction $\Psi(x) = \Psi^*(x)$ that is real. Its Fourier transform

$$\phi(p) = \int d^4x \; \Psi(x) \; e^{ipx} \tag{10.1}$$

therefore satisfies

$$\phi(-p) = \phi^*(p). \tag{10.2}$$

The negative frequencies then contain the same information as the positive frequencies.

In the formal treatment of neutral bosons, the only notable change that needs to be made is the inclusion of a factor $\frac{1}{2}$ in the Lagrangian. The expression (5.5) for charged bosons is replaced by

$$\mathcal{L}(x) = \tfrac{1}{2}\left[(\partial_\mu \Psi)(\partial^\mu \Psi) - m^2\Psi^2\right]. \tag{10.3}$$

This change involves using the fact that Ψ is real, and noting that there is then only one independent function and not two, so that one needs to include the factor $\frac{1}{2}$ to avoid including its effect twice.

10.2 Photons

Quantization of the electromagnetic field in vacuo presents difficulties. The field A^μ is gauge dependent, and for transverse waves in vacuo it must satisfy $\boldsymbol{k} \cdot \boldsymbol{A} = 0$. This implies that only two of the components of A^μ can describe photons. These two transverse components are to be quantized in order to obtain the two transverse states of polarization for photons, and this must involve some procedure for separating them from the other two components (the longitudinal and time components of A^μ) . One procedure for quantizing the field involves introducing fictitious longitudinal and time-like photons, quantizing all four components, and then arranging so that the physical effects of the ficticious photons cancel each other exactly.

One of my research interests is in synthesizing quantum electrodynamic and the kinetic theory of plasmas, and this leads to quite a different attitude to the problem of quantizing the electromagnetic field. Even in the simplest plasma, longitudinal waves are allowed. Examples of longitudinal waves include Langmuir waves and ion sound waves. In less simple plasmas, specifically in anisotropic or magnetized plasmas, the polarization of waves may be intermediate between transverse and longitudinal. One preliminary problem before quantizing the wave field for waves in a plasma is to identify the properties of the waves, which properties include the polarization. One uses the theory of waves in plasmas, which is usually developed in a non-covariant form. A covariant version of the theory of waves in plasmas is summarized in Appendix D.

A formal problem arises: the wave field includes not only the electromagnetic field but also the response of the plasma, which involves induced motions of the particles. Thus, for example, the energy in the waves includes contributions from the electric field, from the magnetic field and from the induced particle motions. The formal problem is that one needs to define the wave subsystem before quantizing it. To overcome this problem I have developed a procedure for quantizing in momentum space, cf. (5.12) *et seq.* All the wave properties are functions of k and one may deduce the appropriate wave Lagrangian by requiring that the corresponding Euler-Lagrange equations reproduce the wave equation in k-space. This procedure is also outlined in Appendix D.

Quanta of Langmuir waves are often called *plasmons*, and other special names are given to quanta of other classes of waves, e.g., phonons, magnons, and so on. Here I shall not normally use such special names, and shall use *photon* in a generic sense to mean a quantum of any wave field. Different classes of waves are referred to as wave *modes*. Thus one has the Langmuir mode, the ion sound mode, the transverse mode, and so on.

There are three basic properties for each specific wave mode.

(i) The condition for the wave equation to have a solution is an algebraic relation involving the components of k, called the *dispersion equation*. A specific solution of the dispersion equation is a *dispersion relation*. Each distinct dispersion relation defines a distinct wave mode (but when one solution is "distinct" from another is not defined, and is sometimes a matter of choice). Thus a dispersion relation is an algebraic relation amongst the components of k. It is usually convenient to express this as a relation for the frequency ω as a function of the components of the 3-vector $\boldsymbol{k}$. This involves solving the dispersion equation with the components of $\boldsymbol{k}$ being the independent variables, and ω being the dependent variable. Suppose a specific wave mode is labeled by a subscript M. Then the dispersion relation may be written in the form $\omega = \omega_M(\boldsymbol{k})$. An alternative formal way of writing a dispersion relation is in terms of the wave 4-vector, specifically $k = k_M$. The relation between these two forms is identified by writing $k_M^\mu = [\omega_M(\boldsymbol{k}), \boldsymbol{k}]$.

(ii) Once a dispersion relation is found, the wave equation may be solved for $A^\mu(k_M)$. The phase, gauge and normalization of $A^\mu(k_M)$ are arbitrary. After making appropriate choices of these, the solution is written as the polarization 4-vector $e_M^\mu(\boldsymbol{k})$. One needs to choose a specific gauge in order to impose a normalization on $e_M^\mu(\boldsymbol{k})$. The most convenient choice of gauge is the temporal gauge; then one has $e_M^\mu(\boldsymbol{k}) = [0, \boldsymbol{e}_M(\boldsymbol{k})]$, and the normalization $\boldsymbol{e}_M(\boldsymbol{k}) \cdot \boldsymbol{e}_M^*(\boldsymbol{k}) = 1$ implies the normalization $e_M^\mu(\boldsymbol{k}) e_{M\mu}(\boldsymbol{k}) = -1$.

(iii) The remaining property is the ratio $R_M(\boldsymbol{k})$ of electric to total energy in the waves. Physically this property is important because the coupling between a charged particle and a wave field is due only to the electric field in the wave. Hence the rate for any process is proportional to $R_M(\boldsymbol{k})$.

10.3 Normalization to one photon

We wish to normalize our wave field so that it corresponds to one photon in the normalization volume. This is achieved as follows.

The 4-potential $A_M^\mu(x)$ for waves in a mode M may be written in the form

$$A_M^\mu(x) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, a_M(\boldsymbol{k}) \big[e_M^\mu(\boldsymbol{k}) e^{-ik_M x} + e^{*\mu}_M(\boldsymbol{k}) e^{ik_M x} \big], \tag{10.4}$$

where the positive and negative solutions are included separately so that $\omega_M(\boldsymbol{k}) > 0$ is implicit. Suppose we use the temporal gauge so that the scalar potential A^0 is zero. Then the electric field is given by $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t$, which may be evaluated using (10.4) simply by noting that $\partial/\partial t$ operating on $e^{\pm i k_M x}$ gives $\pm i\omega_M(\boldsymbol{k}) e^{\pm i k_M x}$. Then integrating the electric energy density $\frac{1}{2}\varepsilon_0 |\boldsymbol{E}|^2$ over all space (and averaging over time) gives the electrical energy in the wave field. This is

$$\int d^3\boldsymbol{x}\, \tfrac{1}{2}\,\varepsilon_0 |\boldsymbol{E}(x)|^2 = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, \varepsilon_0 |\omega_M(\boldsymbol{k}) a_M(\boldsymbol{k})|^2, \tag{10.5}$$

where the normalization condition for the polarization vectors have been used. The electrical energy in the range $V\, d^3\boldsymbol{k}/(2\pi)^3$ may then be identified as $\varepsilon_0 |\omega_M(\boldsymbol{k}) a_M(\boldsymbol{k})|^2$. The electrical energy may be expressed in terms of the total energy $W_M(\boldsymbol{k})$ in this range by writing the electrical energy as $R_M(\boldsymbol{k}) W_M(\boldsymbol{k})$. Thus one has

$$R_M(\boldsymbol{k}) W_M(\boldsymbol{k}) = \varepsilon_0 |\omega_M(\boldsymbol{k}) a_M(\boldsymbol{k})|^2. \tag{10.6}$$

Normalization to one photon then involves identifying the energy $W_M(\boldsymbol{k})$ as being equal to the energy $\hbar\omega_M(\boldsymbol{k})$, where $\hbar$ is included momentarily for clarity, and is now set equal to unity again. Thus normalization to one photon corresponds to

$$|a_M(\boldsymbol{k})| = \left[\frac{\mu_0 R_M(\boldsymbol{k})}{\omega_M(\boldsymbol{k}) V}\right]^{1/2}, \tag{10.7}$$

where $\mu_0 \varepsilon_0 = 1$ (in natural units) is used.

10.4 Specific wave modes

The only waves that considered here are transverse waves in vacuo and in isotopic media and Langmuir waves in an isotropic plasma. The properties of these waves are summarized here. These properties apply only in the rest frame of the medium (except of course for transverse waves in vacuo). The wave properties in any other frame may be obtained by applying a Lorentz transformation.

Transverse waves in an isotropic medium

In an isotropic medium, such as air, water or glass, electromagnetic waves are transverse with a refractive index $\mu = |\boldsymbol{k}|/\omega$ that is different from unity. The dispersion relation may be written in terms of the refractive index $\mu(\omega)$ as a function of ω. The wave properties are

$$\mu(\omega) = |\boldsymbol{k}|/\omega, \quad \boldsymbol{e} \cdot \boldsymbol{\kappa} = 0, \quad R(\omega) = \frac{1}{2\mu(\omega)(\partial/\partial\omega)\big(\omega\mu(\omega)\big)}. \tag{10.8}$$

In this case the dispersion equation is solved with ω and $\boldsymbol{\kappa}$ as the independent variables, and $|\boldsymbol{k}|$ or μ as the dependent variable. Transverse waves in vacuo have $\mu(\omega) = 1$.

Transverse waves have two degenerate states of polarization, and in general it is not possible to describe the polarization in terms of a single polarization vector. In practice one often sums over final states of polarization and averages over initial states of polarization, with the average being half the sum. The sum over states of polarization may be performed as follows. First note that if the two transverse polarization vectors are chosen to correspond to linear polarizations, then the corresponding two polarization vectors form an orthonormal set of basis vector with $\boldsymbol{\kappa}$. By representing the unit tensor δ_{ij} in terms of this basis, one infers the relation

$$\sum e_i e_j^* = \delta_{ij} - \boldsymbol{\kappa}_i \boldsymbol{\kappa}_j, \tag{10.9}$$

where the sum is over the two linear polarization vectors. One then notes that the sum on the left applies to any two orthonormal vectors that span the 2-dimensional space orthogonal to $\boldsymbol{\kappa}$, so that (10.9) is completely general.

Waves in an isotropic plasma

A simple model for a plasma is that of an isotropic thermal plasma. There are three relevant plasma parameters in this model: the electron number density n_e, the electron temperature T_e and the ion mass m_i. The electron plasma frequency, usually called simply the *plasma frequency* ω_p, and the ion plasma frequency ω_{pi} are defined by

$$\omega_p = \left(\frac{e^2 n_e}{\varepsilon_0 m_e}\right)^{1/2}, \quad \omega_{pi} = \left(\frac{e^2 n_i}{\varepsilon_0 m_i}\right)^{1/2}, \tag{10.10}$$

with $n_i = n_e$. Other relevant plasma parameters are *electron thermal speed* V_e, the *Debye length* λ_{De}, and the *ion sound speed* v_s:

$$V_e = \left(\frac{T_e}{m_e}\right)^{1/2}, \quad \lambda_{De} = \frac{V_e}{\omega_p}, \quad v_s = \omega_{pi}\lambda_{De} = \left(\frac{m_e}{m_i}\right)^{1/2} V_e, \tag{10.11}$$

where we use units in which Boltzmann's constant is set equal to unity.

There are three wave modes in such a plasma. The transverse mode has the properties summarized in (10.8) with

$$\mu(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}. \tag{10.12}$$

Note that in this case one has $R(\omega) = \frac{1}{2}$, and that transverse wave exist only for $\omega > \omega_p$. The dispersion relation (10.12) can be written in the standard form $\omega = \omega_T(\boldsymbol{k})$ with

$$\omega_T(\boldsymbol{k}) = [\omega_p^2 + \boldsymbol{k}^2 c^2]^{1/2}, \tag{10.13}$$

where $M = T$ denotes the transverse mode.

The other modes are longitudinal with polarization vector

$$\boldsymbol{e} = \boldsymbol{\kappa}. \tag{10.14}$$

One is the Langmuir mode ($M = L$) with dispersion relation $\omega = \omega_L(\boldsymbol{k})$, with

$$\omega_L(\boldsymbol{k}) = \omega_p + \frac{3\boldsymbol{k}^2 V_e^2}{2\omega_p}, \qquad R_L(\boldsymbol{k}) = \frac{\omega_p^2}{2\omega_L^2(\boldsymbol{k})}. \tag{10.15}$$

The Langmuir mode exists only for $|\boldsymbol{k}|\lambda_{De} \ll 1$. The other longitudinal mode is the ion sound mode $M = S$ with

$$\omega_S(\boldsymbol{k}) = \frac{|\boldsymbol{k}| v_s}{\sqrt{1 + \boldsymbol{k}^2 \lambda_{De}^2}}, \qquad R_S(\boldsymbol{k}) = \frac{\omega_S^2(\boldsymbol{k})}{2\omega_{pi}^2}. \tag{10.16}$$

The ion sound mode exists only at $\omega < \omega_{pi}$. Although these properties are independent of the temperature of the ions, in fact the mode exists only if the ions are much cooler than the electrons.

10.5 Neutrinos

Neutrinos are massless spin-$\frac{1}{2}$ particles. For $m = 0$ the covariant form of Dirac's equation (3.5) reduces to

$$i\partial\!\!\!/\Psi(x) = 0, \quad \text{or} \quad \hat{p\!\!\!/}\Psi(x) = 0. \tag{10.17}$$

A plane wave solution

$$\Psi(x) = u(p)\, e^{-ipx} \tag{10.18}$$

satisfies

$$\hat{p}^\mu \Psi(x) = p^\mu \Psi(x). \tag{10.19}$$

The plane wave solutions (10.18) are eigenfunctions of the 4-momentum operator by construction. To construct spin eigenfunctions we need to introduce an appropriate spin operator. A suitable operator is the *helicity* operator $\hat{W}^\mu$ which is introduced in (C.13) of Appendix C:

$$\hat{W}^\mu = -\tfrac{1}{2}\epsilon^{\mu\alpha\beta\gamma}\hat{S}_{\alpha\beta}\hat{p}_\gamma, \tag{10.20}$$

where

$$\hat{S}^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta]. \tag{10.21}$$

is the spin 4-tensor in the Dirac theory. The helicity operator $\hat{W}^\mu$ satisfies the identities

$$\hat{W}^\mu = \frac{1}{4}[\gamma^\mu, \hat{\not{p}}]\gamma^5, \quad \hat{W}^\mu\gamma^5 = \frac{1}{4}[\gamma^\mu, \hat{\not{p}}], \tag{10.22}$$

which enable one to derive eigenfunctions of $\hat{p}^\mu$ and $\hat{W}^\mu$ as follows.

Using (10.17), (10.19) and (10.22) one finds

$$\hat{W}^\mu\Psi(x) = -\tfrac{1}{2}p^\mu\gamma^5\Psi(x), \qquad \hat{W}^\mu\gamma^5\Psi(x) = -\tfrac{1}{2}p^\mu\Psi(x). \tag{10.23}$$

It follows that

$$\Psi_L(x) = \tfrac{1}{2}(1+\gamma^5)\Psi(x), \qquad \Psi_R(x) = \tfrac{1}{2}(1-\gamma^5)\Psi(x) \tag{10.24}$$

are simultaneous eigenfunctions of $\hat{p}^\mu$ and $\hat{W}^\mu$ with eigenvalues $-\frac{1}{2}p^\mu$ and $\frac{1}{2}p^\mu$ of $\hat{W}^\mu$, respectively. These are the helicity eigenfunctions. The operators $\frac{1}{2}(1 \pm \gamma^5)$ may be regarded as projection operators. They satisfy

$$[\tfrac{1}{2}(1 \pm \gamma^5)]^2 = \tfrac{1}{2}(1 \pm \gamma^5), \tag{10.25}$$

and project onto the two helicity states. The Dirac conjugates of (10.24) are

$$\overline{\Psi}_L(x) = \tfrac{1}{2}\overline{\Psi}(x)(1-\gamma^5), \qquad \overline{\Psi}_R(x) = \tfrac{1}{2}\overline{\Psi}(x)(1+\gamma^5). \tag{10.26}$$

It is found that neutrinos are left handed, corresponding to $\Psi_L(x)$, and that antineutrinos are right handed, corresponding to $\Psi_R(x)$.