Chapter 2

4-Tensor Notation

The 4-tensor notation used here has greek indices running over $0, 1, 2, 3$ or t, x, y, z. The signature is -2, i.e., the metric tensor is diagonal with components $1, -1, -1, -1$. Where appropriate latin indices are used to denote the space components $1, 2, 3$ or x, y, z. To introduce this notation in a formal way, let us define what is meant by a 4-tensor equation.

2.1 4-Tensor Equations

A 4-tensor equation involves elements which are either kernel symbols, or products of kernel symbols, with each symbol having zero, one or more indices. The indices are written in space (one space per index) after the kernel symbol, and any index may be either raised (i.e., as a superscript), denoting a *contravariant component*, or lowered (i.e., as a subscript), denoting a *covariant component.* The indices may have affices, e.g., primes or numerical or other subscripts, and two indices are the same only if they have the same affices. In each element of a 4-tensor equation, an index occurs only either once, when it is called a *free index*, or twice, when it is called a *dummy index.* Each pair of dummy indices must consist of one raised and one lowered index. The *summation convention* is that the sum (from 0 to 3) over each pair of dummy indices is implied.

Each kernel symbol may be regarded as describing a tensor. The *rank* of a tensor is defined as the number of its free indices; 4-tensors of rank zero are called invariants and 4-tensors of rank one are called 4-vectors. Similarly the rank of a tensor equation is equal to the number of the free indices in each of its elements. The number and kind of free indices must be the same in all elements of a tensor equation.

There are three elementary manipulations which may be performed on any tensor equation: raising or lowering an index, relabeling indices and contracting over two indices. A lowered index ν may be converted into a raised index μ by using the contravariant form $g^{\mu\nu}$ of the metric tensor, and a raised index ν may be converted into a lowered index μ by using the covariant form $g_{\mu\nu}$ of the metric tensor. For a 4-vector a, these operations are

$$a^\mu = g^{\mu\nu} a_\nu, \qquad a_\mu = g_{\mu\nu} a^\nu. \tag{2.1}$$

One may relabel any free index, provided the relabeling is made in every element of the tensor equation. One may relabel any pair of dummy indices, and also interchange the raised and lowered index. A *contraction* may be performed on any tensor equation of rank two or higher. It involves converting two free indices into a pair of dummy indices, thereby reducing the rank of the equation by two.

It is convenient to write a 4-vector a in terms of its time component a^0 and its space components in the form of a 3-vector $\mathbf{a}$:

$$a^\mu = [a^0, \mathbf{a}], \qquad a_\mu = [a^0, -\mathbf{a}] \tag{2.2}$$

Note that the three Cartesian components of the 3-vector $\mathbf{a}$ are identified with the contravariant space components a^1, a^2, a^3 of the 4-vector, and that the covariant space components a_1, a_2, a_3 of the 4-vector are equal to minus the Cartesian components of $\mathbf{a}$.

2.2 Important 4-Vectors

The basic 4-vector is a space-time point $x^\mu = [t, \mathbf{x}]$, also called an *event*. (Note that natural units with $c = 1$ are used here, so that the basic unit of distance is a light second.) The following 4-vectors appear frequently below:

$$\text{event:} \qquad x^\mu = [ct, \mathbf{x}], \tag{2.3}$$
$$\text{4-velocity:} \qquad u^\mu = [\gamma, \gamma\boldsymbol{\beta}], \tag{2.4}$$
$$\text{4-momentum:} \qquad p^\mu = [\varepsilon/c, \mathbf{p}], \tag{2.5}$$
$$\text{wave 4-vector:} \qquad k^\mu = [\omega/c, \mathbf{k}], \tag{2.6}$$
$$\text{4-current density:} \qquad J^\mu = [\rho c, \mathbf{J}], \tag{2.7}$$
$$\text{4-potential:} \qquad A^\mu = [\phi/c, \mathbf{A}]. \tag{2.8}$$

where $\mathbf{v}$ is the 3-velocity, $\gamma = (1 - v^2)^{-1/2}$ is the *Lorentz factor*, $\varepsilon = \gamma m$ is the energy, $\mathbf{p}$ is the 3-momentum, ω is the frequency, $\mathbf{k}$ is the wave vector, ρ is the charge density, $\mathbf{J}$ is the 3-current density, ϕ is the electric potential and $\mathbf{A}$ is the vector potential. Another important 4-vector quantity is the operator

$$\text{4-gradient:} \qquad \partial_\mu := \partial/\partial x^\mu = [\partial/\partial t, \partial/\partial \mathbf{x}], \tag{2.9}$$

which appears naturally in terms of covariant components. (The symbol ":=" denotes a definition.)

The invariant formed from two 4-vectors a and b is denoted ab:

$$ab := a^\mu b_\mu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}. \tag{2.10}$$

Similarly, the invariant formed from a single 4-vector a is $a^2 = (a^0)^2 - \mathbf{a}^2$.

2.3 Lorentz Transformations

The 4-tensor character of a physical quantity is defined in terms of its transformation properties under Lorentz transformations. Let K and K' be two inertial frames. An event in K is described by x^μ and the same event is described in K' by $x^{\mu'}$. These components are related by an equation of the form

$$x^{\mu'} = a^{\mu'} + L^{\mu'}{}_\mu \, x^\mu. \tag{2.11}$$

The constant 4-vector $a^{\mu'}$ relates the origins in space and time in K and K'; it corresponds to a generalization of a translational motionfrom three to four dimensions. The other term in (2.1) corresponds to a generalization of a rotation from three to four dimensions. In most cases below, a Lorentz transformation refers only to the rotational part. The translational part is of interest here only for some formal purposes, and is otherwise neglected.

The coefficients $L^{\mu'}{}_\nu$ may be written in matrix form. Note however, that they are not tensors: they have two indices but these two indices refere to different frames of reference. Irrespective of whether the indices are raised or lowered, the matrix convention is that the first-written index labels rows and the second-written index labels columns. In order to preserve the invariant a^2 the determinant of this matrix must be equal to unity to within a sign. For *proper* Lorentz transformations this sign is positive; improper Lorentz transformations involve either reflection of a coordinate axis (parity transformation) or of the time axis (time reversal transformation). The inverse of the transformation matrix $L^{\mu'}{}_\mu$ is is written $L^\mu{}_{\mu'}$. Then for any 4-vector a^μ the transformation properties of the contravariant ant covariant components are

$$a^{\mu'} = L^{\mu'}{}_\mu \, a^\mu, \qquad a^\mu = L^\mu{}_{\mu'} \, a^{\mu'}, \tag{2.12}$$

$$a_{\mu'} = L^\mu{}_{\mu'} \, a_\mu \qquad a_\mu = L^{\mu'}{}_\mu \, a_{\mu'}. \tag{2.13}$$

The transformation matrices satisfy

$$L^\mu{}_{\mu'} \, L^{\mu'}{}_\nu = \delta^\mu{}_\nu, \qquad L^{\mu'}{}_\mu \, L^\mu{}_{\nu'} = \delta^{\mu'}{}_{\nu'}, \tag{2.14}$$

where

$$\delta^\mu{}_\nu = \begin{cases} 1 & \text{for } \mu = \nu, \\ 0 & \text{for } \mu \neq \nu. \end{cases} \tag{2.15}$$

is the unit 4-tensor. (Note that it is conventional to use $\delta^\mu{}_\nu$ rather than the mixed components $g^\mu{}_\nu$ of the metric tensor.)

An equation in 4-tensor form is said to be in a *manifestly covariant* form. This means that the form is obviously unchanged under a Lorentz transformation, so that the equation manifestly satisfies the requirement of the special theory of relativity. Under a transformation from frame K to frame K', a tensor equation transforms simply by adding primes to all the free indices.

In the case in which the axes in K and K' are parallel, and K' is moving along the z-axis of K at velocity $\boldsymbol{\beta}$, the explicit forms for the transformation matrices are

$$L^{\mu'}{}_\mu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \qquad L^\mu{}_{\mu'} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{2.16}$$

with $\gamma = (1-\beta^2)^{-1/2}$.

2.4 Covariant Form of Maxwell's Equations

Maxwell's equations may be written in the covariant form

$$\partial^\mu F^{\nu\rho}(x) + \partial^\rho F^{\mu\nu}(x) + \partial^\nu F^{\rho\mu}(x) = 0, \tag{2.17}$$

$$\partial_\mu F^{\mu\nu}(x) = \mu_0 J^\nu(x), \tag{2.18}$$

where $F^{\mu\nu}(x)$ is the Maxwell tensor, and where the argument x denotes $(t, \mathbf{x})$. The Maxwell tensor is antisymmetric

$$F^{\mu\nu}(x) = -F^{\nu\mu}(x), \tag{2.19}$$

and is related to the Cartesian components of the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ by

$$F^{\mu\nu}(x) := [-\mathbf{E}/c, \mathbf{B}] = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{2.20}$$

(The Cartesian components of the 3-vectors are denoted by x, y, z rather than $1, 2, 3$ to emphasize that relations such as (2.20) are between two different notations; such relations are prescriptions for relating the quantities in the two different notations.) The first of the pair of Maxwell's equations may be written more concisely in terms of the dual of the Maxwell tensor. The dual of any second rank tensor $T^{\mu\nu}$ is defined by

$$T^{D\mu\nu} = \tfrac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,T_{\alpha\beta}, \tag{2.21}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the permutation symbol:

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{for } \alpha\beta\gamma\delta \text{ an even permutation of } 0123, \\ -1 & \text{for } \alpha\beta\gamma\delta \text{ an odd permutation of } 0123, \\ 0 & \text{otherwise.} \end{cases} \tag{2.22}$$

Note that the permutation symbol with lowered indices ϵ_{0123} is numerically equal to minus ϵ^{0123} (proof: $\epsilon^{0123} = g^{00}g^{11}g^{22}g^{33}\epsilon_{0123}$, and $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$). Many authors choose the sign convention $\epsilon_{0123} = 1$, which is opposite to that chosen here.

The dual of the Maxwell tensor is

$$F^{D\mu\nu}(x) = [-\mathbf{B}, -\mathbf{E}/c] = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}. \tag{2.23}$$

Then (2.17) may be replaced by

$$\partial_\mu F^{D\mu\nu}(x) = 0. \tag{2.24}$$

There are two independent invariants that may be constructed from the Maxwell tensor. These are

$$F^{\mu\nu} F_{\mu\nu} = -2\,(\mathbf{E}^2/c^2 - \mathbf{B}^2), \tag{2.25}$$

and

$$F^{\mu\nu} F^D{}_{\mu\nu} = -4\,\mathbf{E}\cdot\mathbf{B}/c. \tag{2.26}$$

2.5 Continuity Equations for Charge and Energy

On operating on (2.18) with ∂_ν, the antisymmetry property (2.19) implies the equation of charge continuity

$$\partial_\mu J^\mu(x) = 0. \tag{2.27}$$

There is another continuity equation that may be derived directly from Maxwell's equations. This is

$$\partial_\mu \Theta^{\mu\nu}(x) = J_\alpha(x) F^{\alpha\nu}(x), \tag{2.28}$$

where

$$\Theta^{\mu\nu}(x) = \varepsilon_0 (F^\mu{}_\alpha F^{\alpha\nu} + \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}) \tag{2.29}$$

is the symmetric energy-momentum tensor. (Note that we have $\mu_0\varepsilon_0 = 1$ in natural units so that either ε_0 or $1/\mu_0$ may be used interchangeably.) On interpreting (2.28) as a continuity equation for energy, one identifies the energy density, W, momentum density, $\mathbf{P}$, energy flux, $\mathbf{F}$ and the stress 3-tensor, $\mathbf{T}$, for the electromagnetic field in vacuo:

$$W = \varepsilon_0 \mathbf{E}^2/2 + \mathbf{B}^2/2\mu_0, \qquad \mathbf{F} = \mathbf{E} \times \mathbf{B}/\mu_0,$$
$$\mathbf{P} = \varepsilon_0 \mathbf{E} \times \mathbf{B}, \qquad \mathbf{T} = W\mathbf{1} - \varepsilon_0 \mathbf{E}\mathbf{E} - \mathbf{B}\mathbf{B}/\mu_0, \tag{2.30}$$

where $\mathbf{1}$ denotes the unit 3-tensor.

2.6 Gauge Transformations

Equation (2.17) may be satisfied exactly by expressing $F^{\mu\nu}$ in terms of the 4-potential A^μ:

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \tag{2.31}$$

The choice $A(x)$ of 4-potential is not unique, and any other choice $A'(x)$ related to $A(x)$ by a gauge transformation is equally acceptable. A gauge transformation is of the form

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \psi(x), \tag{2.32}$$

where $\psi(x)$ is an arbitrary differentiable function.

The freedom to make gauge transformations allows one to impose a gauge condition. All relevant gauge conditions are of the form

$$\hat{G}_\alpha \, A^\alpha(x) = 0, \tag{2.33}$$

where $\hat{G}_\alpha$ is a differential operator in general. Specific gauge conditions correspond to

$$\text{Lorenz gauge:} \qquad \hat{G}^{(\mathrm{Lor})}_\alpha = \partial_\alpha, \tag{2.34}$$
$$\text{Coulomb gauge:} \qquad \hat{G}^{(\mathrm{C})}_\alpha = [0, \partial/\partial\mathbf{x}], \tag{2.35}$$
$$\text{temporal gauge:} \qquad \hat{G}^{(\mathrm{t})}_\alpha = [1, \mathbf{0}]. \tag{2.36}$$

It is desirable to make the general theory independent of the choice of gauge as far as is possible. An equation that maintains its form under a gauge transformation is said to be in a *manifestly gauge-independent* form. It is desirable to develop a form of QED that is manifestly gauge-independent, as well as being manifestly covariant.