

Chapter 5

Field Lagrangians

A Lagrangian formulation of field theory allows one to derive certain properties, such as the energy momentum tensor and the 4-current, in a systematic manner. The specific form of the Lagrangian is to be chosen such that the Euler-Lagrangian equations reproduce the field equations, that is, the Klein Gordon equation or the Dirac equation in the present context.

5.1 Euler-Lagrange equations for a field

The usual definition of a Lagrangian for a field, $\Psi(x)$ say, involves regarding Ψ and Ψ^* as independent “generalized coordinates”, and $\partial\Psi$ and $\partial\Psi^*$ as the corresponding “generalized momenta”. The Lagrangian density $\mathcal{L}(\Psi, \Psi^*, \partial\Psi, \partial\Psi^*)$ is defined such that the action I is given by

$$I = \int d^4x \mathcal{L}(\Psi, \Psi^*, \partial\Psi, \partial\Psi^*). \quad (5.1)$$

The Euler-Lagrange equations are then

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^*)} - \frac{\partial \mathcal{L}}{\partial \Psi^*} = 0, \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} - \frac{\partial \mathcal{L}}{\partial \Psi} = 0. \quad (5.2)$$

The Lagrangian density is to be chosen so that these equations reproduce the required field equations.

The energy momentum tensor for a field is determined by a procedure that is analogous to the derivation of a Hamiltonian from a Lagrangian. Specifically, one has $H = \Sigma p \dot{q} - L$ with L replaced by \mathcal{L} , the qs replaced by Ψ and Ψ^* , the corresponding $\dot{q}s$ replaced by $\partial_0 \Psi$ and $\partial_0 \Psi^*$, the corresponding ps replaced by $\partial \mathcal{L} / \partial(\partial_0 \Psi)$ and $\partial \mathcal{L} / \partial(\partial_0 \Psi^*)$, and H replaced by T^{00} . The time derivatives are replaced by covariant derivatives ($\partial^0 \rightarrow \partial^\mu$), leading to the following prescription for the identification of the energy-momentum tensor:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^*)} \partial^\nu \Psi^* + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \partial^\nu \Psi - g^{\mu\nu} \mathcal{L}. \quad (5.3)$$

The energy in the field is the spatial integral of T^{00} :

$$\int d^3\mathbf{x} T^{00} = \int d^3\mathbf{x} \left[\frac{\partial \mathcal{L}}{\partial(\partial\Psi^*/\partial t)} \frac{\partial\Psi^*}{\partial t} + \frac{\partial \mathcal{L}}{\partial(\partial\Psi/\partial t)} \frac{\partial\Psi}{\partial t} - \mathcal{L} \right]. \quad (5.4)$$

The Lagrangian may be modified to determine the 4-current by including an electromagnetic field. This involves making the minimal coupling replacement (??), *viz.* $\partial^\mu \rightarrow \partial^\mu + iqA^\mu(x)$. The resulting term in the Lagrangian that is proportional to $A^\mu(x)$ is written as $J_\mu(x)A^\mu(x)$, with $J^\mu(x)$ identified as the 4-current.

5.2 Klein Gordon Lagrangian

The Klein Gordon Lagrangian is to be chosen such that the Euler-Lagrange equations (5.2) reproduce the Klein Gordon equation (??) and its hermitian conjugate, respectively. A choice of Lagrangian that achieves this is

$$\mathcal{L}(x) = (\partial_\mu\Psi^*)(\partial^\mu\Psi) - m^2\Psi^*\Psi. \quad (5.5)$$

Given the form of the Lagrangian (5.5), the energy momentum tensor may be constructed using (5.3). For the Klein Gordon field one finds

$$T^{\mu\nu} = (\partial^\mu\Psi^*)(\partial^\nu\Psi) + (\partial^\nu\Psi^*)(\partial^\mu\Psi) - g^{\mu\nu}\mathcal{L}, \quad (5.6)$$

which is used in (??). The 4-current is identified as the coefficient of the terms proportional to A in $\mathcal{L} \rightarrow [(\partial_\mu - iqA_\mu)\Psi^*][(\partial^\mu + iqA^\mu)\Psi] - m^2\Psi^*\Psi$. This prescription gives

$$J^\mu(x) = iq[\Psi^*(\partial^\mu\Psi) - (\partial^\mu\Psi^*)\Psi]. \quad (5.7)$$

The continuity relation $\partial_\mu J^\mu(x) = 0$ is implied by the Klein Gordon equation.

5.3 Dirac Lagrangian

For the Dirac equation it is convenient to choose the independent fields as Ψ and $\bar{\Psi}$. Then the Euler-Lagrange equations (5.4) are replaced by

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} - \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = 0, \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} - \frac{\partial \mathcal{L}}{\partial \Psi} = 0. \quad (5.8)$$

A choice of Lagrangian such that (5.8) reproduces the Dirac equation (??) and adjoint is

$$\mathcal{L} = \frac{1}{2}i \left(\bar{\Psi}\gamma^\mu\partial_\mu\Psi - (\partial_\mu\bar{\Psi})\gamma^\mu\Psi \right) - m\bar{\Psi}\Psi. \quad (5.9)$$

The energy momentum tensor for the Dirac field is

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})} \partial^\nu \bar{\Psi} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} \partial^\nu \Psi - g^{\mu\nu} \mathcal{L} \\ &= -\frac{1}{2}i(\partial^\nu \bar{\Psi})\gamma^\mu \Psi + \frac{1}{2}i\bar{\Psi}\gamma^\mu \partial^\nu \Psi, \end{aligned} \quad (5.10)$$

which is used in (??). The 4-current is

$$J^\mu = q\bar{\Psi}\gamma^\mu\Psi. \quad (5.11)$$

5.4 Lagrangian in momentum space

For some formal purposes it is useful to define a Lagrangian density in momentum space. This is achieved by writing the space time average of the action integral in terms of an integral over momentum space. One has

$$\frac{I}{VT} = \frac{1}{VT} \int d^4x \mathcal{L}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathcal{L}(p). \quad (5.12)$$

The Lagrangian density $\mathcal{L}(x)$ is bilinear in $\Psi(x)$ and its adjoint, and the integral over it in (5.12) may be written in terms of Fourier transforms by using the power theorem (A.4). The Fourier transforms of the wavefunctions are written in terms of $\varphi(P)$ using (5.5), specifically

$$\Phi(P) = \sum_{\epsilon=\pm} \Phi^\epsilon(\epsilon\mathbf{p}) (2\pi)^4 \delta^4(P - \epsilon p), \quad (5.13)$$

and the square of the δ function is rewritten using (A.9). This allows one to identify $\mathcal{L}(P)$.

For the Klein Gordon field the expression (5.5) for $\mathcal{L}(x)$ implies

$$\mathcal{L}(P) = \sum_{\epsilon=\pm} (p^2 - m^2) \varphi^{\epsilon*}(\epsilon\mathbf{p}) \varphi^\epsilon(\epsilon\mathbf{p}) (2\pi)^3 \delta^4(\mathbf{P} - \epsilon\mathbf{p}), \quad (5.14)$$

with $P^0 = \epsilon\varepsilon$. The independent variables may be identified as $\varphi^{\epsilon*}(\mathbf{p})$, $\varphi^\epsilon(\mathbf{p})$ and the phase. Variation of $\mathcal{L}(p)$ with respect to φ^* and φ leads to the wave equation in momentum space $(p^2 - m^2)\varphi^\epsilon(\mathbf{p}) = 0$ and its adjoint respectively.

For the Dirac field the expression (5.9) for $\mathcal{L}(x)$ implies

$$\mathcal{L}(P) = \sum_{\epsilon=\pm} \bar{\varphi}^\epsilon(\mathbf{p})(\not{P} - m)\varphi^\epsilon(\mathbf{p}) (2\pi)^3 \delta^3(\mathbf{P} - \epsilon\mathbf{p}), \quad (5.15)$$

with $P^0 = \epsilon\varepsilon$. The independent variables may be identified as $\bar{\varphi}^\epsilon(\mathbf{p})$, $\varphi^\epsilon(\mathbf{p})$ and the phase. Variation of $\mathcal{L}(P)$ with respect to φ^* and φ leads to the wave equation in momentum space $(\not{p} - m)\varphi^\epsilon(\mathbf{p}) = 0$ and its adjoint respectively.

5.5 Particle action and occupation numbers

The occupation numbers for particles may be derived from the Lagrangian in momentum space. The action is the variable conjugate to the phase, and on dividing the action by \hbar (with $\hbar = 1$ here), the action is reinterpreted in terms of the occupation number, at least to within a sign. The essential idea is to regard $\mathcal{L}(p)$ as a function of the phase ψ of the field, with

$$p^\mu = \partial^\mu \psi. \quad (5.16)$$

The Lagrangian depends on $\partial\psi$ but not on ψ itself, implying the conservation law

$$\partial_\mu \left(\frac{\partial \mathcal{L}(\partial\psi)}{\partial(\partial_\mu \psi)} \right) = 0. \quad (5.17)$$

The conserved quantity is the action

$$\tilde{n}(\mathbf{P}) = \frac{\partial \mathcal{L}(P)}{\partial p^0}, \quad (5.18)$$

which, apart from a possible sign for antiparticles, is also the occupation number.

Let the occupation number be written as $n^\epsilon(\mathbf{p})$, with $\epsilon = 1$ for particles and with $\epsilon = -1$ for antiparticles. It is desirable to define the occupation numbers to be non-negative. Inspection of (5.14) and (5.15) shows that the particle action (5.18) is an odd function of P^0 for spin 0 and an even function of P^0 for spin $\frac{1}{2}$. Consequently for $P = \epsilon p$ the particle action is related the occupation numbers are identified as

$$n^\epsilon(\mathbf{p}) = \begin{cases} \epsilon \tilde{n}(\epsilon \mathbf{p}) & \text{for spin 0,} \\ \tilde{n}(\epsilon \mathbf{p}) & \text{for spin } \frac{1}{2}. \end{cases} \quad (5.19)$$

In this way the occupation numbers are identified as

$$n^\epsilon(\mathbf{p}) = \varphi^{\epsilon*}(\mathbf{P}) \varphi^\epsilon(\mathbf{P}) (2\pi)^3 \delta^3(\mathbf{P} - \epsilon \mathbf{p}), \quad (5.20)$$

for spin 0, and as

$$n_s^\epsilon(\mathbf{p}) = \varphi_s^{\epsilon\dagger}(\mathbf{P}) \varphi_s^\epsilon(\mathbf{P}) (2\pi)^3 \delta^3(\mathbf{P} - \epsilon \mathbf{p}), \quad (5.21)$$

for spin $\frac{1}{2}$.

The normalization condition to one particle in the normalization volume may be imposed trivially in terms of the occupation numbers. One requires

$$V \sum_s \int \frac{d^3 \mathbf{p}}{(2\pi)^3} n_s^\epsilon(\mathbf{p}) = 1, \quad (5.22)$$

where the sum over s is not relevant only for spin 0. On inserting the expressions (5.20) and (5.21) in (5.22) one finds

$$\varphi_s^{\epsilon*}(\epsilon\mathbf{p})\varphi_s^\epsilon(\epsilon\mathbf{p}) = \frac{1}{2\varepsilon V}. \quad (5.23)$$

for spin 0, and

$$\varphi_s^{\epsilon\dagger}(\epsilon\mathbf{p})\varphi_s^\epsilon(\epsilon\mathbf{p}) = \frac{1}{V}, \quad (5.24)$$

for spin $\frac{1}{2}$, respectively. The results (5.23) and (5.24) reproduce (??) and (??), respectively.