

Chapter 6

Motion in a Magnetostatic Field

One can solve Dirac's equation exactly in the presence of a magnetostatic field \mathbf{B} . The form of the solution depends on a choice of gauge for the vector potential $\mathbf{A}(\mathbf{x})$ describing the field.

6.1 Dirac equation in the Landau gauge

When an electromagnetic field with 4-potential A^μ is included, Dirac's equation is modified by the minimal coupling procedure of replacing \hat{p}^μ by $\hat{p}^\mu - q\hat{A}^\mu$, where q is the charge on the particle. One has $q = -e$ for an electron. The Dirac Hamiltonian is then replaced by

$$\hat{H} = \boldsymbol{\alpha} \cdot (\mathbf{p} + e\mathbf{A}) + \beta m - e\phi. \quad (6.1)$$

Here we set $\phi = 0$.

Let the magnetostatic field be along the z -axis. One choice of gauge is

$$\mathbf{A} = (0, Bx, 0), \quad (6.2)$$

which is called the *Landau gauge*. Other choices of gauge are related to this by adding the gradient of a scalar to \mathbf{A} . Other choices include

$$\mathbf{A} = (-By, 0, 0), \quad (6.3)$$

$$\mathbf{A} = \frac{1}{2}(-By, Bx, 0) = \frac{1}{2}B\varpi(-\sin\varphi, \cos\varphi, 0), \quad (6.4)$$

with $\varpi = (x^2 + y^2)^{1/2}$ and $x = \varpi \cos\varphi$, $y = \varpi \sin\varphi$.

An important point to note is that with the choice (6.2), on writing $\hat{p} = -i \text{grad}$, Dirac's equation depends on only one coordinate, namely x . The other coordinates y and z are *ignorable* or *cyclic*. The corresponding momenta p_y and p_z are then constants of the motion, and may be chosen as (continuous) quantum numbers. Also the Hamiltonian does not depend on time and so the energy is also a constant of the motion. Dirac's equation

may then be reduced to an ordinary differential equation in x . Similarly with the other two choices of gauge (6.3) and (6.4) Dirac's equation may be reduced to an ordinary differential equation, in y and ϖ respectively. The latter case is treated in Appendix B.

6.2 Construction of the wave functions

Let us assume a trial wave function of the form

$$\Psi(t, \mathbf{x}) = f(x) \exp(-i\epsilon\epsilon t + i\epsilon p_y y + i\epsilon p_z z), \quad (6.5)$$

where $\epsilon = \pm$ is the sign of the energy whose magnitude is ϵ . The wave function Ψ is a column matrix, and so is $f(x)$. The components of the column matrix are denoted by $f_1 \dots f_4$.

On inserting the trial solution (6.5) into Dirac's equation in the form

$$\left(i \frac{\partial}{\partial t} - \hat{H}\right)\Psi(t, \mathbf{x}) = 0, \quad (6.6)$$

one requires

$$\begin{pmatrix} -\epsilon\epsilon + m & 0 & \epsilon p_z & \hat{O}_1 \\ 0 & -\epsilon\epsilon + m & \hat{O}_2 & -\epsilon p_z \\ \epsilon p_z & \hat{O}_1 & -\epsilon\epsilon - m & 0 \\ \hat{O}_2 & -\epsilon p_z & 0 & -\epsilon\epsilon - m \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{pmatrix} = 0, \quad (6.7)$$

with

$$\hat{O}_1 = -i \left(\frac{\partial}{\partial x} + \epsilon p_y + eBx \right), \quad \hat{O}_2 = -i \left(\frac{\partial}{\partial x} - \epsilon p_y - eBx \right). \quad (6.8)$$

It is convenient to write

$$\xi = (eB)^{1/2} \left(x + \frac{\epsilon p_y}{eB} \right), \quad (6.9)$$

and then (6.7) reduces to

$$\begin{aligned} (-\epsilon\epsilon + m)f_1 + \epsilon p_z f_3 - i(eB)^{1/2}(\xi + d/d\xi)f_4 &= 0, \\ (-\epsilon\epsilon + m)f_2 - \epsilon p_z f_4 + i(eB)^{1/2}(\xi - d/d\xi)f_3 &= 0, \\ (-\epsilon\epsilon - m)f_3 + \epsilon p_z f_1 - i(eB)^{1/2}(\xi + d/d\xi)f_2 &= 0, \\ (-\epsilon\epsilon - m)f_4 - \epsilon p_z f_2 + i(eB)^{1/2}(\xi - d/d\xi)f_1 &= 0. \end{aligned} \quad (6.10)$$

Operating on the first of these equations with $(\xi - d/d\xi)$ and on the second with $(\xi + d/d\xi)$ gives

$$\begin{aligned} \left[\frac{d^2}{d\xi^2} + \frac{\epsilon^2 - m^2 - p_z^2}{eB} - (\xi^2 + 1) \right] f_{1,3} &= 0, \\ \left[\frac{d^2}{d\xi^2} + \frac{\epsilon^2 - m^2 - p_z^2}{eB} - (\xi^2 - 1) \right] f_{2,4} &= 0. \end{aligned} \quad (6.11)$$

Equations (6.11) are of the same form as Schrödinger's equation for a simple harmonic oscillator, the solution of which is given in any elementary textbook on quantum mechanics. The differential equations (6.11) have normalizable solutions only if the constant

$$n = \frac{\varepsilon^2 - m^2 - p_z^2}{eB} \quad (6.12)$$

in (6.11) has integral values. Specifically, let $\ell = 0, 1, 2, \dots$ be the integer; then there are normalizable solutions only for

$$2n \mp 1 = 2\ell + 1. \quad (6.13)$$

The normalized solutions are the harmonic oscillator wave functions

$$v_\ell(\xi) = \frac{1}{(\sqrt{\pi}2^\ell \ell!)^{1/2}} H_\ell(\xi) e^{-\xi^2/2}, \quad (6.14)$$

where H_ℓ is a hermite polynomial. The solutions give

$$f(x) = \begin{pmatrix} C_1 v_{n-1}(\xi) \\ C_2 v_n(\xi) \\ C_3 v_{n-1}(\xi) \\ C_4 v_n(\xi) \end{pmatrix}, \quad (6.15)$$

where C_1, \dots, C_4 are constants.

6.3 Spin eigenfunctions

The constants C_1, \dots, C_4 in (6.15) are to be determined by constructing eigenfunctions of specific spin operators, and normalizing the solutions. The procedure is closely analogous to that for the unmagnetized case. This follows from the properties

$$\begin{aligned} \left(\xi + \frac{d}{d\xi} \right) v_n(\xi) &= \sqrt{2n} v_{n-1}(\xi), \\ \left(\xi - \frac{d}{d\xi} \right) v_n(\xi) &= \sqrt{2(n+1)} v_{n+1}(\xi), \end{aligned} \quad (6.16)$$

of the harmonic oscillator wave functions implying that the wave functions may be eliminated when (6.15) is inserted in (6.11). Thus one requires

$$\begin{pmatrix} -\varepsilon\varepsilon_q + m & 0 & \varepsilon p_z & -ip_n \\ 0 & -\varepsilon\varepsilon_q + m & ip_n & -\varepsilon p_z \\ \varepsilon p_z & -ip_n & -\varepsilon\varepsilon_q - m & 0 \\ ip_n & -\varepsilon p_z & 0 & -\varepsilon\varepsilon_q - m \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = 0, \quad (6.17)$$

with

$$p_n = (2neB)^{1/2} \quad \varepsilon_q = (m^2 + p_z^2 + 2neB)^{1/2}, \quad (6.18)$$

where q denotes the quantum numbers collectively.

Now one may proceed as in the derivation of the solution (??) in the unmagnetized case. One finds four different solutions from the first two columns for $\epsilon = \pm$:

$$\begin{aligned} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} &= \frac{1}{\sqrt{2\varepsilon\varepsilon_q(\varepsilon\varepsilon_q + m)V}} \begin{pmatrix} \varepsilon\varepsilon_q + m \\ 0 \\ \varepsilon p_z \\ ip_n \end{pmatrix}, \\ \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} &= \frac{1}{\sqrt{2\varepsilon\varepsilon_q(\varepsilon\varepsilon_q + m)V}} \begin{pmatrix} 0 \\ \varepsilon\varepsilon_q + m \\ -ip_n \\ -\varepsilon p_z \end{pmatrix}, \end{aligned} \quad (6.19)$$

where the normalization is to one particle in the volume V . Let these be labeled as the $s = +1$ and $s = -1$ solutions respectively. Our four solutions (denoted by $\Psi_q^\epsilon(t, \mathbf{x})$ with q denoting p_z, n, s) are then

$$\begin{aligned} \Psi_{q+}^\epsilon(t, \mathbf{x}) &= \frac{\exp[-i\epsilon\varepsilon t + i\epsilon p_y y + i\epsilon p_z z]}{\sqrt{2\varepsilon\varepsilon_q(\varepsilon\varepsilon_q + m)V}} \begin{pmatrix} (\varepsilon\varepsilon_q + m)v_{n-1}(\xi) \\ 0 \\ \varepsilon p_z v_{n-1}(\xi) \\ ip_n v_n(\xi) \end{pmatrix}, \\ \Psi_{q-}^\epsilon(t, \mathbf{x}) &= \frac{\exp[-i\epsilon\varepsilon t + i\epsilon p_y y + i\epsilon p_z z]}{\sqrt{2\varepsilon\varepsilon_q(\varepsilon\varepsilon_q + m)V}} \begin{pmatrix} 0 \\ (\varepsilon\varepsilon_q + m)v_n(\xi) \\ -ip_n v_{n-1}(\xi) \\ -\varepsilon p_z v_n(\xi) \end{pmatrix}, \end{aligned} \quad (6.20)$$

where $q+$ and $q-$ denote the quantum numbers with $s = +1$ and $s = -1$ respectively.

6.4 Normalization and orthogonality relations

The orthogonality relation between the states may be written in the general form

$$\int d^3\mathbf{x} [\Psi_q^\epsilon(t, \mathbf{x})]^\dagger \Psi_{q'}^{\epsilon'}(t, \mathbf{x}) = \delta^{\epsilon\epsilon'} \delta_{qq'}, \quad (6.21)$$

where q and q' include s and s' . The completeness relation for the states is

$$\sum_q \Psi_q(t, \mathbf{x}) [\Psi_q(t, \mathbf{x}')]^\dagger = \delta^3(\mathbf{x} - \mathbf{x}'). \quad (6.22)$$

The explicit forms of these relations in the present case involves the interpretation of the sum and the δ -function involving the state q and q' .

The normalization condition may be understood by considering the case where the particle is confined to a large but finite box. Let the sides of the box be of length L_x, L_y, L_z in the x, y, z directions respectively. The eigenvalues p_y and p_z are then discrete rather than continuous, specifically one has $p_y = n_y 2\pi/L_y$ and $p_z = n_z 2\pi/L_z$ with $n_y, n_z = 0, \pm 1, \pm 2, \dots$. The sum over states is then trivial, involving sums over these discrete eigenvalues (as well as over n and s). The sums may be approximated by integrals in the limit where the sides of the box becomes arbitrarily large. Then one has

$$\sum_{n_y, n_z = -\infty}^{\infty} \rightarrow L_y L_z \int \frac{dp_y}{2\pi} \int \frac{dp_z}{2\pi}.$$

Thus the sum over states becomes

$$\sum_q = \sum_{s=\pm} \sum_{n=0}^{\infty} L_y L_z \int \frac{dp_y}{2\pi} \int \frac{dp_z}{2\pi}. \quad (6.23)$$

Similarly the δ function expressing orthogonality of the states becomes

$$\delta_{qq'} = \delta_{ss'} \delta_{nn'} \frac{2\pi}{L_y} \delta(p_y - p'_y) \frac{2\pi}{L_z} \delta(p_z - p'_z). \quad (6.24)$$

6.5 Center of gyration

The variable p_y is related to the center of gyration of the particle through (6.9). The limits of integration of p_y are such that p_y is less than eBL_x , where L_x is the range of x . In many cases one is not interested in the spatial distributions of the centers of gyration, which are assumed homogeneous. One then wishes to perform the sum or integral explicitly. In such cases, (6.23) may be further replaced by

$$\sum_q = \sum_{s=\pm} \sum_{n=0}^{\infty} \frac{eBV}{2\pi} \int \frac{dp_z}{2\pi}, \quad (6.25)$$

with $V = L_x L_y L_z$.

Note that the range L_x of x can be fixed in a natural way (although one is not compelled to do so). This follows from the fact that eB has the dimensions (in natural units) of an inverse area, and hence $(eB)^{-1/2}$ is a natural length. Thus the natural normalization corresponds to

$$L_x = \frac{1}{(eB)^{1/2}}, \quad \text{or} \quad V = \frac{L_y L_z}{(eB)^{1/2}}. \quad (6.26)$$

The description of the center of gyration is gauge-dependent. If one chooses the gauge (6.3) in place of (6.2), then an analogous discussion to

the one given here involves a quantum number p_x which is related to the y -coordinate of the center of gyration. In terms of the “cylindrical” gauge (6.4) there is a “radial” quantum number related to the position of the center of gyration.

6.6 Magnetic moment of the electron

It is apparent from (6.13) and the solutions (6.20) that one has

$$n = \ell + \frac{1}{2}(1 + s). \quad (6.27)$$

The energy involves n in the form

$$\varepsilon_q = [m^2 + p_z^2 + 2neB]^{1/2}, \quad (6.28)$$

and separation (6.27) may be regarded as a separation into an orbital part described by ℓ and a spin part. The orbital part describes the perpendicular motion, which is simple harmonic motion. (Circular motion is simple harmonic motion.) The energy states of the simple harmonic oscillator correspond to $(\ell + \frac{1}{2})$ times $\hbar\Omega_0$, where Ω_0 ($\Omega_0 = eB/m$ here) is the frequency of the oscillator. The remaining part corresponds to an energy $\frac{1}{2}s\hbar\Omega_0$ which may be interpreted as a contribution $\boldsymbol{\mu} \cdot \mathbf{B}$, where $\boldsymbol{\mu}$ is the magnetic moment. Dirac’s theory thus predicts a magnetic moment

$$\mu_B = \frac{\hbar e}{2m}. \quad (6.29)$$

In the Pauli-Schrödinger theory the gyromagnetic ratio of the electron, that is the ratio of the magnetic moment to the spin, has to be specified. Dirac’s theory correctly predicts that the gyromagnetic ratio of the electron has the correct value of two, and this was one of the initial major successes of Dirac’s theory. (Radiative corrections in QED correctly predict small difference from two.)

6.7 Magnetic moment tensor

A covariant version of the foregoing discussion enables one to identify a magnetic moment 4-tensor for the Dirac theory. This derivation proceeds as follows.

The minimal coupling replacement for including the effect of an electromagnetic field in *any* covariant wave equation is equivalent to the replacement

$$\partial^\mu \rightarrow D^\mu = \partial^\mu + iqA^\mu, \quad (6.30)$$

with $q = -e$ for the Dirac theory. Then Dirac’s equation becomes

$$(i\cancel{\partial} + e\cancel{A} - m)\Psi = 0. \quad (6.31)$$

On introducing the ansatz

$$\Psi = (i\cancel{D} + e\mathbf{A} + m)\chi, \quad (6.32)$$

Dirac's equation becomes

$$(i\cancel{D} - m)(i\cancel{D} + m)\chi = 0. \quad (6.33)$$

Then using

$$\gamma^\mu \gamma^\nu D_\mu D_\nu = D^\mu D_\mu + \frac{1}{4}[\gamma^\mu, \gamma^\nu][D_\mu, D_\nu]$$

and

$$[D_\mu, D_\nu] = -ieF_{\mu\nu}, \quad S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu],$$

one finds

$$(D^\mu D_\mu + m^2 - eS^{\mu\nu}F_{\mu\nu})\chi = 0. \quad (6.34)$$

The term $-eS^{\mu\nu}F_{\mu\nu}$ in (6.34) contains the magnetic moment interaction term in covariant form. One has

$$S^{\mu\nu} = \frac{1}{2} \begin{pmatrix} 0 & i\alpha_x & i\alpha_y & i\alpha_z \\ -i\alpha_x & 0 & \sigma_z & -\sigma_y \\ -i\alpha_y & -\sigma_z & 0 & \sigma_x \\ -i\alpha_z & \sigma_y & -\sigma_x & 0 \end{pmatrix}, \quad (6.35)$$

and hence

$$S^{\mu\nu}F_{\mu\nu} = i\boldsymbol{\alpha} \cdot \mathbf{E} - \boldsymbol{\sigma} \cdot \mathbf{B}. \quad (6.36)$$

The $\boldsymbol{\sigma} \cdot \mathbf{B}$ term in (6.34) with (6.36) reproduces Pauli term (1.20) which is introduced into the Schrödinger theory in an *ad hoc* manner.

The actual definition of the magnetic moment operator involves further manipulations of the spin 4-tensor $S^{\mu\nu}$, as outlined in Appendix C.