Chapter 9

Formal Aspects of the Dirac Equation

Certain properties of the Dirac matrices are needed both for formal purposes and for performing detailed calculations. In the "Dirac algebra" discussed here, equations are matrix equation and the components of the matrices are implicit. The notation is similar to that of the Pauli matrices in nonrelativistic quantum mechanics, where σ_x, σ_y and σ_z denote 2×2 matrices, whose matrix components are not labeled explicitly. In "spinor algebra" one does label the matrix components explicitly, but this is not done here.

9.1 Independent 4×4 matrices

The Dirac matrices γ^μ may be represented by 4×4 matrices which have either real or imaginary entries. Thus the matrices are not intrinsically complex, and the algebra of the matrices is equivalent to that of real 4×4 matrices. There are 16 independent such matrices.

A suitable choice for the 16 matrices involves introducing γ^5, defined here by

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{9.1}$$

In the literature there are several different definitions of γ^5, involving replacing $-i$ by i or ± 1 in (9.1). One finds that γ^5 satisfies the following relations:

$$\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0, \qquad (\gamma^5)^2 = 1, \qquad (\gamma^5)^\dagger = \gamma^5. \tag{9.2}$$

The standard representation of γ^5 is

$$\gamma^5 = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{9.3}$$

A convenient choice of 16 independent matrices consists of the set

$$\gamma^A = \Big[1, \quad \gamma^\mu, \quad S^{\mu\nu}, \quad \gamma^\mu\gamma^5, \quad \gamma^5\Big], \tag{9.4}$$

where A runs from 1 to 16, and with

$$S^{\mu\nu} = \frac{i}{4}\,[\gamma^\mu, \gamma^\nu]. \tag{9.5}$$

This choice involves a scalar and a pseudo scalar (1, γ^5), a 4-vector and a pseudo 4-vector (γ^μ, $\gamma^\mu\gamma^5$) and an antisymmetric second rank 4-tensor ($S^{\mu\nu}$), these having 1, 1, 4, 4, and 6 components respectively.

In principle any product or sum of products of γ-matrices may be re-expressed as a sum of terms involving only these 16 independent matrices. For example one finds

$$\begin{aligned}
\gamma^\mu\gamma^\nu &= g^{\mu\nu} - 2iS^{\mu\nu},\\
\gamma^\mu\gamma^\nu\gamma^\rho &= g^{\mu\nu}\gamma^\rho - g^{\mu\rho}\gamma^\nu + g^{\nu\rho}\gamma^\mu - i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5,\\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - 2i\Big\{g^{\mu\nu}S^{\rho\sigma} - g^{\mu\rho}S^{\nu\sigma} + g^{\mu\sigma}S^{\nu\rho}\\
&\quad + S^{\mu\nu}g^{\rho\sigma} - S^{\mu\rho}g^{\nu\sigma} + S^{\mu\sigma}g^{\nu\rho}\Big\} + i\epsilon^{\mu\nu\rho\sigma}\gamma^5,
\end{aligned} \tag{9.6}$$

and so on. Also one has

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}S_{\rho\sigma} &= 2i\,S^{\mu\nu}\,\gamma^5,\\
\epsilon^{\mu\nu\rho\sigma}\gamma_\nu\gamma_\rho\gamma_\sigma &= -\,i\,3!\,\gamma^\mu\gamma^5\\
\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma &= -\,i\,4!\,\gamma^5,
\end{aligned} \tag{9.7}$$

Note that terms in (9.6) and (9.7) that do not contain a γ-matrix explicitly are implicitly multiplied by the unit Dirac matrix.

9.2 Traces of products of γ-matrices

The trace (Tr) of the unit 4×4 tensor is equal to four, and the traces of the remaining 15 matrices (9.4) are all zero. The traces of products of γs are important in detailed calculations in QED.

Consider

$$T^{\alpha_1\alpha_2\ldots\alpha_n} = \mathrm{Tr}\left(\gamma^{\alpha_1}\gamma^{\alpha_2}\ldots\gamma^{\alpha_n}\right). \tag{9.8}$$

One has the following results:

(i) The trace of an odd number of γs is zero,

$$T^{\alpha_1\alpha_2\ldots\alpha_n} = 0, \qquad \text{for } n \text{ odd}. \tag{9.9}$$

A derivation of this result involves premultiplying by $\gamma^5\gamma^5$ inside the trace in (9.8). According to (9.2) this is equivalent to multiplying by unity. One may move one γ^5 to the end of the other γs by consecutive interchanges; an odd number of changes in sign occurs in view of the anticommutation relations (9.2). The other γ^5 may then be moved directly to the end using the invariance of the trace under cyclic permutations. Then using $\gamma^5\gamma^5 = 1$, the trace is equal to minus itself, for n odd, completing the proof of (9.9).

(ii) The trace of $\gamma^\mu\gamma^\nu$ follows directly from the definition (**??**) and the invariance of the trace under cyclic permutations:

$$T^{\mu\nu} = 4g^{\mu\nu}. \tag{9.10}$$

(iii) To evaluate $T^{\mu\nu\rho\sigma}$ one uses two results. The first follows from $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2g^{\mu\nu}$, which implies $T^{\mu\nu\rho\sigma} = -T^{\nu\mu\rho\sigma} + 2g^{\mu\nu}T^{\rho\sigma}$. Similarly one has $T^{\nu\mu\rho\sigma} = -T^{\mu\rho\nu\sigma} + 2g^{\nu\rho}T^{\mu\sigma}$, and one uses this property to move the μ index from first to last. Second, cyclic permutation under the trace implies $T^{\mu\nu\rho\sigma} = T^{\nu\rho\sigma\mu}$. On using (9.10), the resulting identity implies

$$T^{\mu\nu\rho\sigma} = 4\Big[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\Big]. \tag{9.11}$$

(iv) The values of $T^{\alpha_1\alpha_2\ldots\alpha_n}$ for $n = 6$ may be derived sequentially by the same procedure, using (9.11), giving

$$\begin{aligned} T^{\mu\nu\rho\sigma\alpha\beta} = 4\Big[&g^{\mu\nu}T^{\rho\sigma\alpha\beta} - g^{\mu\rho}T^{\nu\sigma\alpha\beta} \\ &+ g^{\mu\sigma}T^{\nu\rho\alpha\beta} - g^{\mu\alpha}T^{\nu\rho\sigma\alpha} + g^{\mu\beta}T^{\nu\rho\sigma\alpha}\Big]. \end{aligned} \tag{9.12}$$

Values for higher even n are evaluated in the same manner.

Another set of relations involves contractions over γ-matrices:

$$\begin{aligned} \gamma^\mu \not{a} \gamma_\mu &= -2\not{a}, \\ \gamma^\mu \not{a}\not{b} \gamma_\mu &= 4ab, \\ \gamma^\mu \not{a}\not{b}\not{c} \gamma_\mu &= -2\not{c}\not{b}\not{a}, \\ \gamma^\mu \not{a}\not{b}\not{c}\not{d} \gamma_\mu &= 2(\not{c}\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}). \end{aligned} \tag{9.13}$$

9.3 Other representations of the Dirac matrices

So far only the standard representation has been mentioned. An arbitrary representation may be obtained by introducing a 4×4 transformation matrix S with unit determinant:

$$\det S = 1. \tag{9.14}$$

The wavefunctions and the γ matrices are transformed according to

$$\Psi'(x) = S\,\Psi(x), \qquad \Psi'^\dagger(x) = \Psi^\dagger(x)S^{-1}, \qquad \gamma'^A = S\gamma^A S^{-1}. \tag{9.15}$$

Besides allowing changes in representation, such transformations need to be considered when making Lorentz transformations and other transformations such as the parity, time reversal and charge conjugation transformations.

One alternative that is often used is the spinor representation. In this representation, the wavefunction is written

$$\Psi(x) = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{9.16}$$

where the dependence of the spinors ξ and η on x is implicit. The Dirac equation becomes the two coupled equations

$$(p^0 + \boldsymbol{p}\cdot\boldsymbol{\sigma})\eta = m\xi, \qquad (p^0 - \boldsymbol{p}\cdot\boldsymbol{\sigma})\xi = m\eta. \tag{9.17}$$

Thus in the *spinor* representation one has

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{9.18}$$

A suitable choice for the transformation matrix is

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}. \tag{9.19}$$

In the *Majorana* representation all the matrices are real. This may be obtained from the standard representation using the transformation matrix

$$S = \frac{1}{\sqrt{2}} (\alpha_y + \beta). \tag{9.20}$$

9.4 Lorentz transformation of the Dirac equation

The Klein Gordon equation (**??**) is in a manifestly covariant form because the wavefunction is a Lorentz invariant for spin 0. The Dirac equation (**??**) is in a covariant notation, but before the form can be said to be covariant it is necessary to specify how the wavefunction and the γ matrices transform.

An arbitrary Lorentz transformation is of the form

$$x^{\mu'} = L^{\mu'}{}_{\mu}\, x^{\mu} + a^{\mu'}. \tag{9.21}$$

One has

$$\partial_{\mu'} = L^{\mu}{}_{\mu'} \partial_{\mu}. \tag{9.22}$$

On applying the transformation (9.21) to the γ matrices, it is obvious that one has

$$\gamma^{\mu'} \partial_{\mu'} = L^{\mu'}{}_{\mu} \gamma^{\mu} L^{\nu}{}_{\mu'} \partial_{\nu} = \gamma^{\mu} \partial_{\mu}. \tag{9.23}$$

However the new γs are not in the same representation as the old γs.

To restore the original representation one needs to introduce a transformation matrix $S(L)$ in the 4-dimensional spin space. The transformed wavefunction is

$$\Psi'(x') = S(L)\,\Psi(L^{-1}x'), \tag{9.24}$$

where $x = L^{-1}x'$ denotes the inverse of (9.21). The transformation of the γ matrices is awkward to write. In words, $S(L)$ needs to be such that the new

$$S(L)\,\gamma^{\mu'} S^{-1}(L)$$

are term by term equal to the old γ^μ. That is, $S(L)$ must be such that the new $\gamma^{0'}$ transforms into the original γ^0, and so on.

Thus one requires two different sets of transformation matrices in discussing Lorentz transformations of the Dirac wavefunction and the Dirac matrices. One set consists of the the $L^{\mu'}{}_{\mu}$ that operate in the 4-dimensional space-time. The other set consists of the $S(L)$ that operate in the 4-dimensional spin space. Before considering the relation between these it is appropriate to discuss the generators of the transformations.

9.5 Explicit forms for the transformation matrices $S(L)$

A Lorentz transformation may be described in terms of a 4-dimensional rotation involving an antisymmetric tensor $\omega_{\alpha\beta}$. The space component of this tensor are related to ordinary rotation, with (to within a sign) ω_{12} being the angle through with the rotation about the 3-axis is made. The other components are related to boosts, as described below, with ω_{01} being related to a boost along the 1-axis.

The transformation matrix for a finite 4-dimensional rotation is

$$S = \exp\left[i\tfrac{1}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right]. \tag{9.25}$$

For a boost along the i-axis by a speed v, all the components of $\omega_{\alpha\beta}$ are zero except $\omega_{0i} = -\omega_{0i} = \zeta$, with

$$v = \tanh\zeta, \qquad p = m\sinh\zeta, \qquad \gamma = \cosh\zeta, \tag{9.26}$$

and for a rotation through an angle ϕ about the i-axis, all the components of $\omega_{\alpha\beta}$ are zero except $\omega_{jk} = -\omega_{kj} = \phi$, where ijk is an even permutation of 123.

In the standard representation the explicit forms for a boost along a unit vector $\boldsymbol{n}$ and a rotation about a unit vector $\boldsymbol{n}$ are

$$S = \exp[-\tfrac{1}{2}\boldsymbol{\alpha}\cdot\boldsymbol{n}\zeta] = \cosh\tfrac{1}{2}\zeta - \boldsymbol{\alpha}\cdot\boldsymbol{n}\sinh\tfrac{1}{2}\zeta, \tag{9.27}$$

$$S = \exp[\tfrac{i}{2}\boldsymbol{\sigma}\cdot\boldsymbol{n}\phi] = \cos\tfrac{1}{2}\phi + i\boldsymbol{\sigma}\cdot\boldsymbol{n}\sin\tfrac{1}{2}\phi, \tag{9.28}$$

respectively.

9.6 Parity, time-reversal and charge-conjugation

Besides the continuous set of Lorentz transformations, there are also discrete transformations. These include the parity, time reversal and charge conjugation transformations. Parity corresponds to a reflection of the coordinate axes $\boldsymbol{x} \to -\boldsymbol{x}$, time reversal corresponds to $t \to -t$, and charge conjugation to the reversal of the roles of particles and antiparticles. It is convenient to define *classical* operators P_0 and T_0 which apply to any classical field:

$$P_0 \psi(t, \boldsymbol{x}) = \psi(t, -\boldsymbol{x}), \quad T_0 \psi(t, \boldsymbol{x}) = \psi(-t, \boldsymbol{x}). \tag{9.29}$$

The parity transformation for the Dirac equation involves a unitary operator U_P such that one has

$$\Psi_P(t, \boldsymbol{x}) = U_P \Psi(t, -\boldsymbol{x}). \tag{9.30}$$

To within an arbitrary phase factor η_P, one finds

$$U_P = i\eta_P P_0 \gamma^0. \tag{9.31}$$

For both time reversal and charge conjugation, one needs to consider the complex conjugate $\Psi^*(x)$ of the wave function. If one writes

$$\Psi^*(x) = K\Psi(x), \tag{9.32}$$

then K is anti-unitary satisfying

$$K^2 = 1, \quad K^\dagger = -K^{-1} = -K. \tag{9.33}$$

The time reversal transformation is satisfied as follows:

$$\Psi_T(t, \boldsymbol{x}) = T\big(\Psi(-t, \boldsymbol{x})\big)^T, \quad T = -\gamma^1\gamma^3 K T_0. \tag{9.34}$$

Charge conjugation involves the transpose of the Dirac wavefunction. It is satisfied for

$$\Psi_C(x) = C\big(\Psi(x)\big)^T, \quad C = i\gamma^2 K. \tag{9.35}$$