

Relativistic Quantum Mechanics
Lecture notes for Physics IV

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Preface

In the lecture course in Relativistic Quantum Mechanics for 2006 I propose to place more emphasis on quantum electrodynamics (QED) than on relativistic quantum mechanics *per se*. In particular, I do not intend to cover the material in chapters 6–8 and chapter 18. I have left these chapters in the notes (from an earlier version) to avoid introducing errors in cross-referencing between chapters. Also, several chapters cover material that is primarily for reference, in particular, chapters 2, 5 and 14, and also parts of chapter 1.

Thus the core of the course consists of chapters 1, 3, 4, 9–13, 15–17, 19 and 20. I hope there will be time to cover additional material, which may be from the other four chapters, or from further chapters that I have not yet included. There are also appendices referred to in the text that have yet to be included.

This version is dated the first day of lectures, and I intend to update the notes, including adding additional chapters and appendices during the course.

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