Exercise Sets

Relativistic Quantum Mechanics Set 1

1.1 Maxwell's equations may be written in the 4-vector form

$$\partial^\mu F^{\nu\rho}(x) + \partial^\rho F^{\mu\nu}(x) + \partial^\nu F^{\rho\mu}(x) = 0, \tag{E1}$$

$$\partial_\mu F^{\mu\nu}(x) = \mu_0 J^\nu(x). \tag{E2}$$

Show explicitly how these equations reproduce the 3-vector familiar form of Maxwell's equation.

1.2 Show how

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \tag{E3}$$

reduces to the familiar relation between $\boldsymbol{E}$, $\boldsymbol{B}$ and the vector and scalar potentials $\boldsymbol{A}$, ϕ:

$$\boldsymbol{E} = -\mathrm{grad}\,\phi - \frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \mathrm{curl}\,\boldsymbol{A}. \tag{E4}$$

Hints: Use natural units in which one has

$$x^\mu = [t, \boldsymbol{x}], \quad J^\mu = [\rho, \boldsymbol{J}], \quad A^\mu = [\phi, \boldsymbol{A}], \tag{E5}$$

$$\partial_\mu = [\partial/\partial t, \partial/\partial \boldsymbol{x}], \quad F^{\mu\nu}(x) = -F^{\nu\mu}(x), \tag{E6}$$

$$F^{\mu\nu}(x) := [\boldsymbol{E}, \boldsymbol{B}] = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{E7}$$

1.3 The permutation symbol $\epsilon^{\mu\nu\rho\sigma}$ is completely antisymmetric in all its indices, and ϵ^{0123} is chosen equal to unity.
(a) Why is ϵ_{0123} necessarily equal to minus ϵ^{0123}?
(b) What is the numerical value of the invariant $\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma}$?

1.4 (a) Apply a Lorentz boost, specifically

$$L^{\mu'}{}_{\mu} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{E8}$$

to the Maxwell tensor $F^{\mu\nu}$ and identify the components of the electric field, $\boldsymbol{E}'$, and the magnetic field, $\boldsymbol{B}'$, in the moving frame.
(b) Apply the same boost to the dual of the Maxwell tensor, and show that the same expressions for $\boldsymbol{E}'$ and $\boldsymbol{B}'$ result.

1.5 (a) Show that the $\nu = 0$ component of

$$\partial_\mu \, \Theta^{\mu\nu} = J_\alpha F^{\alpha\nu}, \tag{E9}$$

$$\Theta^{\mu\nu} = \varepsilon_0 (F^{\mu}{}_{\alpha} \, F^{\alpha\nu} + \tfrac{1}{4} \, g^{\mu\nu} \, F_{\alpha\beta} \, F^{\alpha\beta}) \tag{E10}$$

gives the continuity equation for electromagnetic energy,

$$\frac{\partial W}{\partial t} + \operatorname{div} \boldsymbol{F} = -\boldsymbol{J} \cdot \boldsymbol{E}. \tag{E11}$$

with

$$W = \varepsilon_0 \boldsymbol{E}^2/2 + \boldsymbol{B}^2/2\mu_0, \qquad \boldsymbol{F} = \boldsymbol{E} \times \boldsymbol{B}/\mu_0,$$
$$\boldsymbol{P} = \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}, \qquad \boldsymbol{T} = W\mathbf{1} - \varepsilon_0 \boldsymbol{E}\boldsymbol{E} - \boldsymbol{B}\boldsymbol{B}/\mu_0, \tag{E12}$$

(b) For $\nu = i$, (E9) gives the equation of continuity for momentum. Identify this equation in terms of the momentum density, $\boldsymbol{P}$, and the stress 3-tensor, $\boldsymbol{T}$.

1.6 How many independent invariants can be constructed from
(a) two different 4-vectors p and k,
(b) the Maxwell tensor alone,
(c) the Maxwell tensor and one 4-vector,
(d) the Maxwell tensor and two different 4-vectors?
Comment: Part (d) is difficult: I think the answer is 14. See how many you can identify.

Relativistic Quantum Mechanics Set 2

2.1 Show that the Dirac matrices in the standard representation

$$\gamma^0 = \rho_z, \qquad \boldsymbol{\gamma} = i\rho_y \boldsymbol{\sigma} \tag{E13}$$

satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \tag{E14}$$

2.2 Construct the matrices $\not{P} \pm m$ in the standard representation, and show that the matrices satisfy

$$(\not{P} + m)(\not{P} - m) = P^2 - m^2. \tag{E15}$$

2.3 In the *spinor representation* the Dirac matrices have the following forms:

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{E16}$$

(a) Construct γ^μ in this representation and show that it satisfies

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \tag{E17}$$

(b) Show that if the Dirac wavefunction is written in the form

$$\Psi(x) = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{E18}$$

where the ξ and η are spinors in two dimensions, then the Dirac equation becomes the two coupled equations

$$(p^0 + \boldsymbol{p} \cdot \boldsymbol{\sigma})\eta = m\xi, \qquad (p^0 - \boldsymbol{p} \cdot \boldsymbol{\sigma})\xi = m\eta. \tag{E19}$$

(c) Show that both ξ and η satisfy the Klein-Gordon equation.

2.4 With the definition

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{E20}$$

use $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$ to prove

$$\gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = 0, \qquad (\gamma^5)^2 = 1. \tag{E21}$$

Relativistic Quantum Mechanics Set 3

3.1 In the Klein paradox, electrons are envisaged as approaching a potential step V, as illustrated in Figure 1.1.

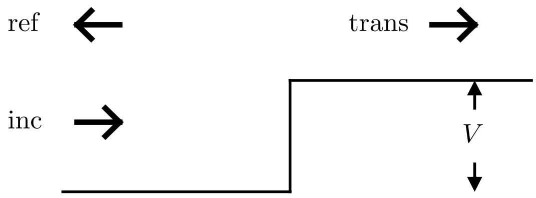

Figure 1.1. An electron is incident from the left onto a potential jump of magnitude V. The wavefunction has incident (inc), reflected (ref) and transmitted (trans) components.

The incident wavefunction is

$$\Psi_{\text{inc}}(x) = \frac{e^{-i\varepsilon t + ip_z z}}{\sqrt{2\varepsilon(\varepsilon + m)}} \begin{pmatrix} \varepsilon + m \\ 0 \\ p_z \\ 0 \end{pmatrix}. \tag{E22}$$

Show that the reflected and transmitted wavefunctions have no component with the opposite spin. (Specifically, show that their wavefunctions have zeros in their columns where $\Psi_{\text{inc}}(x)$ has zeros, i.e., in the second and fourth entries.)

3.2 The Dirac Hamiltonian is

$$\hat{H} = \boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}} + \beta m. \tag{E23}$$

(a) Show that one has

$$\hat{H}^2 = |\hat{\boldsymbol{p}}|^2 + m^2 = \varepsilon^2. \tag{E24}$$

(b) Show that the operator

$$\hat{U} = \frac{\varepsilon + \beta \hat{H}}{\sqrt{2\varepsilon(\varepsilon + m)}} \tag{E25}$$

is unitary.
(c) Establish the identity

$$\hat{H}\beta + \beta \hat{H} = 2m. \tag{E26}$$

(d) Use the operator to make a change of representation, and show that in the new representation the Hamiltonian $\hat{H}'$ reduces to

$$\hat{H}' = \hat{U}\hat{H}\hat{U}^{\dagger} = \beta\varepsilon. \tag{E27}$$

Remark $\hat{H}' = \beta\varepsilon$ is effectively the square root of the relation $\hat{H}^2 = \varepsilon^2$ in this particlular representation.

3.3 Plane wave solutions of Dirac's equation may be written in terms of the functions $u_s(\boldsymbol{p})$

$$u_+(\boldsymbol{p}) = \frac{1}{\sqrt{(\varepsilon+m)}}\begin{pmatrix}\varepsilon+m\\0\\p_z\\p_+\end{pmatrix}, \qquad u_-(\boldsymbol{p}) = \frac{1}{\sqrt{(\varepsilon+m)}}\begin{pmatrix}0\\\varepsilon+m\\p_-\\-p_z\end{pmatrix}, \tag{E28}$$

and

$$v_+(\boldsymbol{p}) = \frac{1}{\sqrt{(\varepsilon-m)}}\begin{pmatrix}\varepsilon-m\\0\\p_z\\p_+\end{pmatrix}, \qquad v_-(\boldsymbol{p}) = \frac{1}{\sqrt{(\varepsilon-m)}}\begin{pmatrix}0\\\varepsilon-m\\p_-\\-p_z\end{pmatrix}, \tag{E29}$$

(a) Show that these functions satisfy the relations

$$\begin{aligned}
&\bar{u}_s(\boldsymbol{p})u_{s'}(\boldsymbol{p}) = \; -\bar{v}_s(\boldsymbol{p})v_{s'}(\boldsymbol{p}) = 2m\delta_{ss'},\\
&u_s^{\dagger}(\boldsymbol{p})u_{s'}(\boldsymbol{p}) = v_s^{\dagger}(\boldsymbol{p})v_{s'}(\boldsymbol{p}) = 2\varepsilon\delta_{ss'},\\
&\bar{u}_s(\boldsymbol{p})v_{s'}(\boldsymbol{p}) = \bar{v}_s(\boldsymbol{p})u_{s'}(\boldsymbol{p}) = 0.
\end{aligned} \tag{E30}$$

(b) Establish the completeness relations

$$\begin{aligned}
&\sum_{s=\pm}[u_s(\boldsymbol{p})\bar{u}_s(\boldsymbol{p}) - v_s(\boldsymbol{p})\bar{v}_s(\boldsymbol{p})] = 2m,\\
&\sum_{s=\pm} u_s(\boldsymbol{p})\bar{u}_s(\boldsymbol{p}) = \tilde{\not{p}} + m,\\
&\sum_{s=\pm} v_s(\boldsymbol{p})\bar{v}_s(\boldsymbol{p}) = \tilde{\not{p}} - m,
\end{aligned} \tag{E31}$$

with $\tilde{p} = [\varepsilon, \boldsymbol{p}]$.

(c) Identify the normalization condition that these wavefunctions satisfy.

3.4 The helicity operator is

$$\hat{h} = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}. \tag{E32}$$

(a) Show that $\hat{h}$ commutes with the Dirac Hamiltonian $\hat{H} = \boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta m$.

(b) Show that the eigenvalues of $\hat{h}$ are sh with $s = \pm$ and $h = |\boldsymbol{p}|$.
(c) Show that, on writing

$$\boldsymbol{p} = (p_\perp \cos\phi,\ p_\perp \sin\phi,\ p_z), \tag{E33}$$

that

$$\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{1}{\sqrt{2h2\varepsilon V}} \begin{pmatrix} \sqrt{\varepsilon + \epsilon m}\,\sqrt{h + \epsilon s p_z}\, e^{-i\phi/2} \\ s\epsilon\sqrt{\varepsilon + \epsilon m}\,\sqrt{h - \epsilon s p_z}\, e^{i\phi/2} \\ s\epsilon\sqrt{\varepsilon - \epsilon m}\,\sqrt{h + \epsilon s p_z}\, e^{-i\phi/2} \\ \sqrt{\varepsilon - \epsilon m}\,\sqrt{h - \epsilon s p_z}\, e^{i\phi/2} \end{pmatrix} \tag{E34}$$

are simultaneous eigenfunctions of the Hamiltonian and of the helicity operator.

Relativistic Quantum Mechanics Set 4

4.1 Establish the identities

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau\gamma^5) = 4i\epsilon^{\mu\nu\rho\tau}, \tag{E35}$$

$$\begin{aligned}&\gamma^\mu \not{a}\gamma_\mu = -2\not{a}, \qquad \gamma^\mu \not{a}\not{b}\gamma_\mu = 4ab, \qquad \gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a},\\ &\gamma^\mu \not{a}\not{b}\not{c}\not{d}\gamma_\mu = 2(\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}).\end{aligned} \tag{E36}$$

4.2 With

$$S^{\mu\nu} = \tfrac{1}{4}i[\gamma^\mu, \gamma^\nu], \quad \hat{W}^\mu = \tfrac{1}{2}\epsilon^{\mu\alpha\beta\gamma}\hat{J}_{\alpha\beta}\hat{p}_\gamma, \tag{E37}$$

$$\hat{\tilde{S}}^{\mu\nu} = \frac{1}{2m}\Big[\hat{H}\gamma^0\hat{S}^{\mu\nu} + \gamma^0\hat{S}^{\mu\nu}\hat{H}\Big], \quad \hat{\tilde{W}}^\mu = \frac{1}{2m}\Big[\hat{H}\gamma^0\hat{W}^\mu + \gamma^0\hat{W}^\mu\hat{H}\Big], \tag{E38}$$

show that one has

$$\hat{\tilde{S}}^{\mu\nu} = \frac{1}{2m}\begin{pmatrix} 0 & -\hat{d}_x & -\hat{d}_y & -\hat{d}_z \\ \hat{d}_x & 0 & \hat{\mu}_z & -\hat{\mu}_y \\ \hat{d}_y & -\hat{\mu}_z & 0 & \hat{\mu}_x \\ \hat{d}_z & \hat{\mu}_z & -\hat{\mu}_x & 0 \end{pmatrix}, \tag{E39}$$

$$\hat{\boldsymbol{d}} = -\rho_z\,\boldsymbol{\sigma}\times\hat{\boldsymbol{p}}, \qquad \hat{\boldsymbol{\mu}} = m\boldsymbol{\sigma} + \rho_y\,\boldsymbol{\sigma}\times\hat{\boldsymbol{p}}. \tag{E40}$$

4.3 The Lorentz transformation operator in the Dirac spin space for a boost is

$$S = \exp[-\tfrac{1}{2}\boldsymbol{\alpha}\cdot\boldsymbol{n}\zeta] = \cosh\tfrac{1}{2}\zeta - \boldsymbol{\alpha}\cdot\boldsymbol{n}\sinh\tfrac{1}{2}\zeta, \tag{E41}$$

where the parameters of the boost are defined by $\boldsymbol{n} = \boldsymbol{p}/|\boldsymbol{p}|$, $p = m\sinh\zeta$, $\gamma = \cosh\zeta$.

(a) Show that one has

$$S = \frac{\varepsilon + m - \boldsymbol{\alpha}\cdot\boldsymbol{p}}{[2m(\varepsilon+m)]^{1/2}}. \tag{E42}$$

(b) Apply a Lorentz tranformation from the "rest frame" K_0, where the wavefunctions for a particle at rest

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

correspond (loosely) to electrons with spin up and down, respectively, to the "laboratory frame" K in which the particle has velocity $\boldsymbol{v}$. (Note that the frame K is moving with velocity $-\boldsymbol{v}$ relative to K_0).

(c) Compare the resulting wavefunctions with the wavefunctions derived in lectures, specifically,

$$\varphi_{+}^{\epsilon}(\epsilon\boldsymbol{p}) = \frac{1}{\sqrt{2\epsilon\varepsilon(\epsilon\varepsilon+m)V}}\begin{pmatrix}\epsilon\varepsilon+m\\ 0\\ \epsilon p_z\\ \epsilon p_+\end{pmatrix},$$

$$\varphi_{-}^{\epsilon}(\epsilon\boldsymbol{p}) = \frac{1}{\sqrt{2\epsilon\varepsilon(\epsilon\varepsilon+m)V}}\begin{pmatrix}0\\ \epsilon\varepsilon+m\\ \epsilon p_-\\ -\epsilon p_z\end{pmatrix}, \qquad \text{(E43)}$$

and comment on the comparison for $\epsilon = 1$ and $p_\pm = 0$.

(d) How might one use a Lorentz transformation to construct wavefunctions (i) with $p_\pm \neq 0$ and (ii) for $\epsilon = -1$ from wavefunctions corresponding to particles at rest?

Relativistic Quantum Mechanics Set 5

5.1 The matrix element for Cerenkov emission of wave quantum with wavevector $\boldsymbol{k}$ in the mode M by an electron is

$$iM_{\rm fi} = ie\, e^*_{M\mu}(\boldsymbol{k})\, \bar{u}_{s'}(\boldsymbol{p}')\gamma^\mu u_s(\boldsymbol{p}), \tag{E44}$$

where $\boldsymbol{p}, s$ and $\boldsymbol{p}', s'$ denote the initial and final 3-momentum and spin of the electron. The transition rate per unit time is

$$w_{i\to f} = V\,(2\pi)^4\,\delta^4(p'+k-p)\,\frac{|a_M(\boldsymbol{k})M_{\rm fi}|^2}{2\varepsilon V 2\varepsilon' V}\,\frac{Vd^3\boldsymbol{p}'}{(2\pi)^3}\,\frac{Vd^3\boldsymbol{k}}{(2\pi)^3},$$

with $|a_M(\boldsymbol{k})| = [\mu_0 R_M(\boldsymbol{k})/\omega_M(\boldsymbol{k})]^{1/2}$. The *probability of Cerenkov emission* is defined by writing

$$w_{\rm i\to f} = w_M(\boldsymbol{p},\boldsymbol{k})\,(2\pi)^3\delta^3(\boldsymbol{p}'-\boldsymbol{p}+\boldsymbol{k})\,\frac{d^3\boldsymbol{p}'}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}}{(2\pi)^3}. \tag{E45}$$

(a) Summarize the argument leading to the result that for *unpolarized electrons* one has

$$\left\langle |M_{\rm fi}|^2 \right\rangle = e^2\, e^*_{M\mu}(\boldsymbol{k}) e_{M\nu}(\boldsymbol{k})\, \tfrac{1}{2}{\rm Tr}\,[(\not{p}'+m)\gamma^\mu(\not{p}+m)\gamma^\nu]\,. \tag{E46}$$

(b) Derive an explicit expression for $w_M(\boldsymbol{p},\boldsymbol{k})$.

(c) Write down the counterparts of (1) for
(i) absorption of a wave quantum by an electron,
(ii) emission of a wave quantum by a positron, and
(iii) annihilation of a pair into a wave quantum.

(d) Write down $w_{\rm i\to f}$ for each of the processes in part (c) in forms analogous to (2).

5.2 Calculate the rate of transition due to gyromagnetic emission in vacuo of an electron in the first excited state ($n=1$, $\ell=0$, $s=1$) to the ground state ($n=0$, $\ell=0$, $s=-1$) due to a spin flip. Show that this rate is smaller than that for the transition without spin flip (from $n=1$, $\ell=1$, $s=-1$ to $n=0$, $\ell=0$, $s=-1$) by a factor B/B_c. More specifically show that the rate is given by

$$R_{\rm spin\,flip} = a\,\frac{e^4B^2}{3\pi\varepsilon_0 m^3}\,\frac{B}{B_c}, \tag{E47}$$

where a is a numerical factor to be identified.
5.3 Double Compton scattering is a process that involves one electron and one photon in the initial state, and one electron and two photons in the final state

(a) Draw all the different Feynman diagrams of third order that contribute to this process.
(b) Write down the contribution to M_{fi} corresponding to one of these diagrams
(c) Identify how M_{fi} depends on the polarization vectors of the three photons, and write down the procedure used in this case for unpolarized photons.
(d) Write down the density of final states factor for this process.

Relativistic Quantum Mechanics Set 6

6.1 The Feynman diagrams for bremsstrahlung due to an electron scattered by a Coulomb field are shown in Figure 1.2.

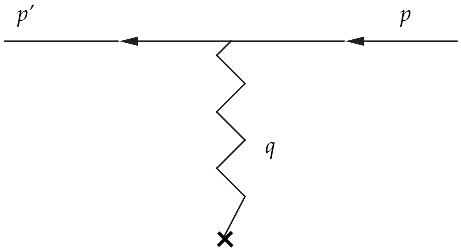

Figure 1.2. The Feynman diagram for scattering of an electron by an external field.

(a) Explain each step in the derivation of the S-matrix element for this process using the rules for Feynman diagrams and amplitudes. Specifically fill in the details of the derivation fo the following expression form

$$S_{\mathrm{fi}} = \frac{iZe^3}{\varepsilon_0|\boldsymbol{p'}+\boldsymbol{k}-\boldsymbol{p}|^2}\,\frac{2\pi\,\delta(\varepsilon'+\omega-\varepsilon)}{\sqrt{2\varepsilon' V}\,\sqrt{2\varepsilon V}}\,e^*_\mu(\boldsymbol{k})\bar{u}_{s'}(\boldsymbol{p}') \times \left[\frac{\gamma^\mu(\not{p}'+\not{k}+m)\gamma^0}{(p'+k)^2-m^2}+\frac{\gamma^0(\not{p}-\not{k}+m)\gamma^\mu}{(p-k)^2-m^2}\right]u_s(\boldsymbol{p}). \tag{E48}$$

(b) Show that for unpolarized electrons the probability of a

$$w_{\mathrm{i}\to\mathrm{f}} = \left|\frac{iZe^3}{\varepsilon_0|\boldsymbol{p'}+\boldsymbol{k}-\boldsymbol{p}|^2}\right|^2\frac{2\pi\delta(\varepsilon'+\omega-\varepsilon)}{2\varepsilon' V\,2\varepsilon V}\,|a_M(\boldsymbol{k})|^2 \times e^*_{M\mu}(\boldsymbol{k})e_{M\nu}(\boldsymbol{k})\,\tfrac{1}{2}\mathrm{Tr}[(\not{p}'+m)M^\mu(\not{p}+m)M^{\dagger\nu}], \tag{E49}$$

with

$$M^\mu := \frac{2p'^\mu+\gamma^\mu\not{k}}{2p'k}\gamma^0-\gamma^0\frac{2p^\mu-\not{k}\gamma^\mu}{2pk}. \tag{E50}$$

(c) For soft photons one has $k \ll p$ and (7) may be approximated by

$$M^\mu \approx \left(\frac{2p'^\mu}{2p'k}-\frac{2p^\mu}{2pk}\right)\gamma^0. \tag{E51}$$

Show that in this case the cross section for bremsstrahlung of transverse waves in vacuo factorizes into that for Mott scattering times an additional factor that involves the photon. In particlular show that this factor is

$$\frac{e^2}{4\pi\varepsilon_0}\,\frac{2}{\pi}\,\left(\ln\frac{\omega_{\max}}{\omega_{\min}}\right)\frac{4}{3}\beta^2\sin^2\tfrac{1}{2}\theta, \tag{E52}$$

where β is the speed of the electron, θ is the angle through which it is scattered and where $\omega_{\max}$ and $\omega_{\min}$ are frequencies at which an integral over frequency is cut off.

6.2 The formation of a pair from two photons is the inverse process corresponding to two-photon pair annihilation.

(a) By considering the change in the incoming flux from annihilation to formation of a pair, argue that the formation cross section is related to that for annihilation, cf. (20.27),

$$d\sigma_{\rm ann} = 8\pi r_0^2 \, \frac{m^2 X \, ds}{t(t-4m^2)}, \tag{E53}$$

by

$$d\sigma_{\rm form} = d\sigma_{\rm ann} \, \frac{t-4m^2}{t}. \tag{E54}$$

(b) Evaluate $(t-4m^2)/t$ in the center-of-mass frame, and show that it is equal to v^2 where v is the speed of the electron or positron.

(c) Show that in any other frame in which the photons propagate in opposite directions with energies ω_1, ω_2, one has $(t-4m^2)/t = 1 - m^2/\omega_1\omega_2$.

(d) By rewriting the result (20.28) for annihilation, and arguing concerning a difference in a factor of two due to the final particles not being identical, show that the total cross section for pair formation is

$$\sigma_{\rm form} = \tfrac{1}{2}\pi r_0^2 (1-v^2) \left\{ (3-v^4) \, \ln \frac{1+v}{1-v} - 2v(2-v^2) \right\}. \tag{E55}$$